Eikonal exponentiation and Universality in ultra-relativistic gravitational scattering

Paolo Di Vecchia

Niels Bohr Institute, Copenhagen and Nordita, Stockholm

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- This talk is based on the work at two loops in gravity and in massive sugra together with

  C. Heissenberg, R. Russo and G. Veneziano
  arXiv:2008.12743 and arXiv:2101.05772

- and at one loop on Einstein gravity with
  A. Koemans-Collado and R. Russo (2019)
  arXiv:1904.02667
  and with
  A. Cristofoli, P. Damgaard and C. Heissenberg
  arXiv:2003.10274

- and on two papers on $N = 8$ massless supergravity with

  S. Naculich, R. Russo, G. Veneziano and C. White (2019)
  arXiv:1908.05603 and arXiv:1911.11716

Plan of the talk

Introduction

- The discovery of gravitational waves at LIGO, generated by black hole merging, poses the problem of computing very precisely the dynamics of binary black hole merging.
- In the past this has mostly been done by solving Einstein's equations in the presence of the two black holes.
- Mostly using the expansion for small velocities: Post-Newtonian (PN) expansion.
- More recently a different approach has been used: extract classical quantities from the quantum scattering amplitude.
- When the two black holes are far away from each other, one can use perturbation theory expanding in powers of the Newton's constant G_N: Post-Minkowskian (PM) expansion.
- When they get closer to each other, their interaction becomes very strong and one must use Numerical Relativity (NR).
- In recent years computations of higher loop orders have been developed both for QCD and for studying the UV properties of supergravity.

- Can we use the methods of quantum field theory to compute in a more efficient way classical quantities at higher orders?
- The KEY IDEA is to extract from the full quantum calculation the EIKONAL that contains the information about classical quantities.
- At the tree level the elastic scattering amplitude of two massless particles is given by the diagram where a single graviton is exchanged:

$$iA_0(s,t) = 8\pi i G_N \hbar \frac{s^2}{-t} + \dots \quad ; \quad t = -(p_1+p_4)^2 \; ; \; s = -(p_1+p_2)^2$$

$\dots$ stand for subleading term at high energy.
- Going to impact parameter space one gets

$$2i\delta_0 = \int \frac{d^{D-2}\vec{q}}{(2\pi\hbar)^{D-2}} \frac{iA_0(s,t=-\vec{q}^2)}{2s} \, e^{\frac{i}{\hbar}\vec{b}\vec{q}} = i\frac{Gs}{\hbar} \frac{\Gamma(\frac{D}{2}-2)}{(\sqrt{\pi}b)^{D-4}}$$

- This quantity is called the (leading) eikonal.
- In the classical limit, it is natural to take b, s and the length scale $R^{D-3} \sim G_N\sqrt{s}$ (in analogy with the Schwarzschild radius) as classical quantities characterising the collision.

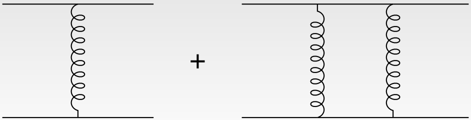


Figure: Examples of ladder diagrams involved in the exponentiation of the leading energy contributions.

- In terms of these classical quantities the (lead.) eikonal becomes

$$2i\delta_0 = i\frac{\Gamma(\frac{D}{2}-2)}{\pi^{\frac{D-4}{2}}}\left(\frac{R}{b}\right)^{D-3}\frac{b\sqrt{s}}{\hbar}$$

- The regime we are describing is the one in which

$$\frac{\hbar}{\sqrt{s}} << R << b$$

corresponding to classical regime on the left and perturbative regime on the right.
- $2\delta_0$ is a big quantity and the factor $1/\hbar$ signals that this quantity should appear in an exponential $\mathrm{e}^{2i\delta_0}$, so it can describe the value of the classical action.
- The exponentiation can be shown by summing (ladder) diagrams with the exchange of many gravitons
  Kabat and Ortiz, hep-th/9203082
- Conversely, the hypothesis that the eikonal exponentiates fixes the high energy behaviour of the multiloop diagrams.

