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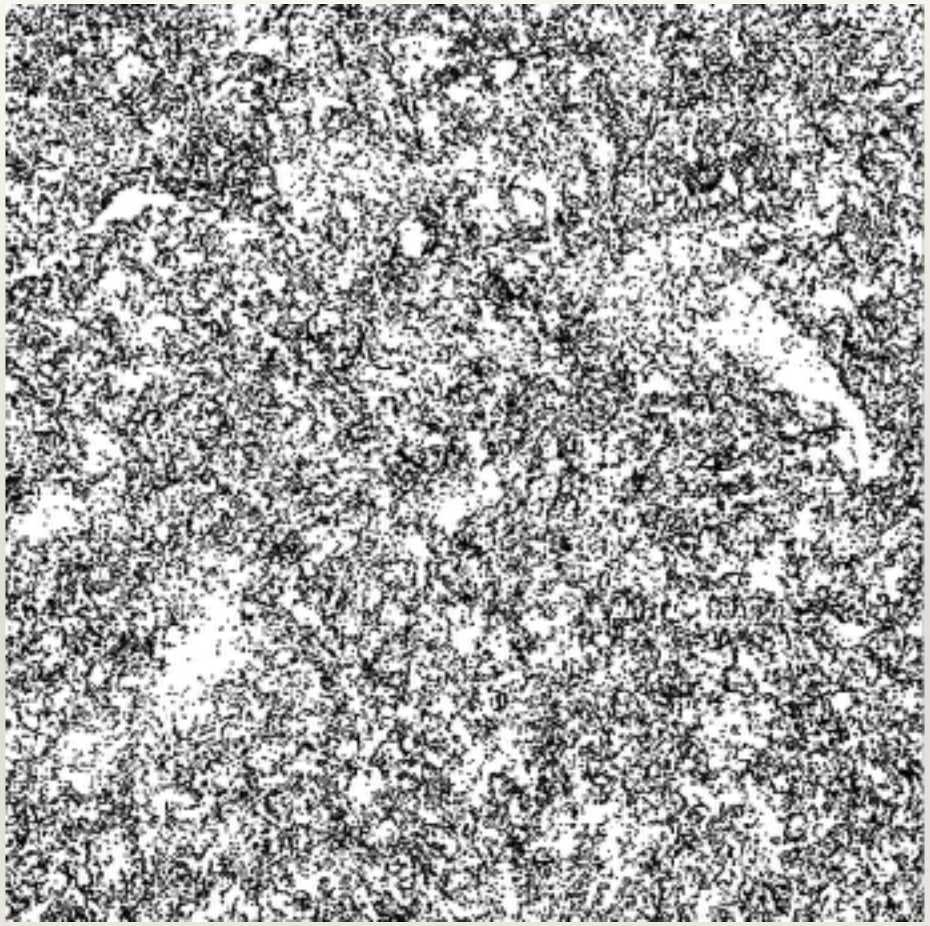
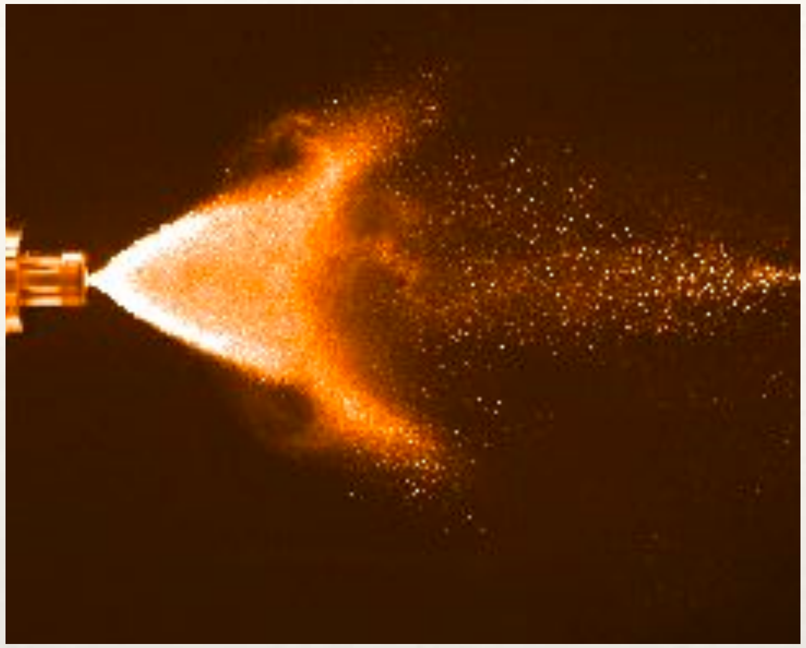
Turbophoresis of heavy inertial particles in statistically homogeneous flow



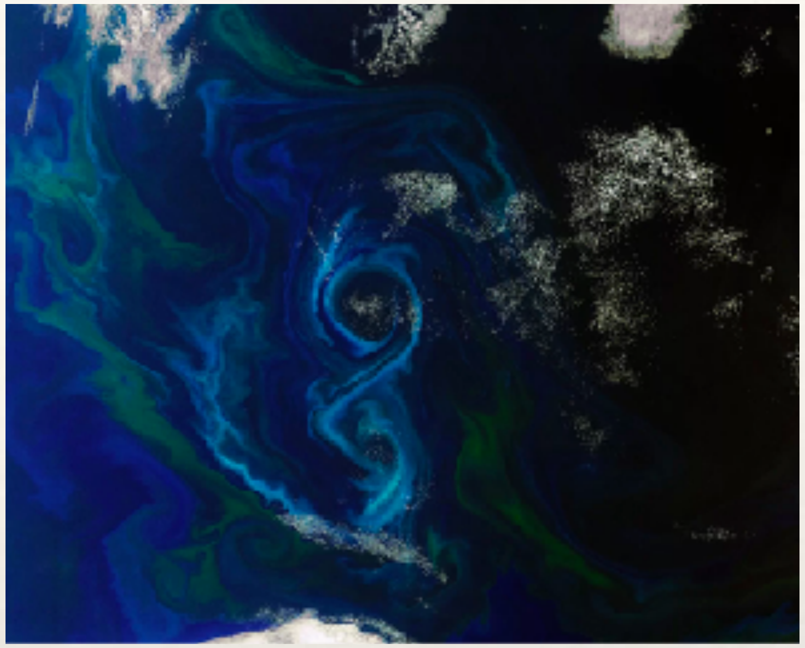
joint work with **_Robin Vallée_**

Particle-laden flows

Spray combustion in engines

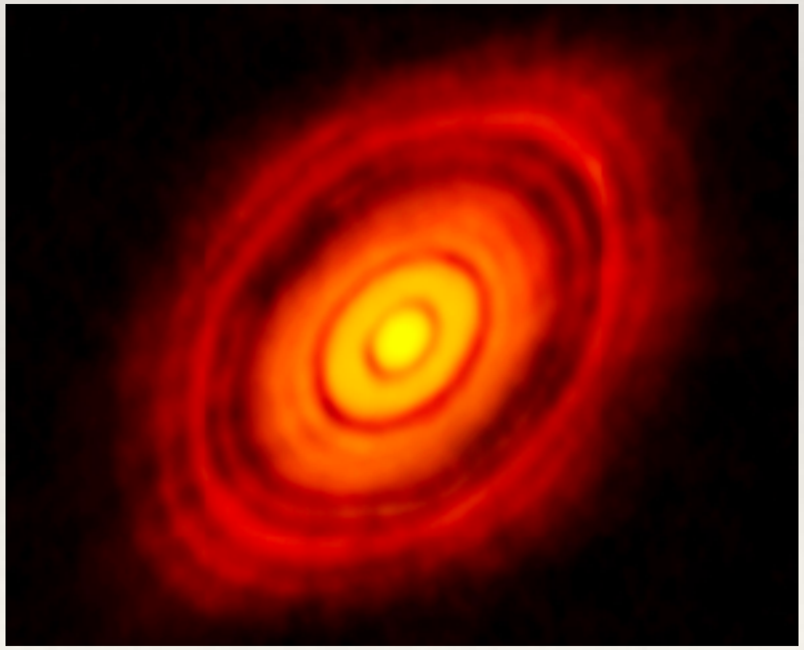


Biomixing in the oceans



Warm clouds

Planet formation



Predicting concentrations in the inertial range of turbulence?

Heavy inertial particles

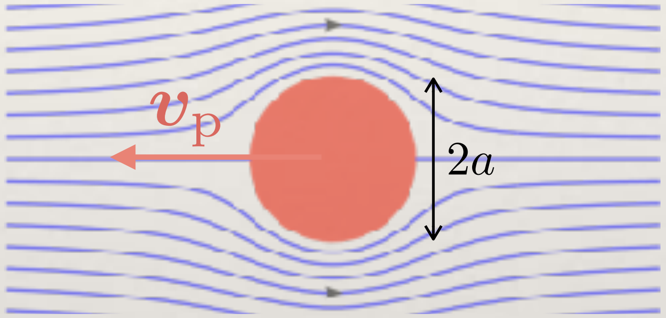
- Incompressible turbulence

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = -\frac{1}{\rho_\mathrm{f}}\nabla p + \nu\,\nabla^2\boldsymbol{u} + \boldsymbol{f}_\mathrm{ext} \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0$$

- **Particles**: small, rigid, heavy, dilute with moderate slip

$$Re_\mathrm{p} = \frac{|\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}|\,\ell}{\nu} \ll 1 \qquad \frac{\mathrm{d}\boldsymbol{v}_\mathrm{p}}{\mathrm{d}t} = -\frac{1}{\tau}\left[\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}, t)\right]$$

Response time $\tau = \dfrac{2\,\rho_p\,a^2}{9\,\rho_f\,\nu}$

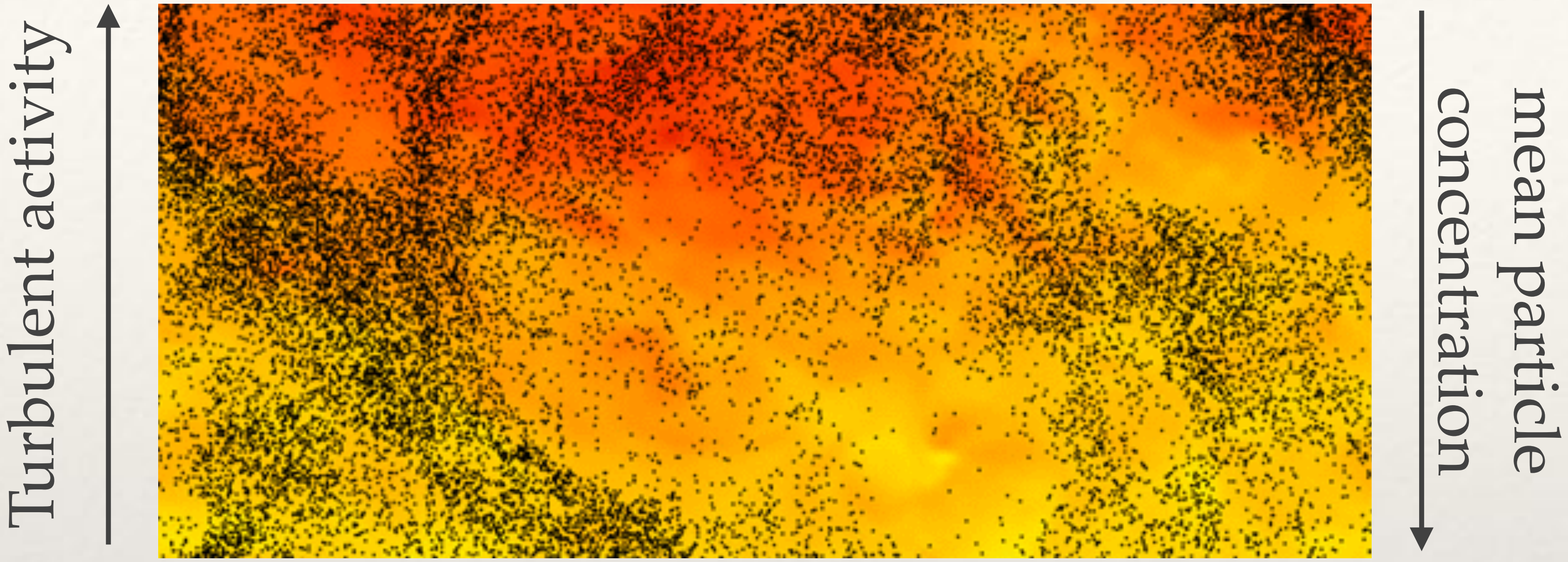


- **Dimension-less parameters:**

Fluid inertia $Re = \dfrac{U\,L}{\nu}$ Particle inertia $St = \dfrac{\tau\,U}{L}$

Turbophoresis

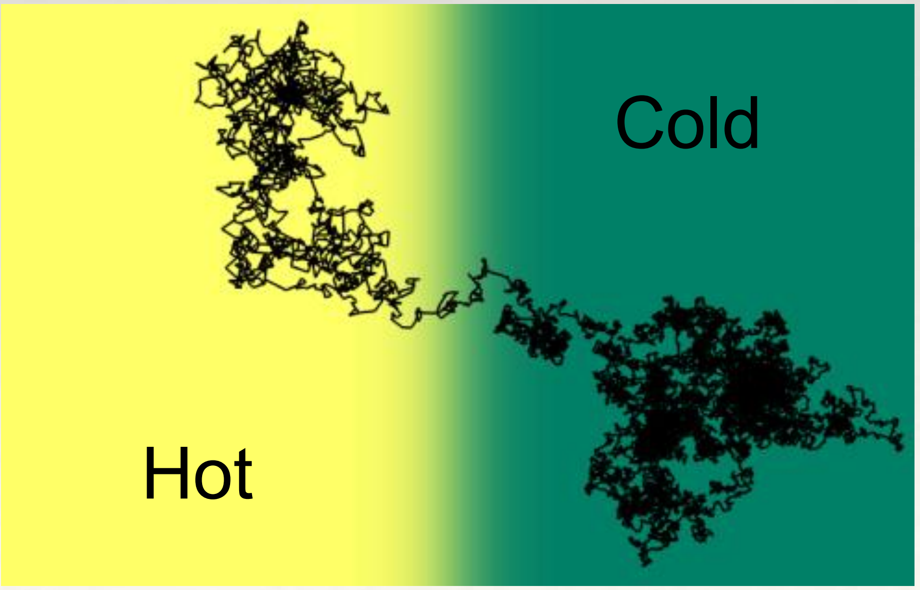
- In **inhomogeneous flow:** (Caporaloni et al. 1975, Reeks 1983)



(from De Lillo *et al.* 2016)

Effective diffusion equation for the average particle concentration

- Analogy with **thermophoresis**: diffusive particles spend more time in colder regions

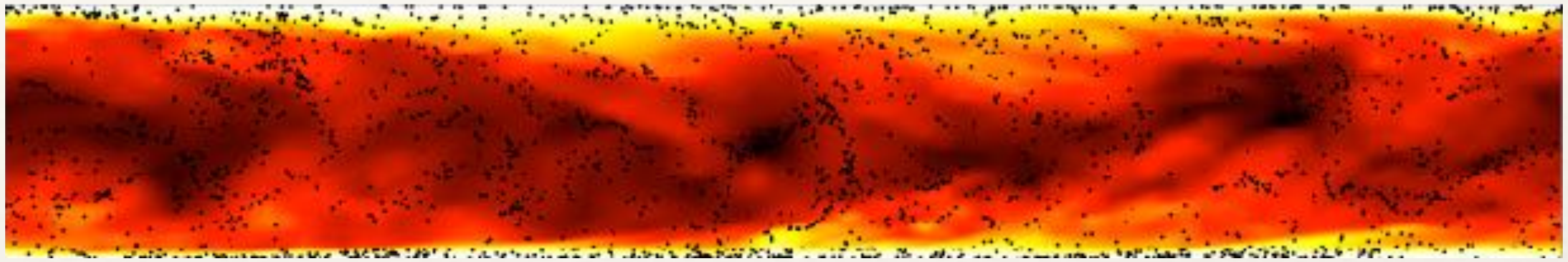


Inhomogeneous turbophoresis

- Turbulent boundary layers: channel flow

particle migrate
toward the walls

(Rouson & Eaton 2001,
Marchioli & Soldati 2002,
Costa *et al.* 2020)

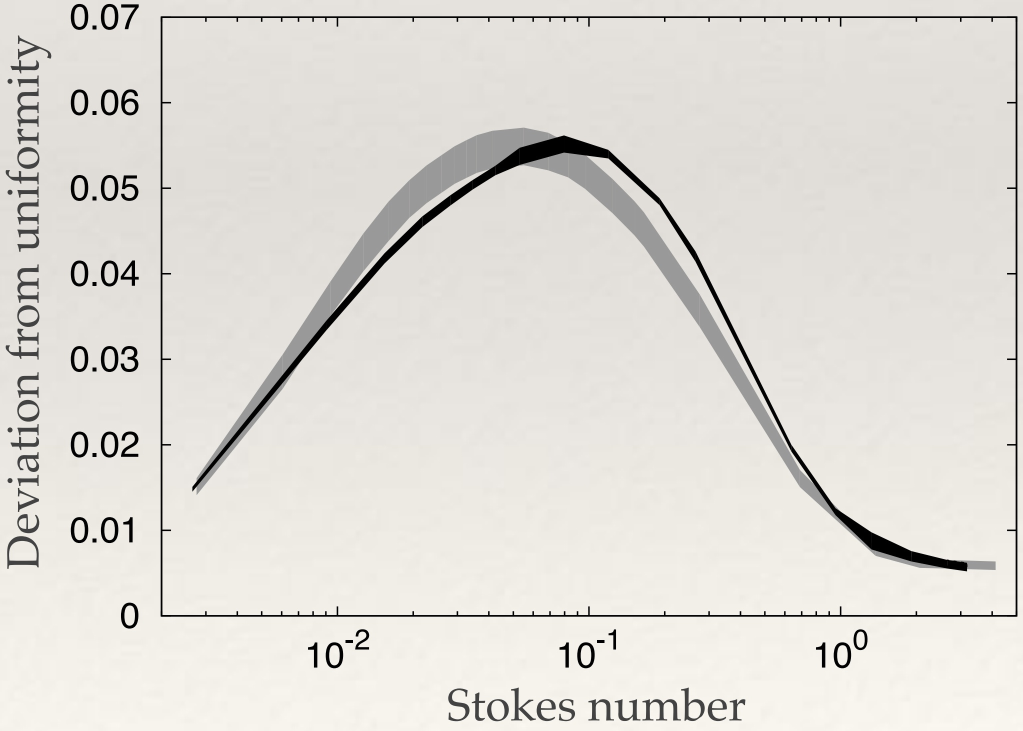


ejection from high-kinetic-energy regions

- Periodic flow with non-uniform forcing

Non-monotonic dependence
upon the particle response time

(De Lillo *et al.* 2016, Mitra *et al.* 2018)



Effective diffusion

$$\kappa(x) \propto \text{temp} \propto \left\langle |V_{\mathrm{p},x}|^2 \right\rangle$$

Do such considerations extend to statistically homogeneous flows?

Direct numerical simulations

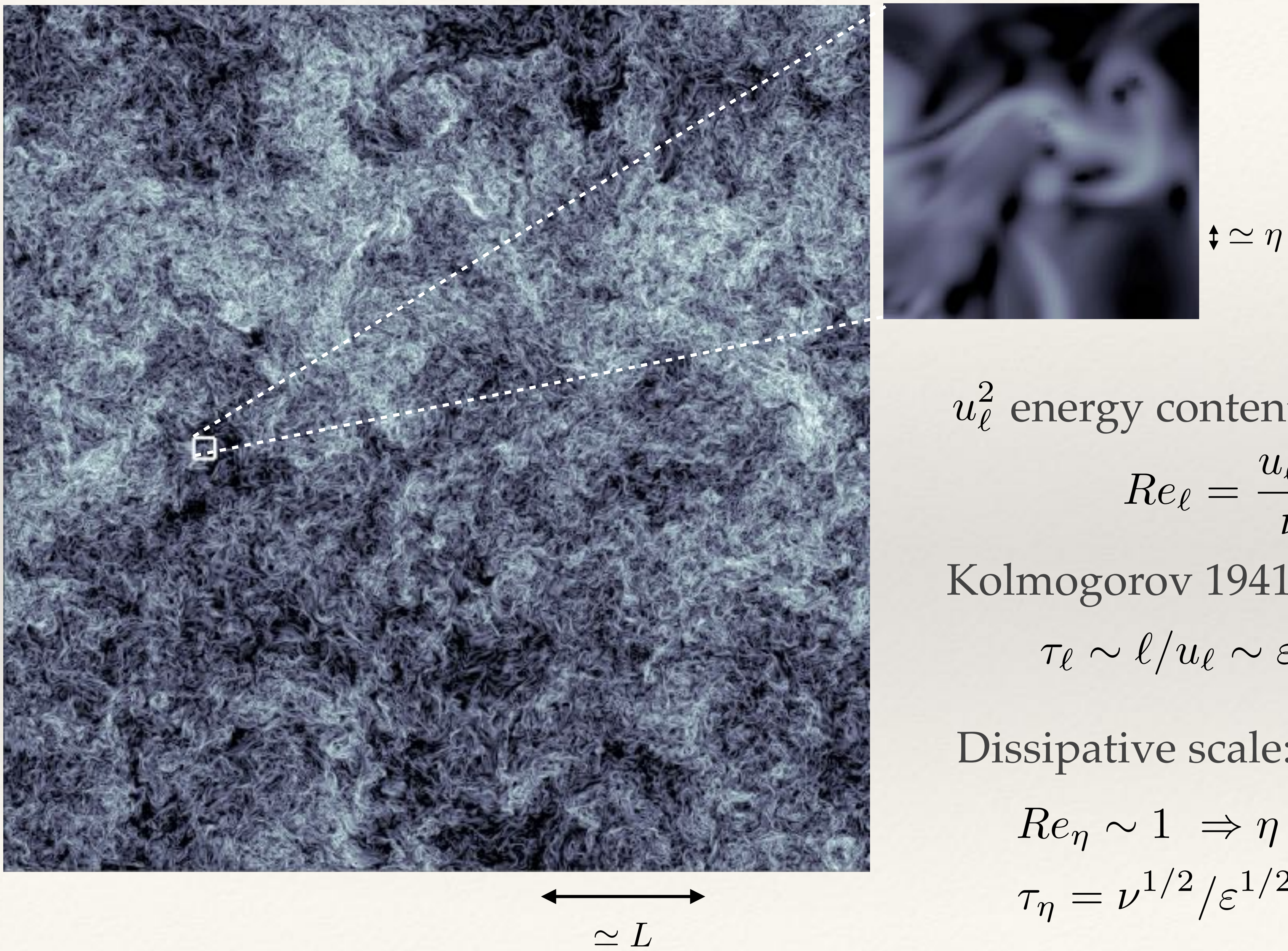
- **Fluid**: Pseudo-spectral code LaTu
  P3DFFT, 3rd order Runge–Kutta, MPI

- **Particles**: Lagrangian approach with tri-linear interpolation

N^3	ν	Δt	ε	u_{rms}	R_λ	N_{p}
1024^3	$6\,10^{-5}$	0.003	$3.47\,10^{-3}$	0.185	290	$1.25\,10^{7}$
2048^3	$2.5\,10^{-5}$	0.0012	$3.61\,10^{-3}$	0.189	460	10^{8}

Time and length scales of turbulence

$$\varepsilon_{\mathrm{loc}}(\boldsymbol{x}) = (\nu/2)\,\mathrm{tr}\,(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}}(\boldsymbol{x}))^2$$



u_ℓ^2 energy content at scale ℓ

$$Re_\ell = \frac{u_\ell\,\ell}{\nu}$$

Kolmogorov 1941: $u_\ell \sim (\varepsilon\,\ell)^{1/3}$

$$\tau_\ell \sim \ell/u_\ell \sim \varepsilon^{-1/3}\,\ell^{2/3}$$

Dissipative scale:

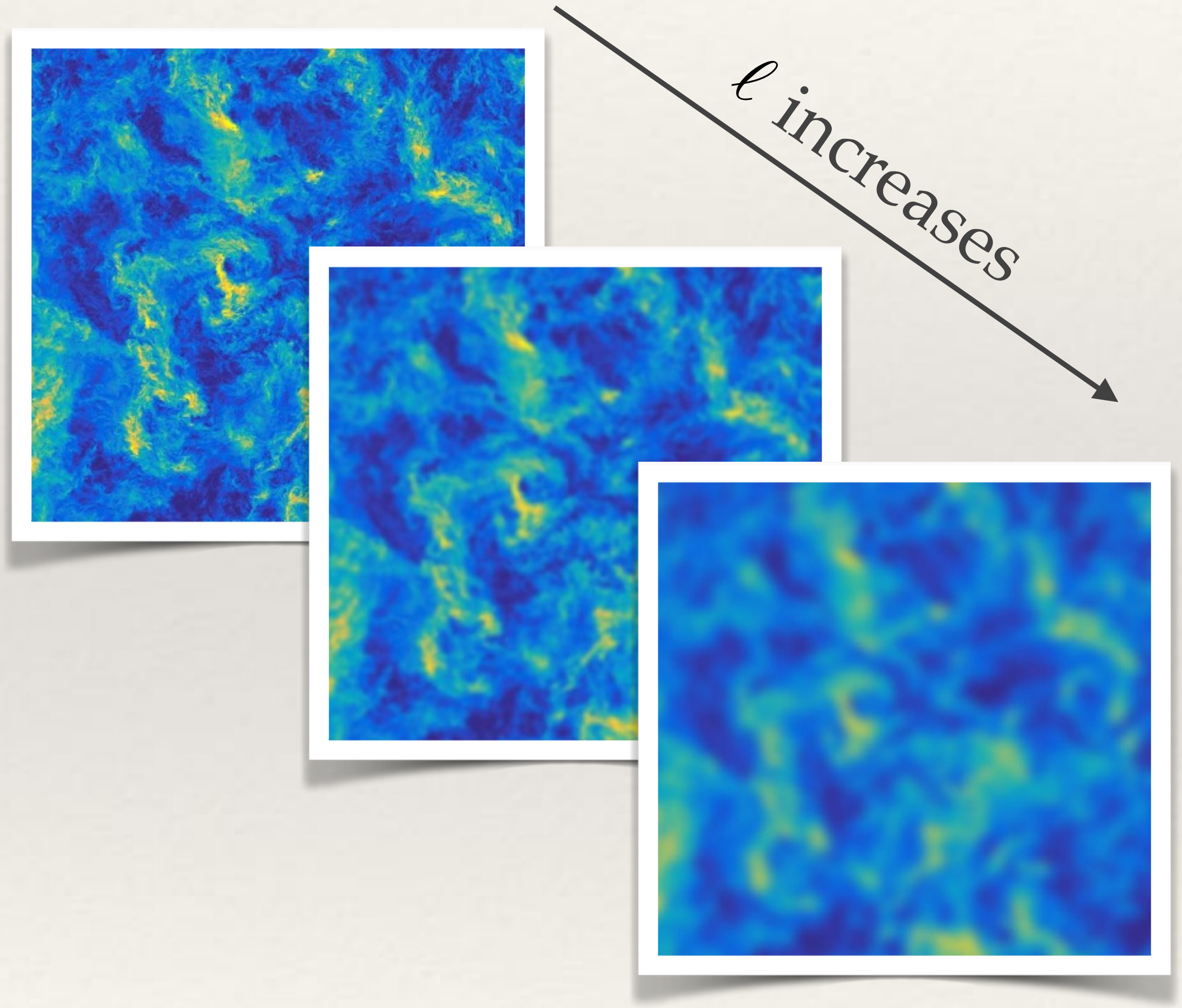
$$Re_\eta \sim 1 \;\; \Rightarrow \eta \sim \nu^{3/4}/\varepsilon^{1/4}$$

$$\tau_\eta = \nu^{1/2}/\varepsilon^{1/2}$$

Refined self-similarity

- Coarse-grained dissipation $\varepsilon_\ell(\boldsymbol{x}) \equiv \frac{1}{|\mathcal{B}_\ell|}\int_{\mathcal{B}_\ell(\boldsymbol{x})} \varepsilon_{\mathrm{loc}}(\boldsymbol{x}')\, d^3x'$ $\langle \varepsilon_\ell \rangle = \varepsilon$

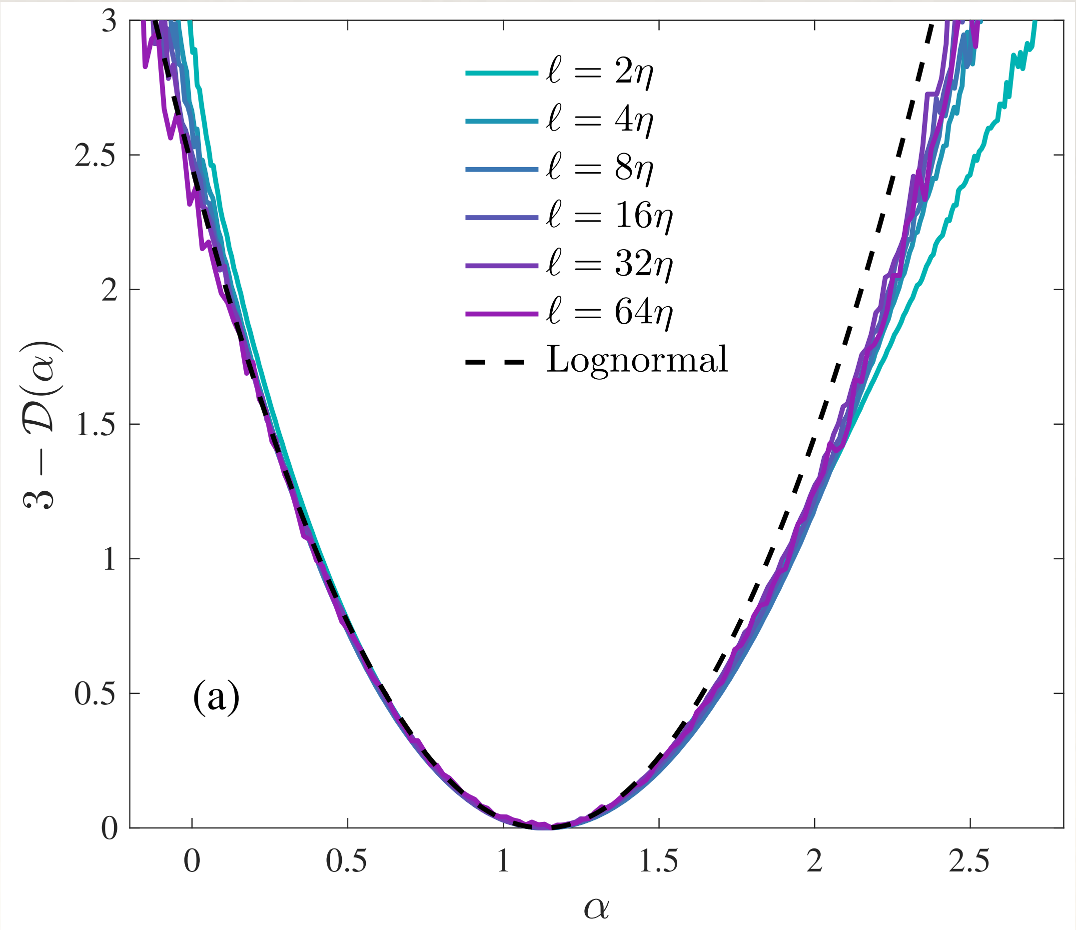
$$u_\ell \simeq \varepsilon_\ell^{1/3}\,\ell^{1/3} \quad \text{(Kolmogorov 1962)}$$



- Multifractal statistics of dissipation

$$\varepsilon_\ell = \varepsilon\,(\ell/L)^{\alpha-1} \quad \text{with}$$

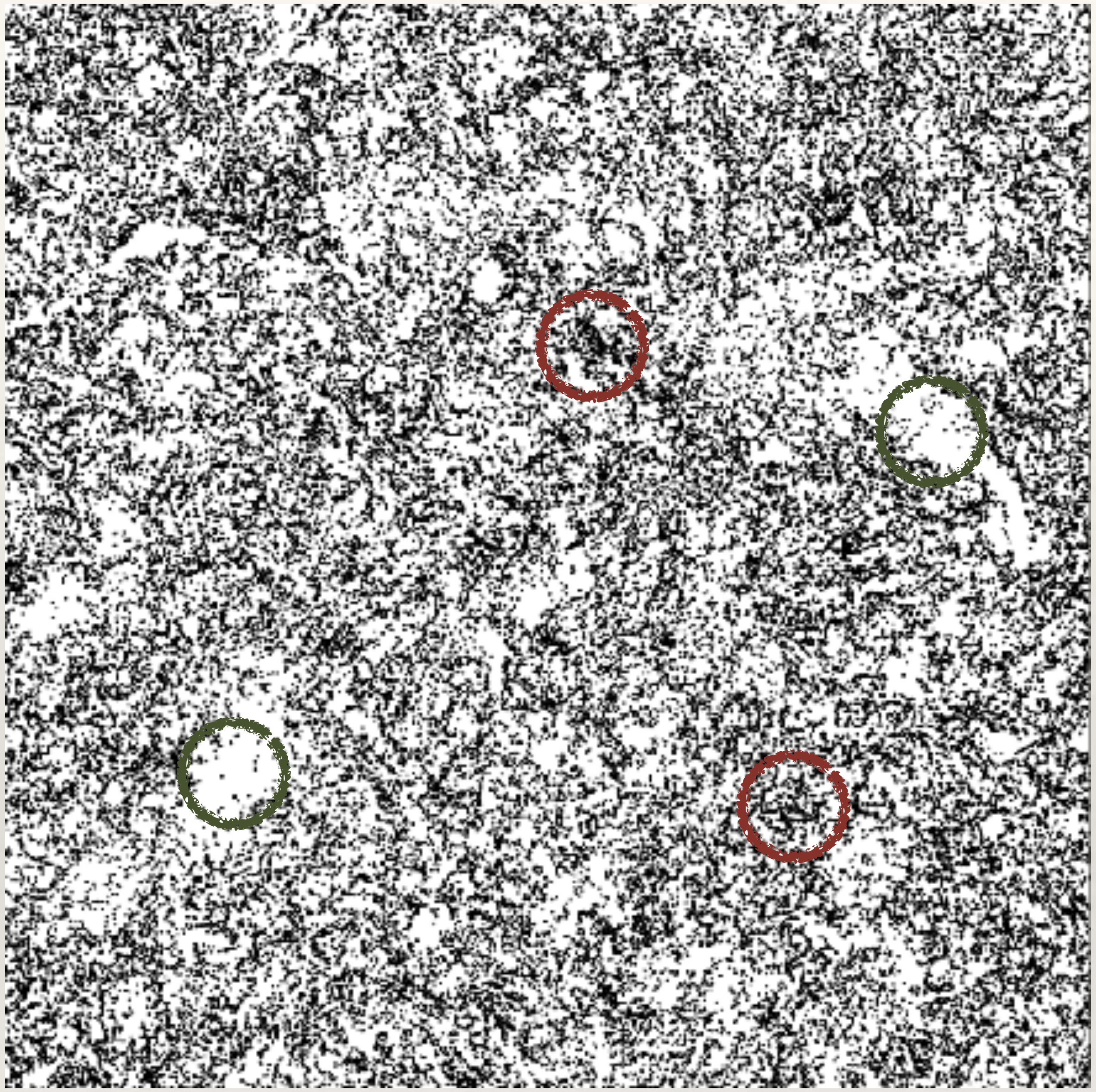
$$p(\varepsilon_\ell)\,\mathrm{d}\varepsilon_\ell = (\ell/L)^{3-\mathcal{D}(\alpha)}\,\mathrm{d}\mu(\alpha)$$



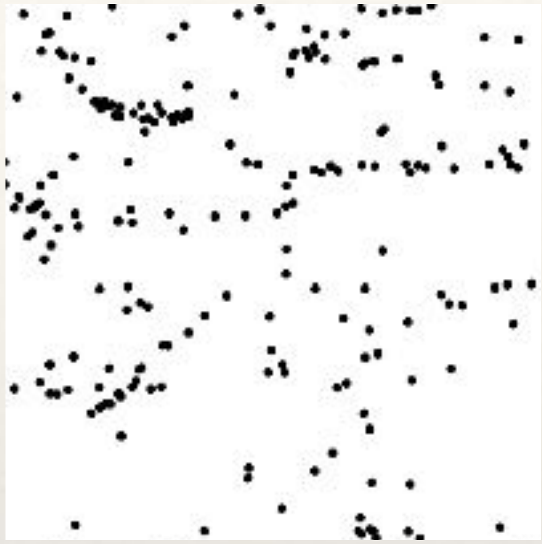
Instantaneous inhomogeneities in turbulent activity