- From the eikonal we can extract the deflection angle:

$$\sin\frac{\theta}{2} = -\frac{\hbar}{\sqrt{s}}\frac{\partial}{\partial b}2\delta_0 = \frac{R}{b} \;\; ; \;\; R = 2G_N\sqrt{s} \quad \textit{for} \quad D = 4$$

- What happens at smaller values of b?
- We have to extract the subleading eikonal.
- It is contained in the scattering amplitude at one loop.
- We have first to check that, in impact parameter space, the leading term at high energy comes from the exponentiation of the leading eikonal.
- We call this term super-classical because it behaves as $\frac{1}{\hbar^2}$.
- Then extract the subleading classical term $2\delta_1$.
- It turns out that in the massless case this term is vanishing.
- There is a quantum term $\hbar^0$ that diverges logarithmically at high energy that is also very important for the exponentiation.
- In order to get the next classical term in the massless case we have to go to two loops (3PM).

- The basic assumption that has been verified in many cases is that, in the Regge limit, the full amplitude is encoded in the expression:

$$i\tilde{A}(s,b) \equiv \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{iA(s,q^2=-t)\mathrm{e}^{ibq}}{4Ep} = (1+2i\Delta(s,b))\,\mathrm{e}^{2i\delta} - 1$$

  where $\delta = \delta_0 + \delta_1 + \delta_2 + \cdots$ is the classical eikonal and Δ encodes the quantum corrections.
- First compute the amplitude and then, from it, extract the classical contribution δ that then can be used to compute the deflection angle.
- Unitarity relation:

$$SS^\dagger = 1 \Longrightarrow 2ImT_{ab} = \sum_n T_{an}T^\dagger_{nb} \;\; ; \;\; S = 1 + iT$$

- In the massless case in GR it was computed by Amati, Ciafaloni and Veneziano (ACV90) getting ($\epsilon = \frac{4-D}{2}$)

$$\text{Re}(2\delta_2) \simeq \frac{4G_N^3 s^2}{\hbar b^2} + \dots \;\; ; \;\; 2\delta_0 = -\frac{Gs}{\epsilon\hbar}\Gamma(1-\epsilon)(\pi b^2)^\epsilon$$

and

$$\text{Im}(2\delta_2) \simeq \frac{1}{2s}\frac{(8G_N s)^3 \log s \Gamma^3(1-\epsilon)}{16(\pi b^2)^{1-3\epsilon}}\left[-\frac{1}{4\epsilon} + \frac{1}{2} + \mathcal{O}(\epsilon)\right]$$

- From the real part one can compute the deflection angle:

$$\sin\frac{\theta}{2} = -\frac{\hbar}{\sqrt{s}}\frac{\partial}{\partial b}(2\delta_0 + Re(2\delta_2)) = \frac{R}{b} + \frac{R^3}{b^3} + \dots$$

where $R \equiv 2G\sqrt{s}$.

- Einstein's light deflection by the Sun is given by the first term if $\sqrt{s} \to M$, M is the mass of the Sun.

- The result for Re$(2\delta_2)$ is confirmed for $\mathcal{N} = 8$ supergravity Naculich, Russo, White, Veneziano, PDV, 1911.11716 and in GR (massless scalars) and $\mathcal{N} \geq 4$ supergravities Bern, Ita, Parra-Martinez, Ruf, 2002.02459.
- Everybody agrees that the result found in ACV90 is universal (valid for any massless theory)
- The two-loop calculation with two arbitrary masses m_1 and m_2 was performed by Bern, Cheung, Roiban, Shen, Solon, Zeng, 1901.04424 and 1908.01493.
- Their result is confirmed in two papers by Kälin, Liu, Porto, 2007.04977 and Cheung and Solon, 2003.08351
- They computed the conservative 3PM contribution to the deflection angle.

- The total deflection angle is equal

$$\frac{1}{2}\chi^{tot} = \frac{\chi_1}{j} + \frac{\chi_2}{j^2} + \frac{\chi_3^{cons}}{j^3} \;\; ; \;\; j = \frac{J}{Gm_1 m_2} \;\; ; \;\; J = pb$$

where

$$\chi^1 = \frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} \;\; ; \;\; \chi^2 = \frac{3\pi}{8}\frac{(m_1 + m_2)(5\sigma^2 - 1)}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \sigma}}$$

in terms of the variable

$$\sigma = -\frac{p_1 p_2}{m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}$$

- χ^1 and χ^2 have been known for long time.
- The deflection angle is a classical (finite at high energy) quantity:

$$\frac{\chi_1}{j} \sim \frac{G_N \sigma}{p} \sim \frac{R\sqrt{\sigma}}{p} \;\; ; \;\; \frac{\chi_2}{j} \sim \frac{G_N^2 \sigma^{3/2} M}{p^2} \sim R^2 \frac{\sigma^{1/2} M}{p^2}$$

- At 3PM Bern et al. computed a new term that Damour wrote in the following suggestive form

$$\chi_3^{cons} = \chi_3^{Schw} - \frac{2\nu\sqrt{\sigma^2 - 1}}{h^2(\nu, \sigma)} \bar{C}^{cons}(\sigma)$$

where

$$h(\nu, \sigma) = \sqrt{1 + 2\nu(\sigma - 1)} = \frac{\sqrt{s}}{m_1 + m_2} \;\; ; \;\; \nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\chi_3^{Schw} = \frac{64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5}{3(\sigma^2 - 1)^{3/2}}$$

together with

$$\bar{C}^{cons}(\sigma) = \frac{2}{3}\sigma(14\sigma^2 + 25) + \frac{4(4\sigma^4 - 12\sigma^2 - 3)}{\sqrt{\sigma^2 - 1}} \sinh^{-1}\sqrt{\frac{\sigma - 1}{2}}$$

- Finite at high energy (without $\sinh^{-1}$ factor)

$$\frac{\chi_3^{cons}}{j^3} \sim \frac{G_N^3 \sigma^3}{p^3} \sim R^3 \frac{\sigma^{3/2}}{p^3}$$

- Since

$$\sinh^{-1}\sqrt{\frac{\sigma-1}{2}} \sim \frac{1}{2}\log\sigma$$

the term with $\sinh^{-1}$ diverges at high energy.

- It comes entirely from the H diagram.
- What does it mean such divergence?
- Damour in 1912.02139 has tried to correct it, but his proposal is also in contradiction with ACV90.
- On the other hand, the previous expression agrees with 6PN calculations, Blümlein, Maier, Marquard, Schäfer, 2003.07145
- This logarithmic divergence has been confirmed in the case of $\mathcal{N}=8$ supergravity where the massive states (describing the black holes) are obtained by Kaluza-Klein reduction of ten-dimensional type IIA string theory Parra-Martinez, Ruf and Zeng, 2005.04236

- This has led many people to conclude that there are two kinds of universalities.
- One for massless states and the other for massive states.
- This was justified by the fact that in one case one takes $q^2 << s, m^2$ and in the other case $m^2 = 0 < q^2 << s$
- Once the choice is made, one cannot go back to the other case.
- This contradicts the idea that at high energy one should have only one universality and a finite deflection angle.
- The calculations in the massive case rely on the approximation of the potential region.
- It consists in approximating the momentum of the gravitons exchanged by $(k^0, \vec{k}) \sim (qv, q)$ where v (the relative velocity in the com frame) is small.
- and then resumming over the expansion in velocity.
- In the soft region one instead approximates $k^\mu \sim q^\mu$.

- We computed the various two loop integrals that appear in massive $\mathcal{N}=8$ supergravity in the soft region using the diff. eq. with boundary conditions given directly in the soft region and in the ultra-relativistic limit we obtained:

$$
\begin{aligned}
& A_2(s,q^2) \\
& \simeq \frac{(8\pi G_N)^3 s^3}{(4\pi)^4}\left(\frac{4\pi\hbar^2 \mathrm{e}^{-\gamma_E}}{q^2}\right)^{2\epsilon}\left\{-\frac{2\pi^2 s}{\hbar^3\epsilon^2\frac{q^2}{\hbar^2}}-\frac{4\pi(i-\pi)}{\hbar\epsilon^2}\right. \\
& \left.+\frac{1}{\hbar\epsilon}\left[4\pi^2+8\pi i\log\frac{s}{m_1 m_2}-8\pi i-i\frac{\pi^3}{3}\right]\right\}+\mathcal{O}(\epsilon^0)
\end{aligned}
$$

whose real part is not divergent at high energy!