Particle clustering

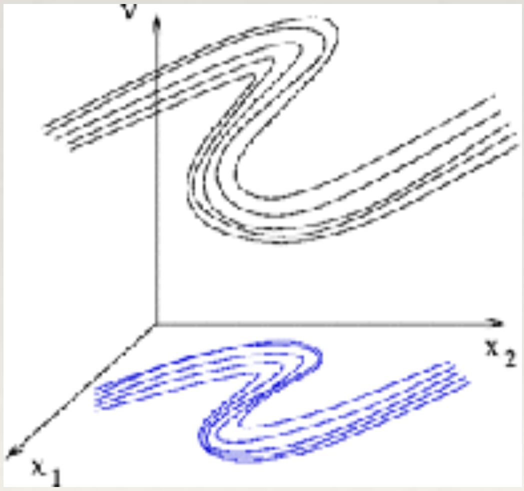
Inertial-range voids and clusters



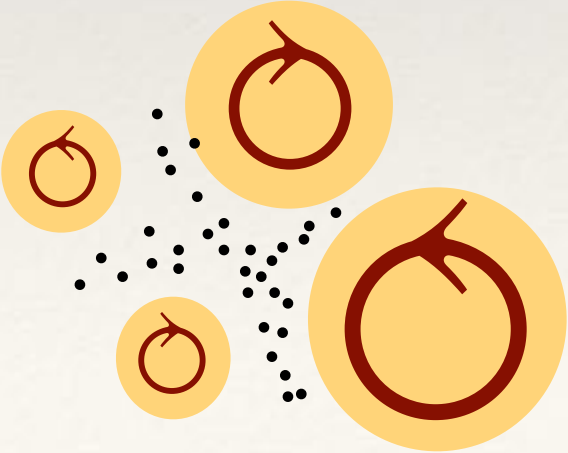
fractal distribution at dissipative scales



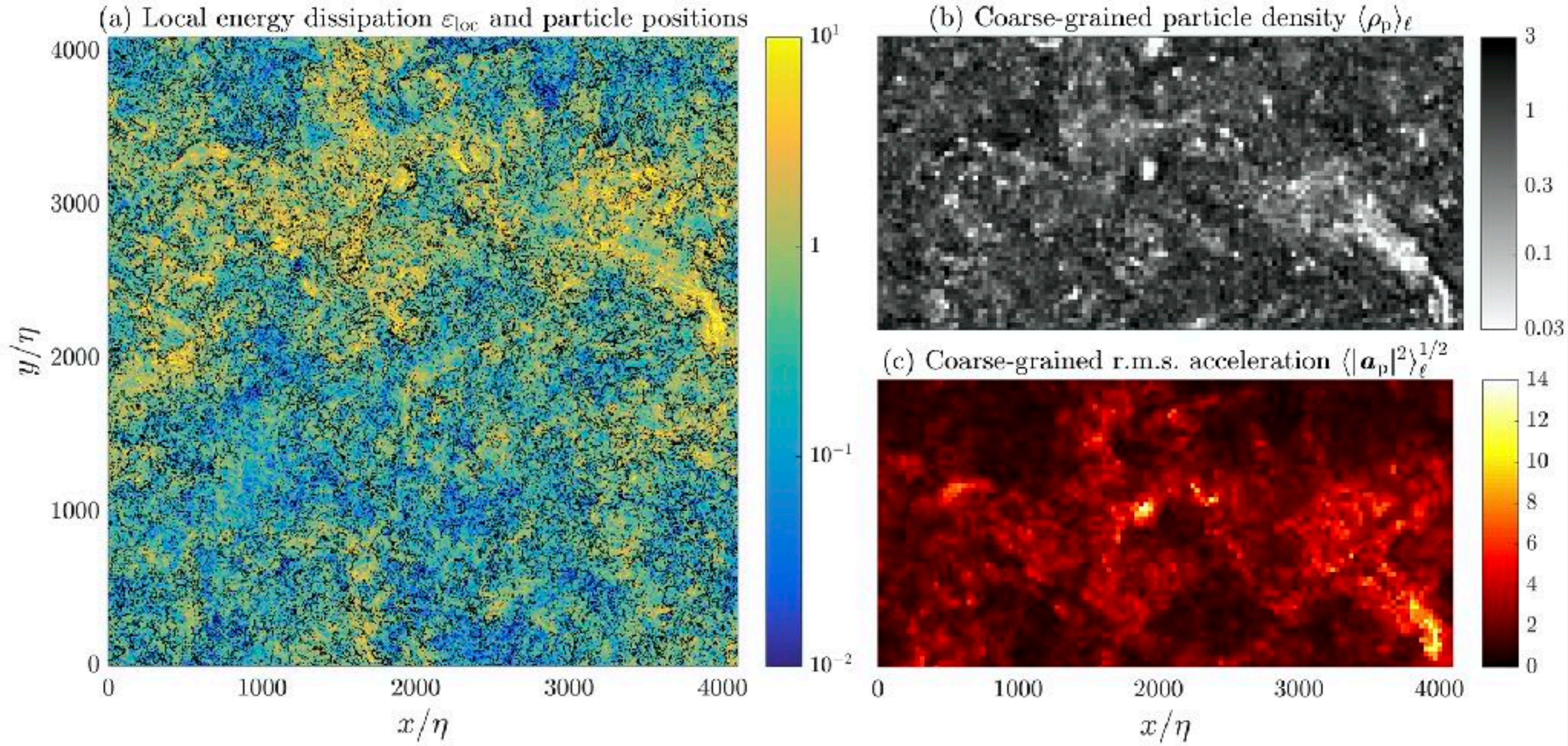
dissipative dynamics: attractor



Ejection from eddies

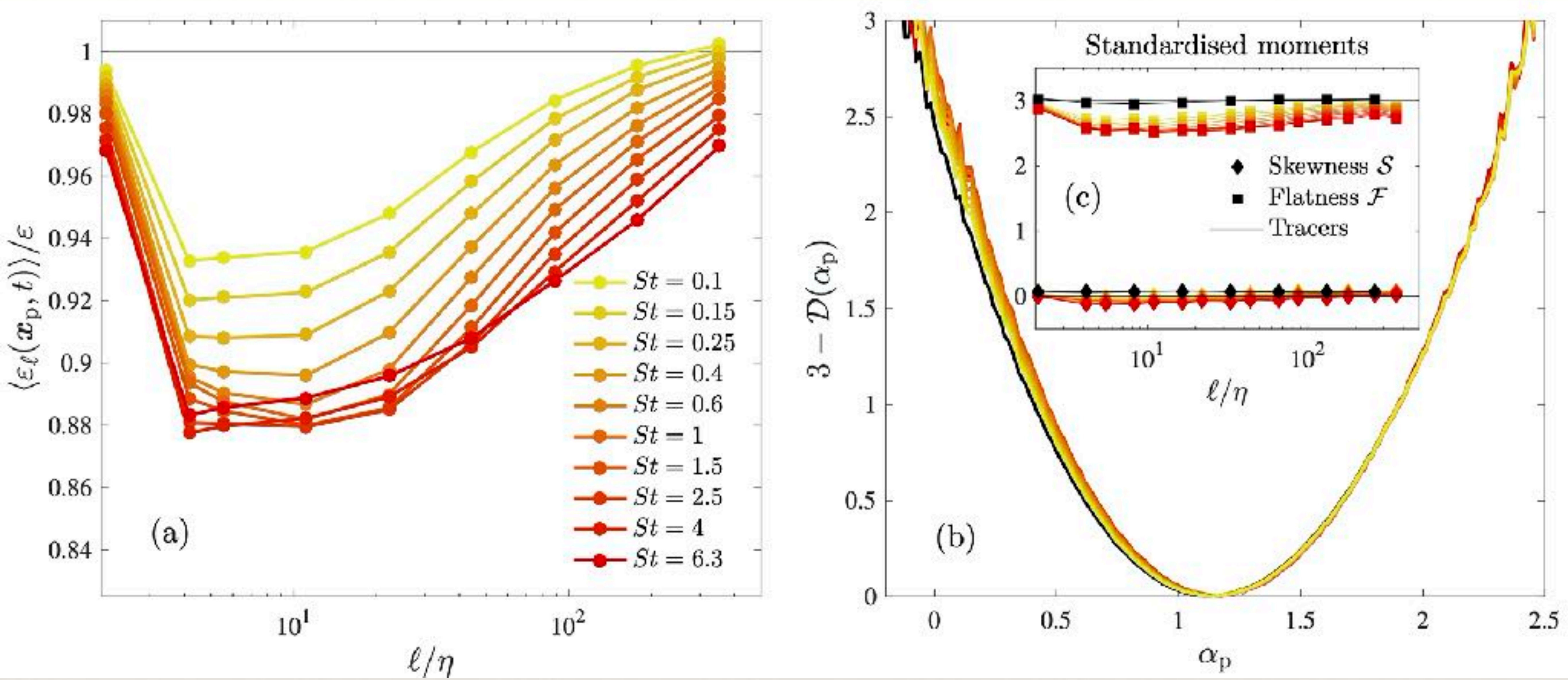


Correlations with flow structures



Clear correlations between particle positions, accelerations, and turbulent activity over inertial-range scales.

Preferential sampling



$$\alpha_\mathrm{p} = \frac{\log \varepsilon_\ell(\boldsymbol{x}_\mathrm{p})/\varepsilon}{\log \ell/L}$$

The biases induced by inertia are not striking !

Diffusive process

- Importance of acceleration to measure deviations from the fluid

$$\boldsymbol{a}_\mathrm{p} = -\frac{1}{\tau}\left[\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}, t)\right] \quad \Rightarrow \quad \boldsymbol{v}_\mathrm{p} = \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}, t) - \tau\, \boldsymbol{a}_\mathrm{p}$$

- Acceleration is moreover:
  - ✓ correlated over (relatively) short timescales
  - ✓ very sensitive to flow activity

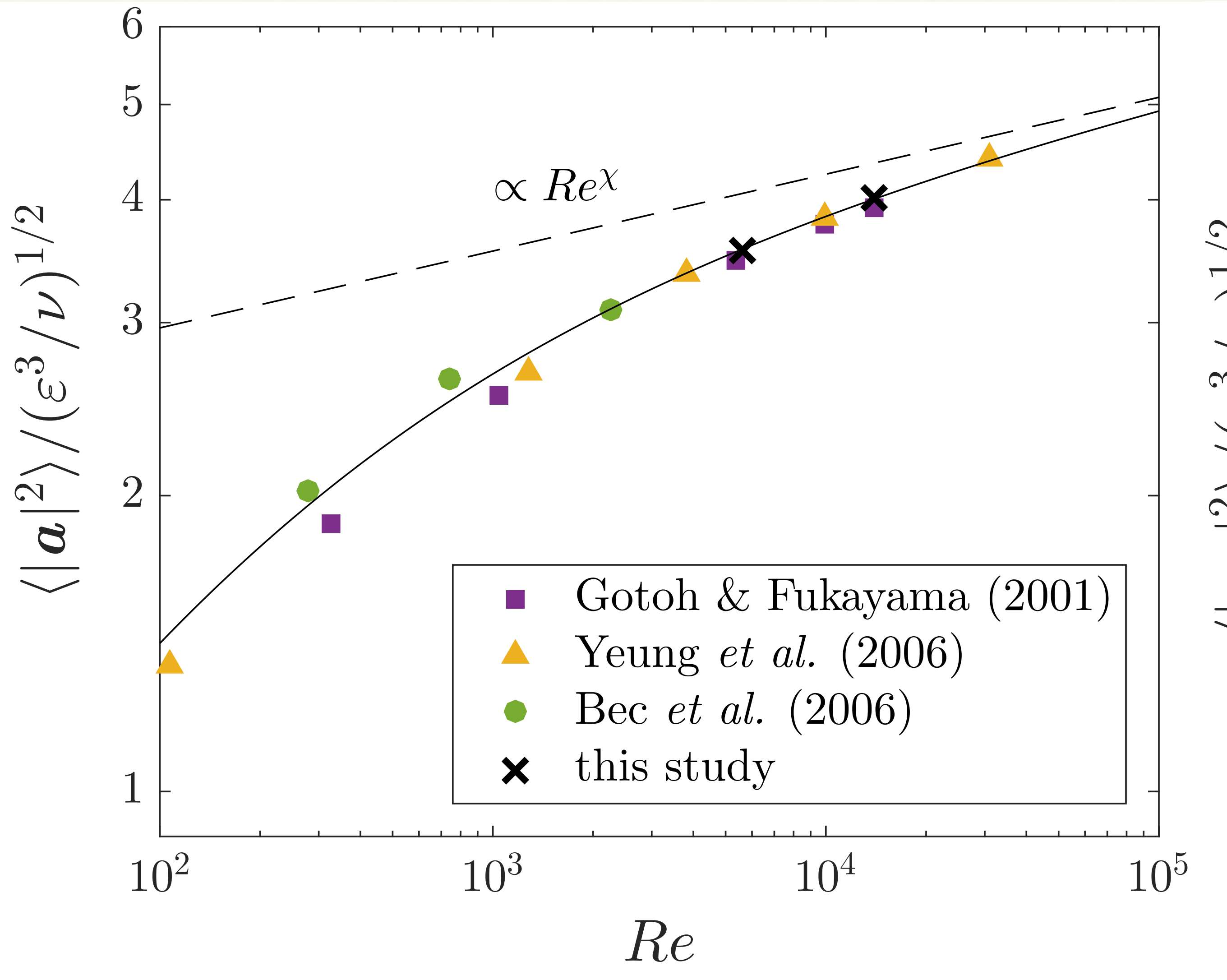
$$\boldsymbol{x}_\mathrm{p}(t+\delta t) - \boldsymbol{x}_\mathrm{p}(t) = \int_t^{t+\delta t} \underset{\substack{\big\downarrow\\ \text{large-scale}\\ \text{quantity}}}{\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(s), s)}\,\mathrm{d}s - \tau \int_t^{t+\delta t} \underset{\substack{\big\downarrow\\ \text{diffusion?}\\ \text{(central-limit theorem)}}}{\boldsymbol{a}_\mathrm{p}(s)}\,\mathrm{d}s$$