- Going to impact parameter space, the real part can be written as follows:

$$\mathrm{Re}\,\tilde{A}_2(s,b) \simeq -\frac{i}{6}(2i\delta_0)^3 - \mathrm{Im}(2\Delta_1)2\delta_0 + \frac{4G_N^3 s^2(\pi b^2)^{3\epsilon}\Gamma^3(1-\epsilon)}{b^2} + \mathcal{O}(\epsilon)$$

- From which we can extract

$$\mathrm{Re}(2\delta_2) \simeq \frac{4G_N^3 s^2}{\hbar b^2} + \mathcal{O}(\epsilon)$$

getting the same result as in ACV90!

- The real part of the eikonal to 3PM for arbitrary masses with $s = m_1^2 + m_2^2 + 2m_1 m_2 \sigma$ (with $\sigma \geq 1$)

$$\mathrm{Re}(\delta_2) = \frac{2G_N^3 (2m_1 m_2 \sigma)^2}{b^2} \times \left[\frac{\sigma^4}{(\sigma^2 - 1)^2} - \cosh^{-1}(\sigma) \left(\frac{\sigma^2}{\sigma^2 - 1} - \frac{\sigma^3 (\sigma^2 - 2)}{(\sigma^2 - 1)^{5/2}} \right) \right]$$

- The corresponding 3PM contribution to the scattering angle as a function of the angular momentum $J = pb$ reads

$$\chi_{\rm 3PM} = -\frac{16 m_1^3 m_2^3 \sigma^6 G_N^3}{3J^3 (\sigma^2 - 1)^{3/2}} + \frac{32 m_1^4 m_2^4 \sigma^6 G_N^3}{J^3 (\sigma^2 - 1) s} - \frac{64 m_1^4 m_2^4 G_N^3 \sigma^4}{J^3 s} \left(1 - \frac{\sigma (\sigma^2 - 2)}{(\sigma^2 - 1)^{3/2}} \right) \mathrm{arcsinh} \sqrt{\frac{\sigma - 1}{2}}$$

that is finite at high energy! ; $\sigma = \frac{s - m_1^2 - m_2^2}{2m_1 m_2} = -\frac{p_1 p_2}{m_1 m_2}$

- The interpretation of the previous result is that the deflection angle computed by Bern et al contains only the conservative part and does not take into account the effect of the radiation-reaction.
- When one takes into account the effect of the radiation reaction then one gets extra terms that eliminate the problem with the divergence in the ultra-relativistic limit.
- After our paper Damour in 2010.01641 has added a term coming from radiation (loss of angular momentum) to the conservative part of Bern et al and again found a finite behaviour at high energy: his result is valid in GR.
- According to the last Damour's paper the 3PM contribution to the deflection angle is equal to

$$\chi_3 = \chi_3^{Schw} - \frac{2\nu\sqrt{\sigma^2-1}}{h^2(\sigma,\nu)} \bar{C}^{Tot}(\sigma) \, ; \chi_3^{Schw} = \frac{64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5}{3(\sigma^2-1)^{3/2}}$$

where

$$\bar{C}^{Tot} = \bar{C}^{cons} + \bar{C}^{rad} \, , \;\; h^2(\sigma,\nu) = 1 + 2\nu(\sigma - 1) \, , \, \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

- where

$$\bar{C}^{cons}(\sigma) = \frac{2}{3}\sigma(14\sigma^2 + 25) + \frac{4(4\sigma^4 - 12\sigma^2 - 3)}{\sqrt{\sigma^2 - 1}} \sinh^{-1}\sqrt{\frac{\sigma - 1}{2}}$$

- and

$$\bar{C}^{rad} = -\frac{(2\sigma^2 - 1)^2}{\sqrt{\sigma^2 - 1}}\left(-\frac{8}{3} + \frac{1}{v^2} + 2\frac{3v^2 - 1}{v^3}\sinh^{-1}\sqrt{\frac{\sigma - 1}{2}}\right)$$

  where

$$v = \frac{\sqrt{\sigma^2 - 1}}{\sigma}$$

- More recently, in 2101.05772, we have included radiation reaction effects, due to the emission of soft gravitons, and we found agreement with Damour.
- The two calculations by Damour and us gave the same result, but it is still not clear, at a more physical level, why.