$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx \left[\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t), t) - \tau\, \langle \boldsymbol{a}_\mathrm{p}\rangle(t)\right]\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{x}_\mathrm{p}(t), t) \circ \mathrm{d}\boldsymbol{W}(t),$$

$$\left(\boldsymbol{\sigma}^\mathsf{T}\boldsymbol{\sigma}\right)_{i,j} = \tau^2\, T_\mathrm{I}\left(\langle a_\mathrm{p}^i a_\mathrm{p}^j\rangle - \langle a_\mathrm{p}^i\rangle\, \langle a_\mathrm{p}^j\rangle\right)$$

Acceleration variance

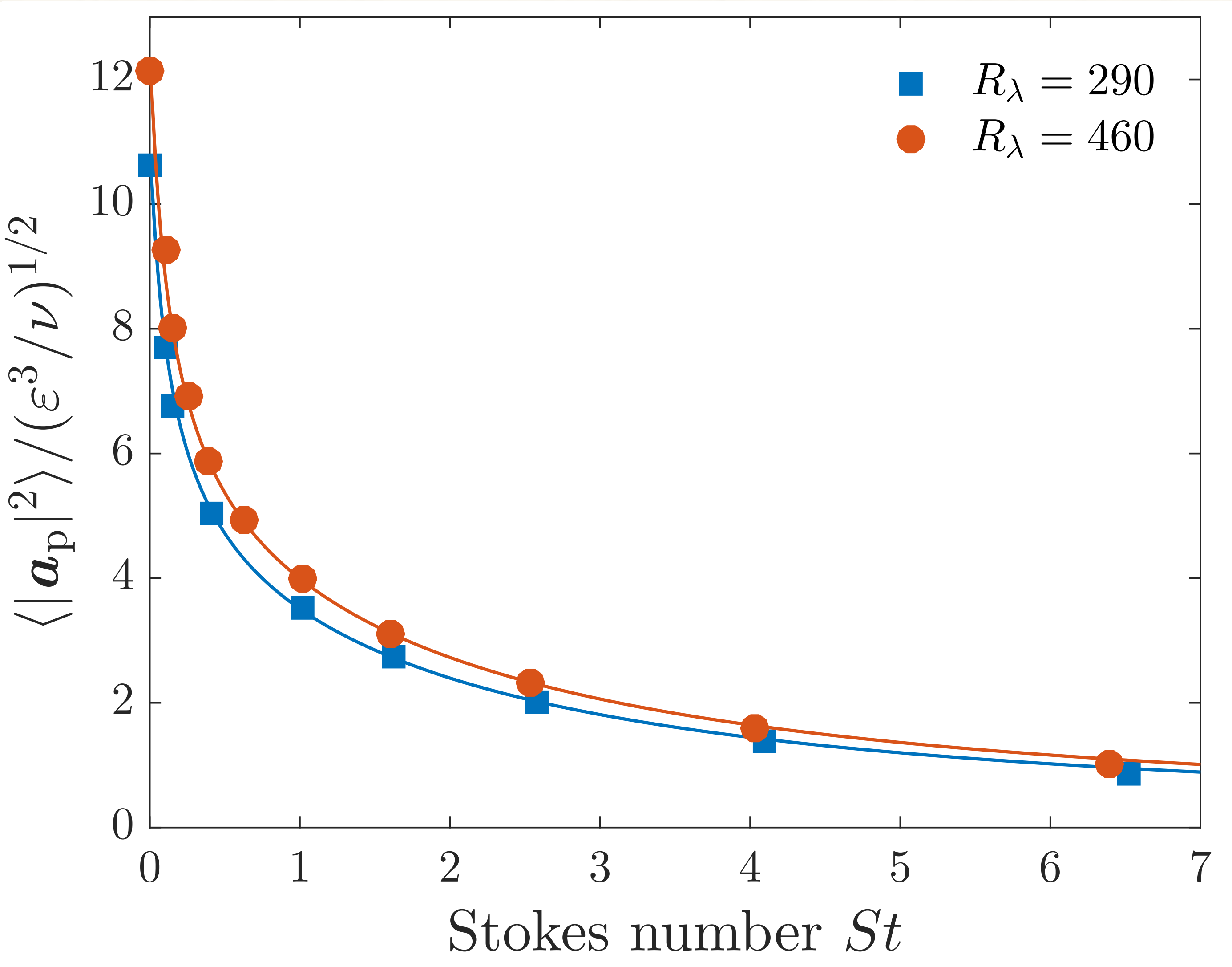
tracers



dependence on the Reynolds number

$$A_2(Re) = \frac{\langle |\boldsymbol{a}|^2\rangle \, \nu^{1/2}}{\varepsilon^{3/2}} \approx \frac{c\, Re^\chi}{[1 + (R_\star/Re)^2]^{1-\chi/2}}$$

particles

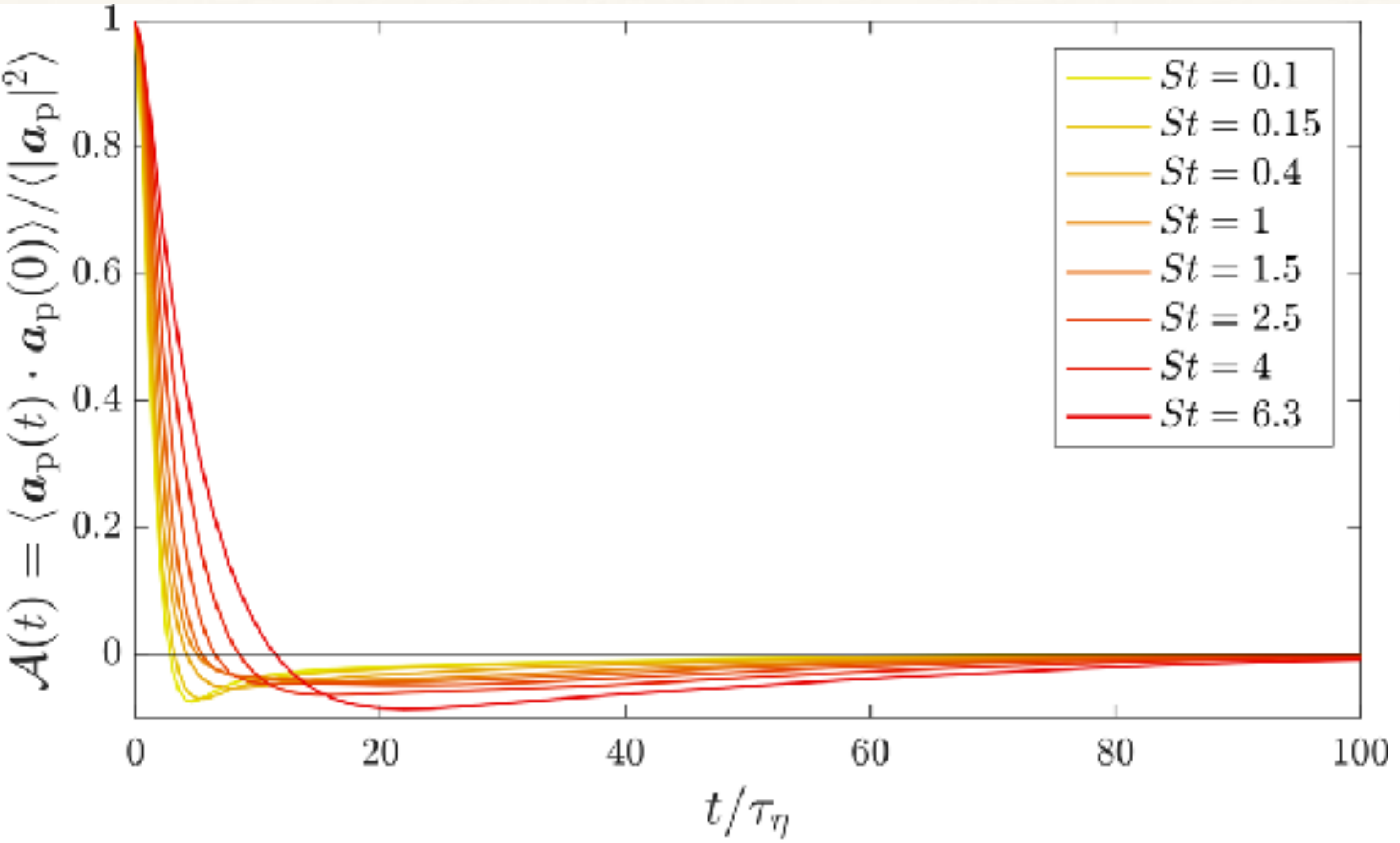


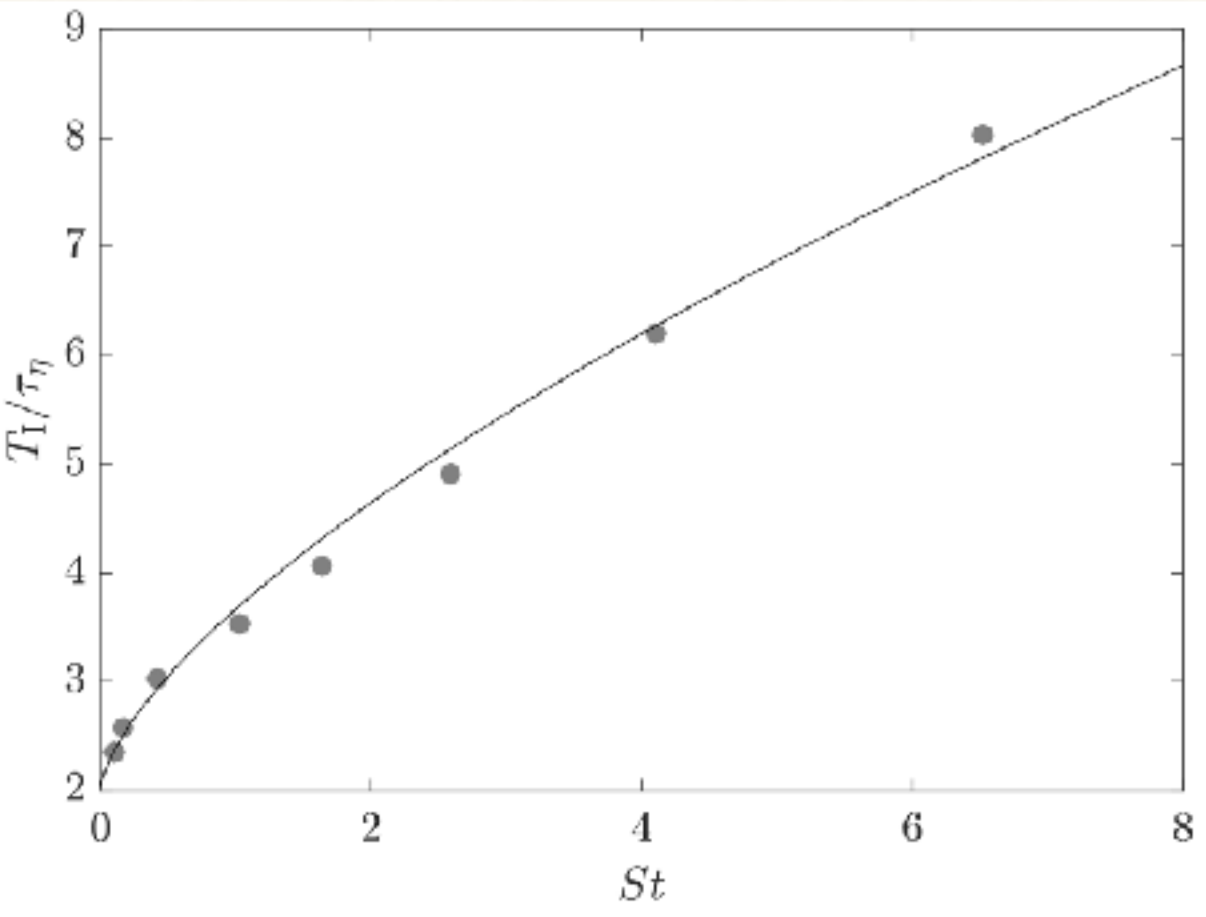
on the Stokes number

$$\langle |\boldsymbol{a}_\mathrm{p}|^2\rangle \approx \frac{A_2(Re)\, \varepsilon^{3/2}}{\nu^{1/2}} \, \frac{1 - \exp\left(-c_1/St^{1/2}\right)}{(1 + c_2\, St^2)^{1/4}}$$