- In conclusion the deflection angle in GR up to 3PM is given by

$$\frac{\chi}{2} = \frac{\chi_1}{j} + \frac{\chi_2}{j^2} + \frac{\chi_3}{j^3} \quad ; \quad j = \frac{J}{G m_1 m_2} \quad ; \quad J = pb$$

where

$$\left(\sigma = -\frac{p_1 p_2}{m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2}, h^2(\nu, \sigma) = 1 + 2\nu(\sigma - 1), \nu = \frac{m_1 m_2}{(m_1+m_2)^2}\right)$$

$$\chi_1 = \frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} \quad ; \quad \chi_2 = \frac{3\pi}{8} \frac{(m_1 + m_2)(5\sigma^2 - 1)}{\sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \sigma}}$$

$$\chi_3 = \frac{64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5}{3(\sigma^2 - 1)^{\frac{3}{2}}} - \frac{2\nu\sqrt{\sigma^2 - 1}}{h^2(\sigma, \nu)}$$

$$\times \left\{ \frac{2\sigma(14\sigma^2 + 25)}{3} - \frac{(2\sigma^2 - 1)^2}{\sqrt{\sigma^2 - 1}} \left(-\frac{8}{3} + \frac{\sigma^2}{\sigma^2 - 1} \right) \right.$$

$$+ \left[\frac{4(4\sigma^4 - 12\sigma^2 - 3)}{\sqrt{\sigma^2 - 1}} - \frac{2(2\sigma^2 - 1)^2 \sigma (2\sigma^2 - 3)}{(\sigma^2 - 1)^2} \right]$$

$$\times \sinh^{-1} \sqrt{\frac{\sigma - 1}{2}}$$

- My talk is divided in two parts.
- In the first part, using analyticity and crossing symmetry of the scattering amplitude, we will show that a logarithmic divergence in the real part cannot be present.
- This argument applies to GR and confirms the result of ACV90.
- In the second part we will consider $\mathcal{N} = 8$ supergravity with massive states obtained by the KK reduction of type II A superstring to $M^4 \times T^6$.
- In this case there are only few (with respect to GR) diagrams at two loop that can, with some effort, be computed using the technique of the diff. eq. involving for each diagram several master integrals.
- We solve the diff. eqs. giving boundary conditions in the soft region.
- The log. divergent part coming from the H diagram is cancelled by an analogous one coming from the other two-loop diagrams and the deflection angle has a finite value in the UR limit.

Analyticity, crossing symmetry and asymptotic(GR)

- We start by extracting the 3PM eikonal using the methods of ACV90 that are based on the following properties:
- 1. Real analyticity of the scattering amplitude $A(s^*, t) = A^*(s, t)$ as a function of the complex variable s at $t \leq 0$. $t = -q^2$ is the exchanged momentum (squared).
  2. $s \leftrightarrow u$ crossing symmetry $A(s, t) = A(u, t)$ with $u = -s - t + 2(m_1^2 + m_2^2)$.
  3. Some information about its high-energy asymptotic behaviour: at ℓ-loop order the leading term at high energy behaves as $s^{\ell+2}$.
  4. Eikonal exponentiation
- Help from an amusing mathematical analogy with high-energy hadron scattering in QCD.
- The elastic hadron amplitude $A^{Had}(s, t)$ is believed to behave, at high energy, as $s \log^p s$.
- An important quantity is $\frac{ReA^{Had}(s,0)}{ImA^{Had}(s,0)}$.
- Such a ratio, for $t \neq 0$, plays also an important role in ACV90 and in the present approach.

- The constraints coming from analyticity and crossing, being linear, apply at each loop order, at each order in ϵ and also to different terms in the high-energy expansion (Pomeron and subleading Regge contributions).
- Assuming 1, 2, 3 and

$$\mathrm{Im} A(s,t) \sim s^n \log^p s \ \ \textit{and} \ \ |A| s^{-n-1} \to 0$$

one can write $n+1$ subtracted dispersion relation:

$$\mathrm{Re}\, A(s,t) = Q_{2m}(s,t) + \frac{2}{\pi} s^{2m+2} \, \mathcal{P} \int_{s_0}^{\infty} ds' \, \frac{\mathrm{Im}\, A(s',t)}{s'^{2m+1}(s'^2 - s^2)}$$

$s_0 = (m_1 + m_2)^2$ is the s-channel threshold, $Q_{2m}(s,t)$ is a polynomial of degree $2m$, $\mathcal{P}$ denotes the principal part, and the integer m is defined by $n = 2m$ for n even or by $n = 2m+1$ for n odd.