Correlations of acceleration

short time correlations (component-wise)





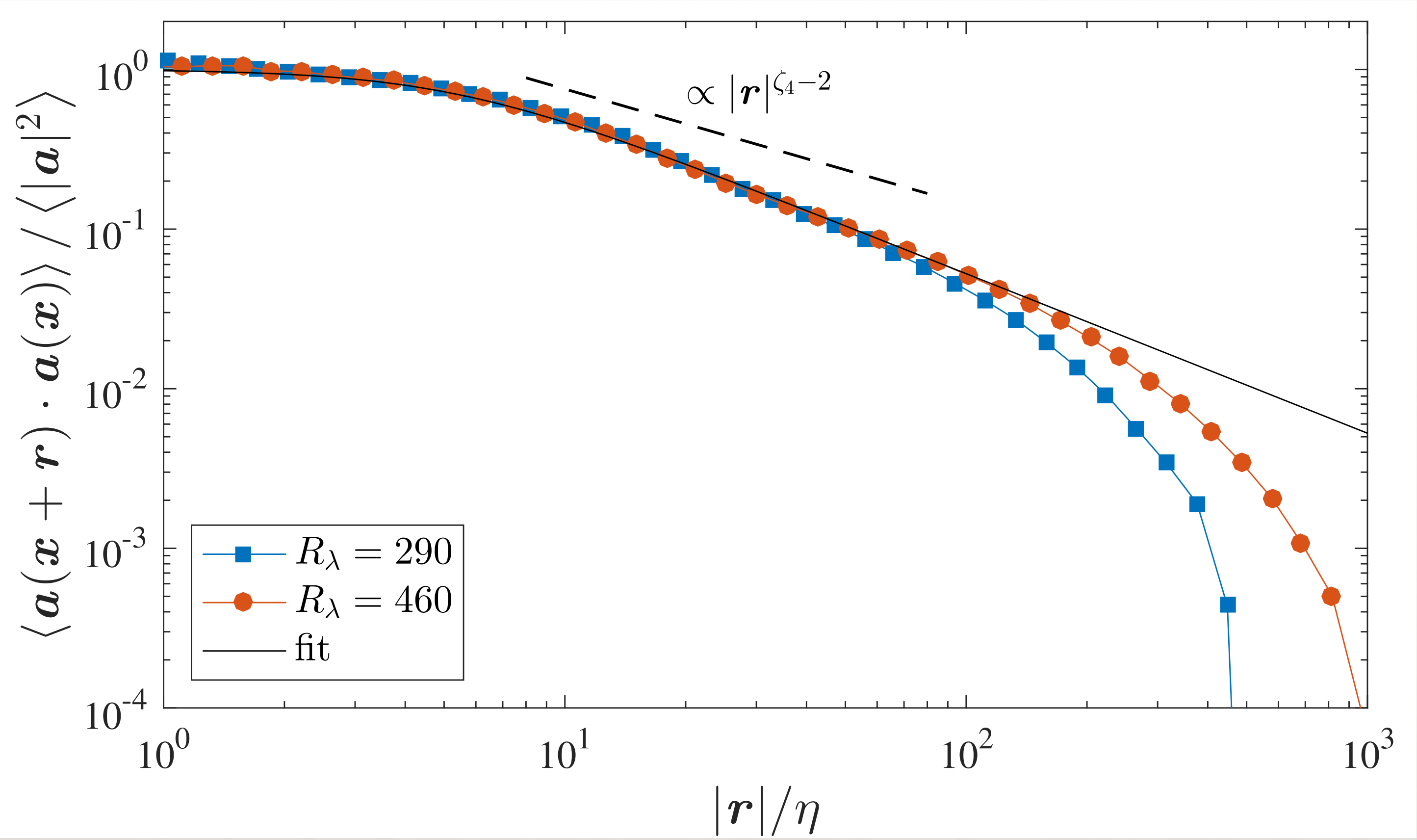
$$\mathcal{A}(t) = \langle \boldsymbol{a}_\mathrm{p}(t) \cdot \boldsymbol{a}_\mathrm{p}(0) \rangle \,/\, \langle |\boldsymbol{a}_\mathrm{p}|^2 \rangle$$

$$T_\mathrm{I} = \int \mathcal{A}(t)\, \mathrm{d}t$$

$$T_\mathrm{I} \approx \tau_\eta (a\, St^{2/3} + b)$$

Correlations of acceleration

long-range spatial correlations



$$C(r) = \langle \boldsymbol{a}(\boldsymbol{r},t)\cdot\boldsymbol{a}(0,t)\rangle$$

Hill-Wilczak (1995), Xu *et al.* (2007)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}C}{\mathrm{d}r}\right) \approx \langle \nabla^2 p(\boldsymbol{x}',t)\,\nabla^2 p(\boldsymbol{x},t)\rangle$$

$$= \partial_{ijkl}\langle u_i(\boldsymbol{r},t)\,u_j(\boldsymbol{r},t)\,u_k(0,t)\,u_l(0,t)\rangle$$

$$\propto \langle \delta_r u^4\rangle \sim r^{\zeta_4} \text{ when } r \gg \eta$$

Inertial-range expectation

$$C(r)\sim r^{\zeta_4-2} \qquad \zeta_4\approx 1.27$$

However, contributions from dissipative scales dominate over a wide range of scales

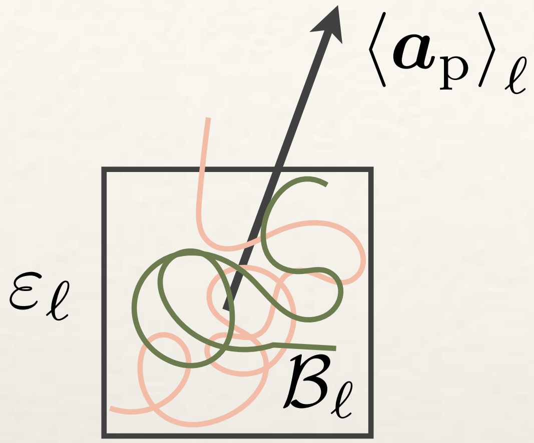
$$C(r)\approx \frac{\langle|\boldsymbol{a}|^2\rangle}{\left(1+(c\,r/\eta)^2\right)^{1/2}} \sim r^{-1}$$

Long-range spatial correlations reflect intrinsic correlations of turbulent activity

Statistics conditioned on the local value $\varepsilon_\ell(\boldsymbol{x})$ of the energy dissipation

Coarse-grained dynamics

$\langle\cdot\rangle \equiv \langle\cdot\rangle_\ell$: average over a coarse-graining box of size ℓ



Residence time ≫ correlation time T_I

⇒ Box average ≈ time integral conditioned on local turbulent fluctuations

Particle statistics conditioned on ε_ℓ depend solely on:

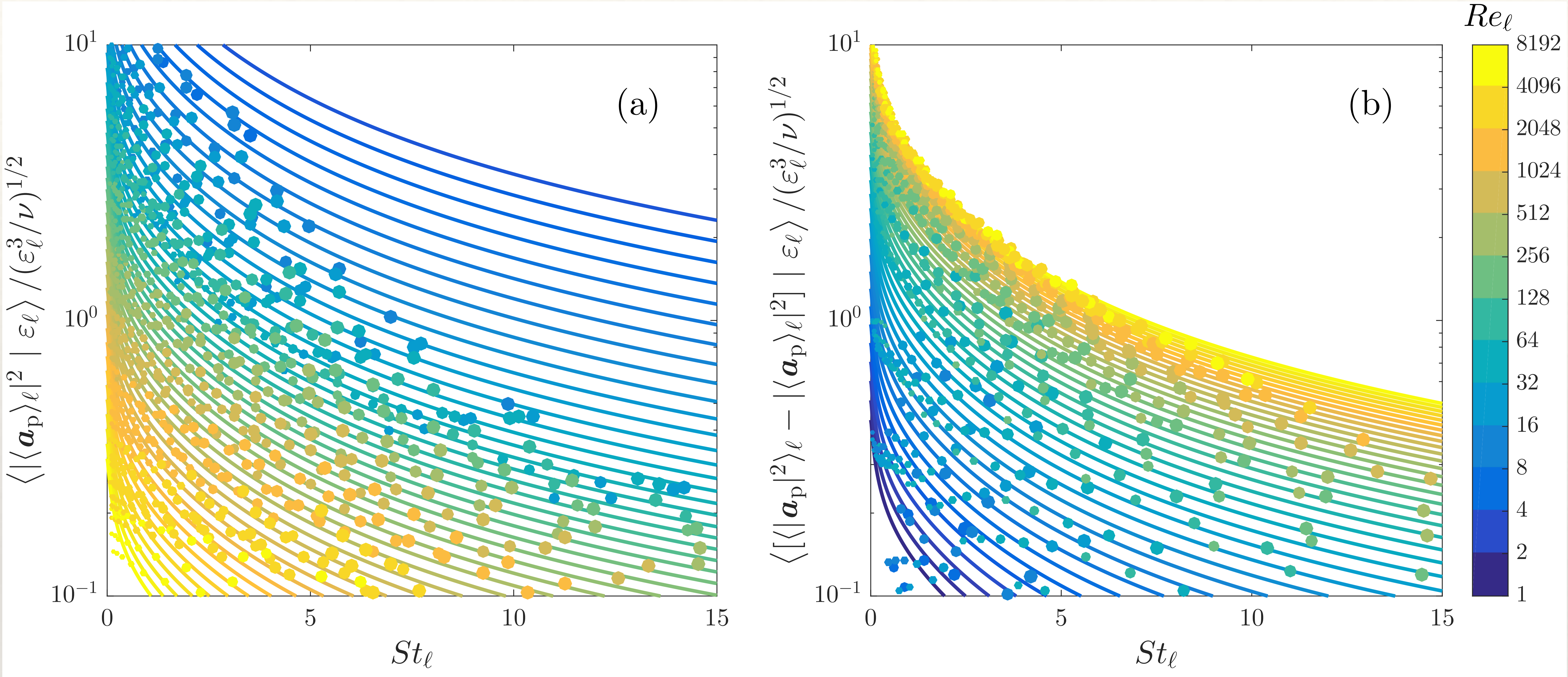
- the local Reynolds number $Re_\ell = \dfrac{\varepsilon_\ell^{1/3}\ell^{4/3}}{\nu}$
- the local Stokes number $St_\ell = \dfrac{\tau\,\varepsilon_\ell^{1/2}}{\nu^{1/2}}$

Effective equation

$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx [\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) - \tau\,\langle\boldsymbol{a}_\mathrm{p}\rangle_\ell(\boldsymbol{x}_\mathrm{p}(t),t)]\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{x}_\mathrm{p}(t),t)\circ\mathrm{d}\boldsymbol{W}(t),$$

$$\left(\boldsymbol{\sigma}^\mathsf{T}\boldsymbol{\sigma}\right)_{i,j} = \tau^2\,T_\mathrm{I}\left(\langle a_\mathrm{p}^i a_\mathrm{p}^j\rangle_\ell - \langle a_\mathrm{p}^i\rangle_\ell\,\langle a_\mathrm{p}^j\rangle_\ell\right)$$