- From the previous dispersion relation one can compute $\rho = \frac{\text{Re}\,A(s,t)}{\text{Im}\,A(s,t)}$.
- For n even one gets:

$$\frac{\text{Re}\,A(s,t)}{s^n} = \frac{2}{\pi} s^2 \mathcal{P} \int_{s_0}^{\infty} ds' \frac{\text{Im}\,A(s',t) s'^{-n}}{s'(s'^2 - s^2)} + \frac{Q_n(s,t)}{s^n}$$

$$\Rightarrow \rho = \frac{2}{\pi} s^2 (\log s)^{-p} \mathcal{P} \int_{s_0}^{\infty} ds' \frac{\log^p s'}{s'(s'^2 - s^2)} + \frac{Q_n(s,t)}{s^n \log^p s}$$

$$\sim -\frac{2\log s}{(1+p)\pi}$$

- while for n odd one gets:

$$\frac{\text{Re}\,A(s,t)}{s^n} = \frac{2}{\pi} s \mathcal{P} \int_{s_0}^{\infty} ds' \frac{\text{Im}\,A(s',t) s'^{-n}}{(s'^2 - s^2)} + \frac{Q_{n-1}(s,t)}{s^n}$$

$$\Rightarrow \rho = \frac{2}{\pi} s (\log s)^{-p} \mathcal{P} \int_{s_0}^{\infty} ds' \frac{\log^p s'}{(s'^2 - s^2)} + \frac{Q_{n-1}(s,t)}{s^n \log^p s} \sim \frac{\pi p}{2\log s}$$

- At the tree level the elastic amplitude is given by one graviton exchange diagram:

$$A_0(s,t) = -\frac{8\pi G_N s^2 \left(1 - \frac{\Sigma}{s} + \mathcal{O}(s^{-2})\right)}{t} + \text{analytic terms in } t$$

where $\Sigma \equiv 2\left(m_1^2 + m_2^2\right)$.

- and correspondingly:

$$2\delta_0 = \tilde{A}_0 \equiv \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \frac{A_0\left(s, t=-\vec{q}^{\,2}\right)\ e^{i\vec{q}\cdot\vec{b}}}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}$$

$$= G_N s\left(1 - \frac{\Sigma}{2s} + \mathcal{O}(s^{-2})\right)\Gamma(-\epsilon)(\pi b^2)^{\epsilon}$$

- $4Ep = 4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}$

- Following ACV90 and 1911.11716, in the Regge limit the full amplitude is encoded in the expression:

$$i\tilde{A}(s,b) \equiv \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{iA(s,q^2=-t)\mathrm{e}^{ibq}}{4Ep} = (1+2i\Delta(s,b))\,\mathrm{e}^{2i\delta} - 1$$

where $\delta = \delta_0 + \delta_1 + \delta_2 + \cdots$ is the classical eikonal and Δ encodes the quantum corrections.

- At one-loop we know that A must include a leading imaginary term, growing like s^3 and responsible for the start of the exponentiation.
- This comes from the box and crossed box in the form:

$$\mathrm{Im}\, A_1^{(1)}(s,t) = s^3\left(1 - \frac{3\Sigma}{2s} + \mathcal{O}\left(s^{-2}\right)\right) F_1(t, m_i^2)$$

$$\text{with } s^3\left(1 - \frac{3\Sigma}{2s}\right)\tilde{F}_1 = \frac{(2\delta_0)^2}{2},$$

which therefore fixes F_1 modulo analytic terms as $t \to 0$.

- Analyticity and crossing symmetry imply

$$A_1^{(1)}(s,t) = -\frac{1}{\pi}\left[\left(s-\frac{\Sigma}{2}\right)^3 \log(-s) + \left(u-\frac{\Sigma}{2}\right)^3 \log(-u)\right] F_1(t,m_i^2)$$

$$\sim s^3\left(1-\frac{3\Sigma}{2s}\right) F_1(t,m_i^2)\left(i+\frac{3t}{\pi s}(\log s + \mathcal{O}(1))\right) + \mathcal{O}(\Sigma^2 s, t^2 s)\,,$$

as confirmed by explicit calculations.

- From explicit calculations it is known that there is a quantum term contributing:

$$A_1^{(2)}(s,t) \sim s^2 G_1(t,m_i^2)(-i\pi + 2\log s) + \mathcal{O}(s^2)$$

in agreement with what one gets from analyticity and crossing for $n=2$ (n even) with $p=0$.

- Both at one and two loops there are additional structures containing the masses.
- They take care of each other and we do not consider them here.

- At two loops, one needs a term $\sim s^4$ to reproduce the third-order term in the exponentiation of $2\delta_0$. From anal. + crossing symmetry

$$A_2^{(1)}(s,t) = \frac{1}{2}\left(s^4 + u^4\right) F_2(t, m_i^2)$$
$$\sim \left(s^4 + 2s^3(t-\Sigma) + \mathcal{O}(t^2 s^2, \Sigma^2 s^2)\right) F_2(t, m_i^2)$$
$$\text{with } s^4 \left(1 - 2\frac{\Sigma}{s}\right) \tilde{F}_2(b, m_i^2) = -\frac{1}{6}(2\delta_0)^3$$

- $F_2(t, m_i^2)$ is known (up to analytic terms at $t = 0$).
- The terms of order s^2 are not classical.
- In order to have a classical term we need a term that behaves as s^3 apart from possible logs.
- Let us parametrize the latter in the form

$$\mathrm{Im}\, A_2^{(2)}(s,t) = G_2(t, m_i^2) s^3 \log^p(s) \ \text{ with some } p > 0$$

- Then from analyticity and crossing symmetry we get

$$\mathrm{Re}\, A_2^{(2)}(s,t) = \frac{\pi p}{2 \log s}\, \mathrm{Im}\, A_2^{(2)}(s,t) \left(1 + \mathcal{O}\left(\frac{1}{\log^2 s}\right)\right)$$

- Before proceeding further let us check the previous analysis on massless $\mathcal{N} = 8$ supergravity where the two loop amplitude has been computed [Henn and Mistlberger,1902.07221].
- Two-loop amplitude for $s - u$ symmetric case:

$$A_2(s, q^2) = \frac{(8\pi G_N)^3}{(4\pi)^4} \left(\frac{4\pi e^{-\epsilon\gamma_E}}{q^2} \right)^{2\epsilon} B^2(\epsilon) \left\{ -\frac{2\pi^2}{\epsilon^2 q^2} \left(s^4 - 2q^2 s^3 \right) \right.$$
$$-\frac{4\pi i s^3}{\epsilon^2} - \frac{2\pi i s^3}{\epsilon^3}$$
$$\left. +\frac{s^3}{\epsilon} \left[4\pi \left(\pi + 2i \log s \right) + 8\pi i + \frac{7i\pi^3}{12} \right] + \mathcal{O}(\epsilon^0) \right\}$$

where

$$B(\epsilon) = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \quad ; \quad q^2 \equiv -t$$

- We use exponentiation to argue

$$
\begin{aligned}
&\operatorname{Re}\tilde{A}_2(s,b) = \operatorname{Re}\tilde{A}_2^{(1)}(s,b) + \operatorname{Re}\tilde{A}_2^{(2)}(s,b) \\
&= -\frac{4}{3}\delta_0^3 - 4\delta_0 \operatorname{Im}\Delta_1 + 2\operatorname{Re}\delta_2 \\
&\Rightarrow \operatorname{Re}\tilde{A}_2^{(2)}(s,b) = 2\operatorname{Re}\delta_2 - 2s^3\widetilde{(tF_2)} - 4\delta_0 \operatorname{Im}\Delta_1 \\
&2\operatorname{Im}\Delta_1 = -\pi s^2\,\tilde{G}_1\,,
\end{aligned}
$$

- and

$$
\begin{aligned}
&\operatorname{Im}\tilde{A}_2(s,b) = 2\operatorname{Im}\delta_2 + 4\delta_0 \operatorname{Re}\Delta_1 \\
&2\operatorname{Re}\Delta_1 = 2s^2\log s\,\tilde{G}_1 + \frac{3}{\pi}s^2\log s\,\widetilde{(tF_1)}\,,
\end{aligned}
$$

- The term $4\delta_0\,\operatorname{Re}(\Delta_1)$ represents the full elastic contribution to the s-channel discontinuity of the amplitude while $2\,\operatorname{Im}(\delta_2)$ represents the inelastic (3 particle) contribution.