Coarse-grained acceleration



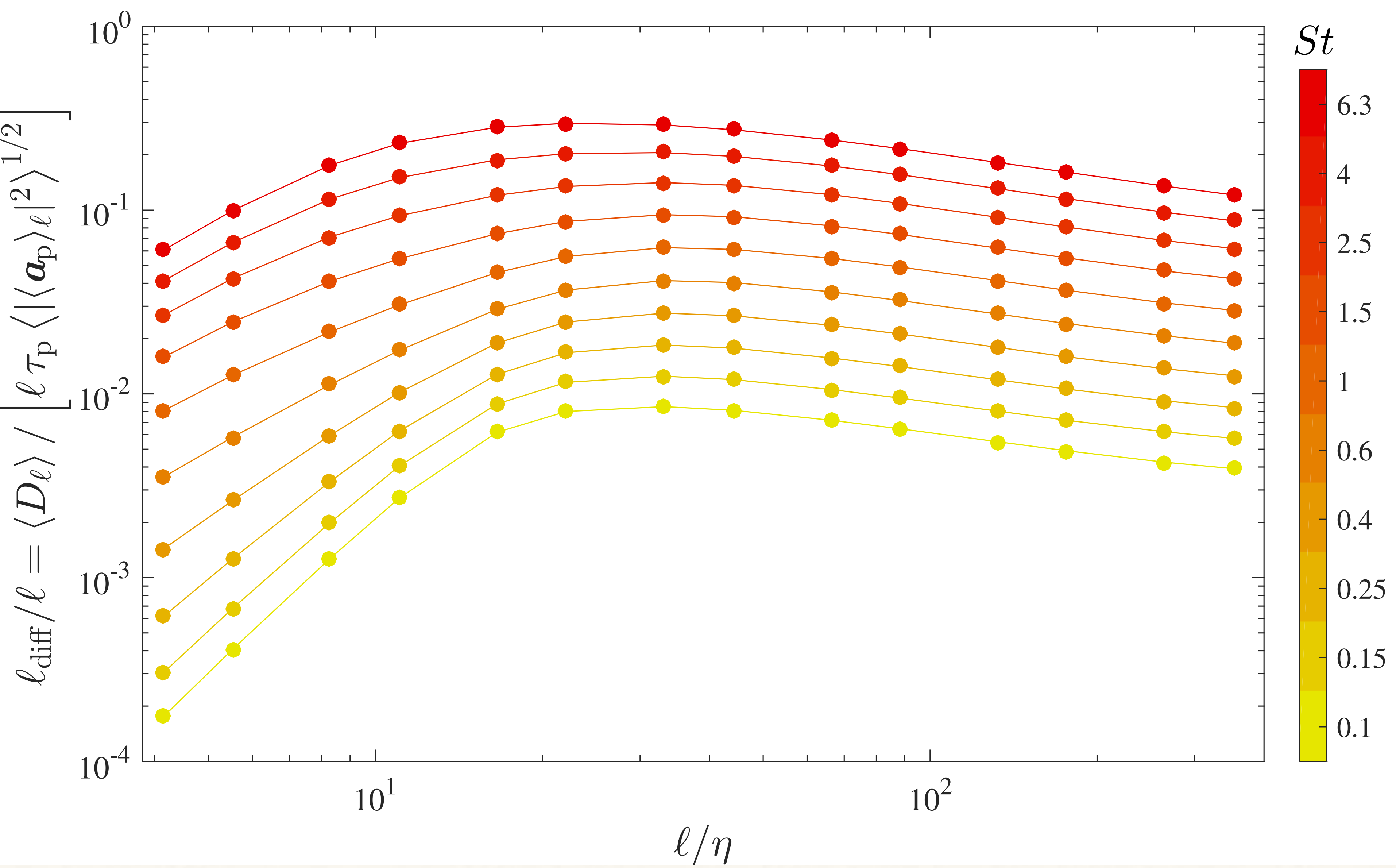
$$\langle |\langle \boldsymbol{a}_\mathrm{p}\rangle_\ell|^2 \mid \varepsilon_\ell\rangle \simeq \frac{A_2(Re_\ell)\,\varepsilon_\ell^{3/2}}{\nu^{1/2}} \frac{\left[1 + c' Re_\ell^{-3/2}\right]^{1/2}}{Re_\ell^{3/4}} \frac{1 - \exp\left(-c_1/St_\ell^{1/2}\right)}{\left(1 + c_2\, St_\ell^2\right)^{1/4}}$$

$$\langle \langle |\boldsymbol{a}_\mathrm{p}|^2\rangle_\ell - |\langle \boldsymbol{a}_\mathrm{p}\rangle_\ell|^2 \mid \varepsilon_\ell\rangle \simeq \frac{A_2(Re_\ell)\,\varepsilon_\ell^{3/2}}{\nu^{1/2}} \frac{1 - \exp\left(-c_1/St_\ell^{1/2}\right)}{\left(1 + c_2\, St_\ell^2\right)^{1/4}}$$

Effect of diffusion

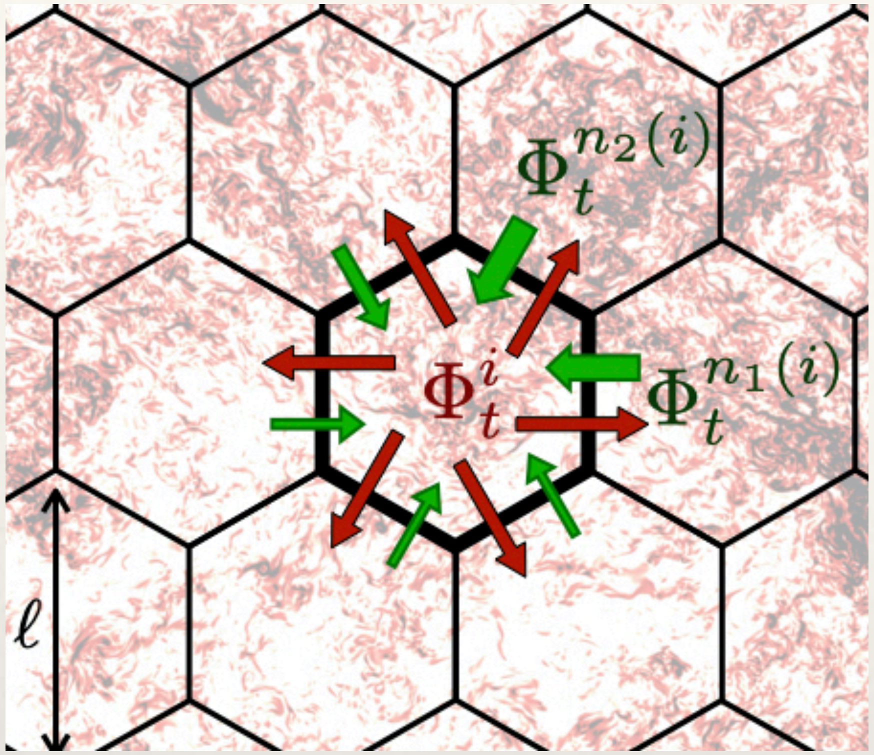
$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx [\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) - \tau\,\langle\boldsymbol{a}_\mathrm{p}\rangle_\ell(\boldsymbol{x}_\mathrm{p}(t),t)]\,\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{x}_\mathrm{p}(t),t)\circ\mathrm{d}\boldsymbol{W}(t)$$

dominates at scales $\gg \ell_\mathrm{diff}$ (Batchelor's scale)



Drift prevails at moderate Stokes numbers and inertial-range scales

Effective inertial-range dynamics

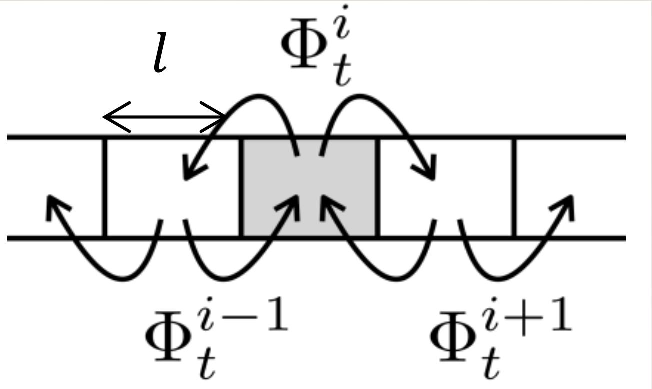


$\langle \rho_\mathrm{p} \rangle_\ell$ coarse-grained particle density

$$\boldsymbol{v}_p^{\mathrm{eff}}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) - \tau \underset{\substack{\uparrow \\ \text{depends on past}}}{\langle \boldsymbol{a}_\mathrm{p} \rangle_\ell(\boldsymbol{x},t)}$$

Particle fluxes = transport by the fluid velocity + ejection due to inertia

Outgoing flux from the cell *i*: $\Phi_t^i \approx \int_{\partial\mathcal{B}_\ell} \tau \langle \rho_\mathrm{p} \boldsymbol{a}_\mathrm{p} \rangle \cdot \mathrm{d}\boldsymbol{S} \propto \tau \ell^2 \langle \rho_\mathrm{p} \rangle_\ell \, | \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell |$



$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = (1/2)\Phi_t^{i+1} - \Phi_t^i + (1/2)\Phi_t^{i-1}$$

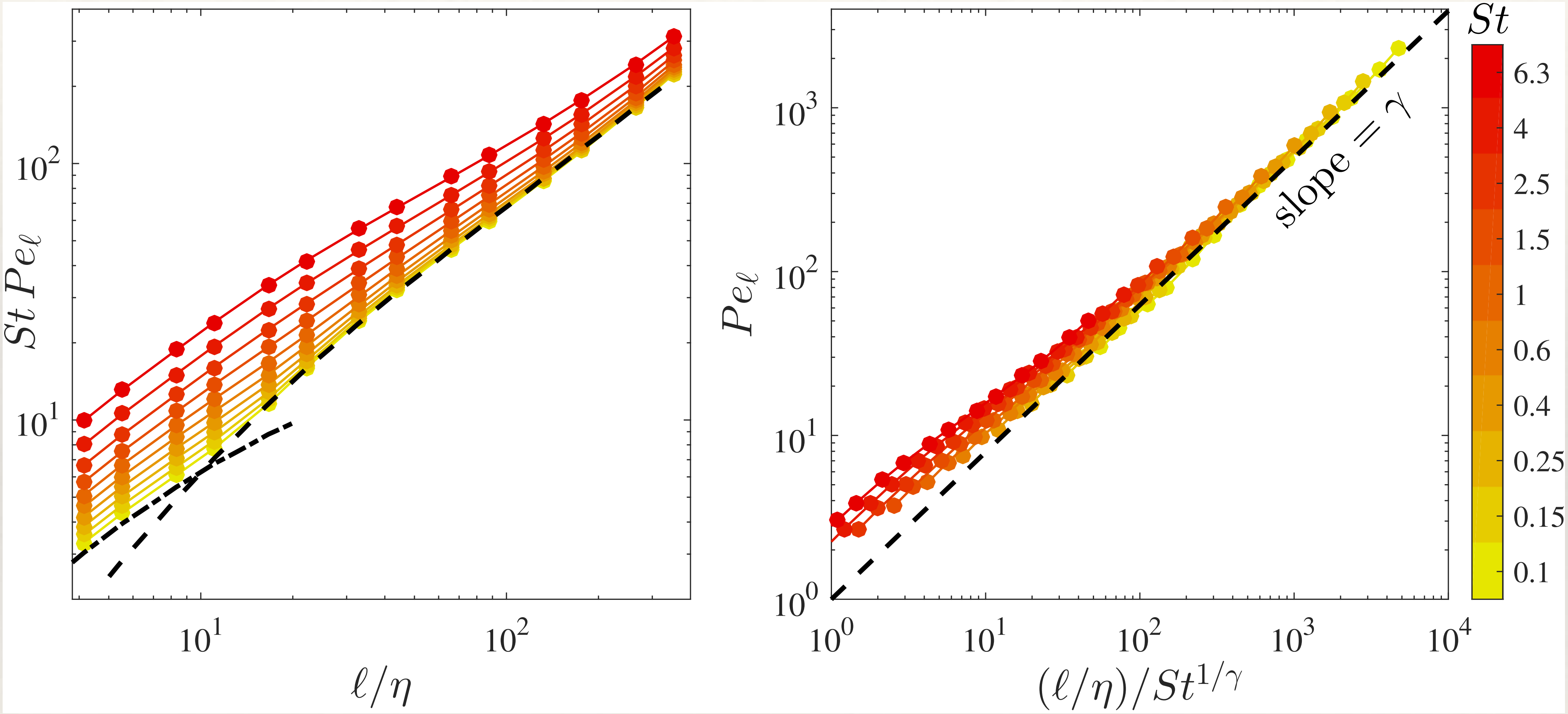
$$\partial_t \langle \rho_\mathrm{p} \rangle_\ell + \boldsymbol{u} \cdot \nabla \langle \rho_\mathrm{p} \rangle_\ell \approx \nabla^2 (\kappa_\ell \, \langle \rho_\mathrm{p} \rangle_\ell)$$

$$\kappa_\ell(\boldsymbol{x},t) \propto \tau \ell \, |\langle \boldsymbol{a}_\mathrm{p} \rangle_\ell|$$

Scale-dependent Peclet number

$$\partial_t \langle \rho_{\mathrm{p}} \rangle_\ell + \boldsymbol{u} \cdot \nabla \langle \rho_{\mathrm{p}} \rangle_\ell \approx \nabla^2 (\kappa_\ell \langle \rho_{\mathrm{p}} \rangle_\ell)$$

$$Pe_\ell = \frac{\delta_\ell u}{\ell\, \kappa_\ell} = \frac{\text{diffusive time}}{\text{advective time}}$$

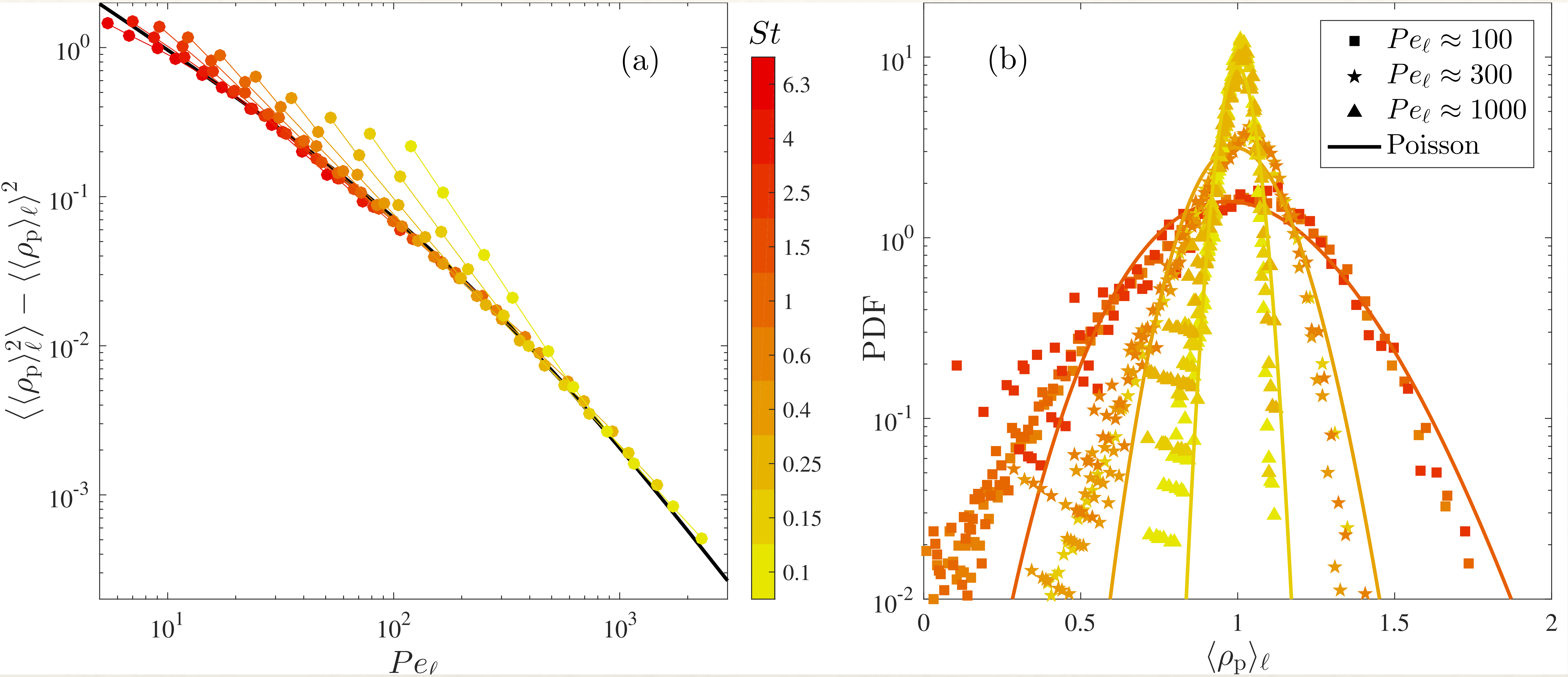


$Pe_\ell \propto \ell^\gamma / \tau$ with $\gamma \approx 0.898$

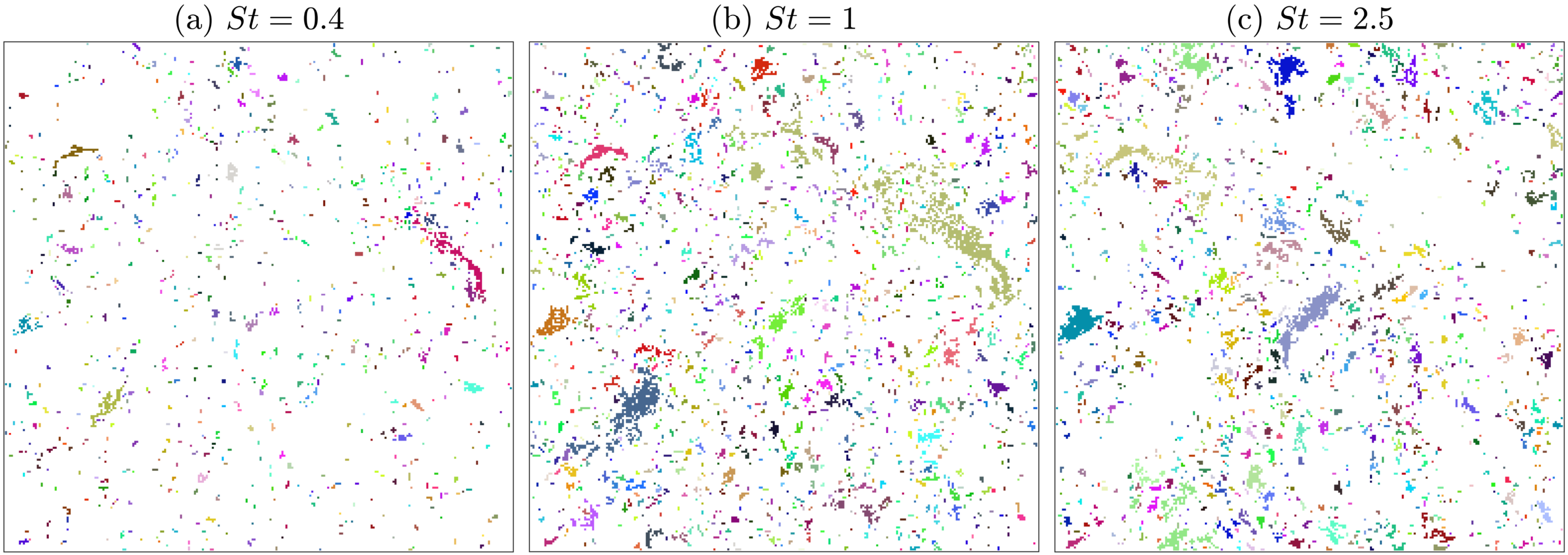
Inertial-range distribution depends solely on Pe_ℓ

Coarse-grained distributions

mass-density variance probability density



Distribution of voids



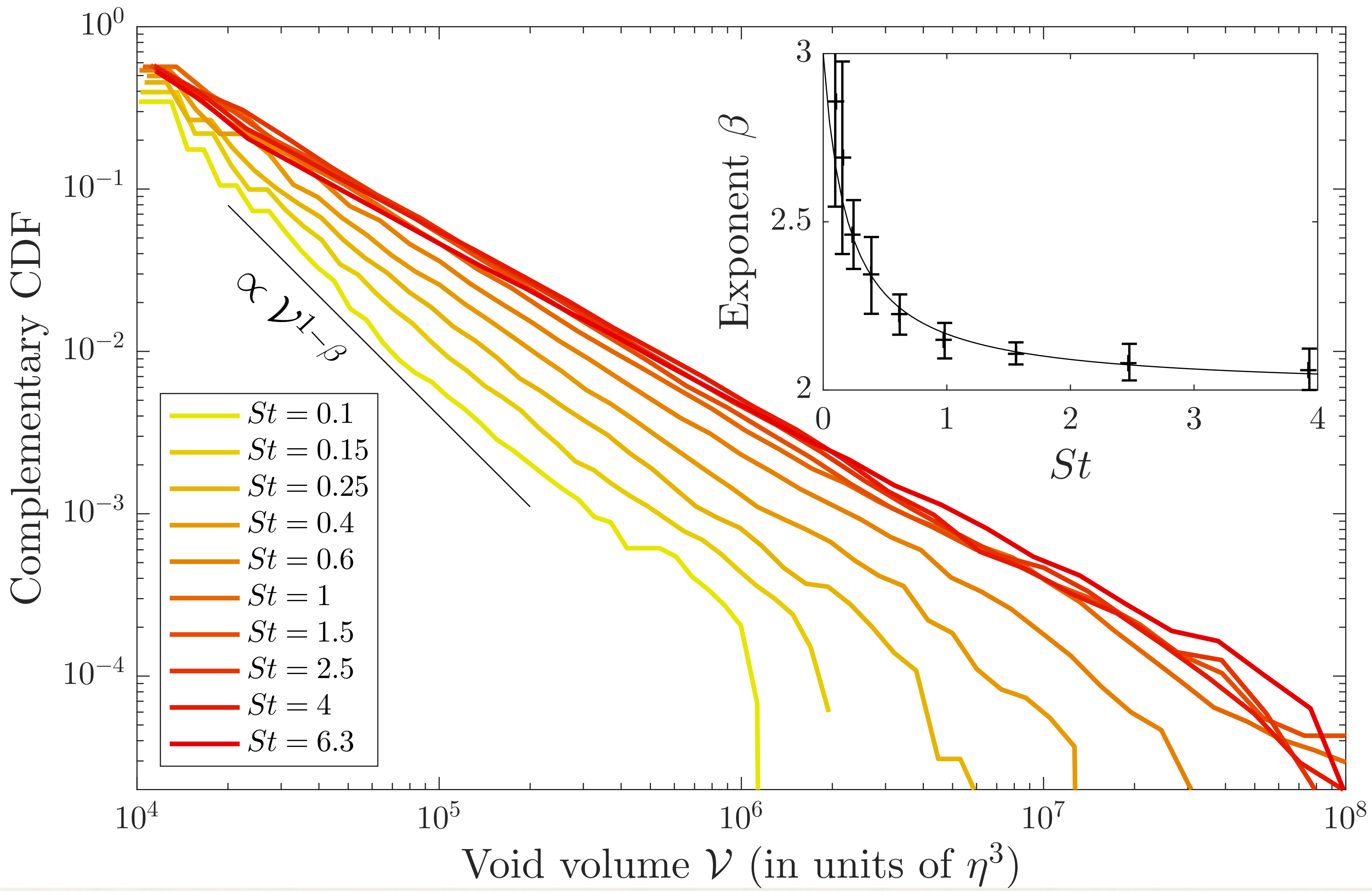
Connected boxes of size ℓ that do not contain any particle

Statistics of volumes $\mathcal{V}$ independent of the coarse-graining size ℓ

$\Rightarrow$ involve correlations between neighbouring boxes

Distribution of voids

Power-law distribution: $p(\mathcal{V}) \propto \mathcal{V}^{-\beta}$ with β depending on St



Summary / Conclusions

- Turbophoresis acts in statistically homogeneous flows because they display instantaneous non-uniformities
- Inertial-range particles dynamics can be described in terms of an effective diffusion equation with a space-dependent diffusivity that is determined by the local turbulent activity
- The inertial-range distribution of particles can be inferred from such a model. However, statistics that span different scales (e.g. voids) require accounting for spatial correlations of the diffusion coefficient.