- Use the previous equation to connect $\text{Re}\,\delta_2$ to $2\,\text{Im}\,\delta_2$ that is a quantity easy to compute

$$
\begin{aligned}
2\,\text{Re}(\delta_2) &= \frac{\pi p}{2\log s}(2\,\text{Im}\,\delta_2) + \pi(p-1)2\delta_0 s^2 \tilde{G}_1 \\
&+ \frac{3p}{2} s^2 (2\delta_0)\widetilde{(tF_1)} + 2s^3\widetilde{(tF_2)} + \mathcal{O}\left(\frac{1}{\log s}\right) \\
&= \frac{\pi p}{2\log s}(2\,\text{Im}\,\delta_2) - \frac{4-3p}{s}\delta_0(2\nabla\delta_0)^2 \\
&-(p-1)(2\delta_0)(2\,\text{Im}\,\Delta_1) + \mathcal{O}\left(\frac{1}{\log s}\right) ,
\end{aligned}
$$

where we have expressed $\widetilde{(tF_1)}$ and $\widetilde{(tF_2)}$ in terms of δ_0, and $\tilde{G}_1$ in terms of $\text{Im}\,\Delta_1$.
- $\tilde{G}_1$ is not a universal quantity, but the dependence on $\tilde{G}_1$ drops out if $p = 1$!

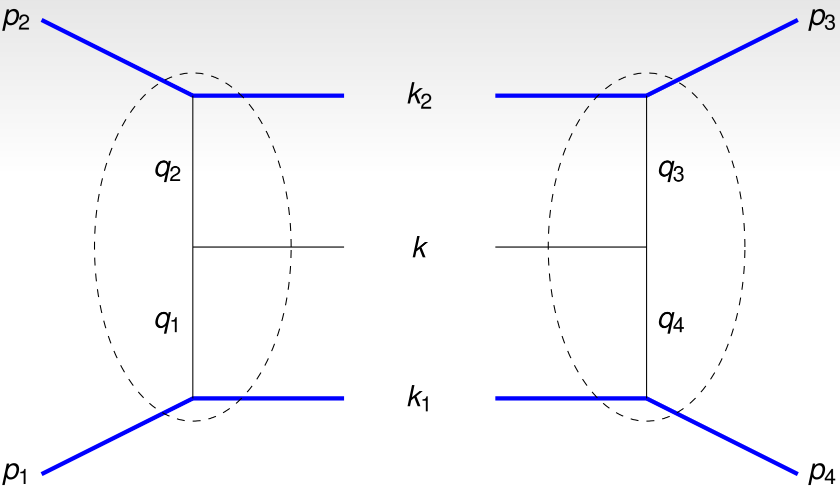
- We will show that $p = 1$ and therefore we obtain

$$\text{Re}(2\delta_2) = \frac{\pi}{2\log s}\,\text{Im}(2\delta_2) - \frac{\delta_0}{s}(\nabla 2\delta_0)^2 + \mathcal{O}\left(\frac{1}{\log s}\right)$$

- Both $\text{Im}(\delta_2)$ and $\delta_0(\nabla\delta_0)^2$ are IR divergent, but these divergences cancel so that physical observables derived from $\text{Re}(\delta_2)$, such as the deflection angle, are finite.

Computation of the 3-particle cut

- In order to extract $\mathrm{Im} 2\delta_2$ we compute the three-particle discontinuity represented by the following diagram:



- The five-point amplitude involving four scalars and one graviton is given by

$$
\begin{aligned}
M^{\mu\nu} = 2(8\pi G_N)^{\frac{3}{2}} \Bigg\{ & (k_1 p_2)(k_2 p_1) \left(-\frac{k_{1\mu}}{k_1 k} + \frac{k_{2\mu}}{k_2 k} \right) \left(-\frac{p_{2\nu}}{p_2 k} + \frac{p_{1\nu}}{p_1 k} \right) \\
& + 4 q_1^2 q_2^2 \left[\frac{q_1^\mu (p_1 p_2) + p_2^\mu (p_1 k) - p_1^\mu (p_2 k)}{q_1^2 q_2^2} + \frac{k_2^\mu}{2 k_2 k} \left(\frac{p_1 p_2}{q_1^2} + \frac{1}{2} \right) \right. \\
& \left. - \frac{k_1^\mu}{2 k_1 k} \left(\frac{p_1 p_2}{q_2^2} + \frac{1}{2} \right) \right] \\
& \times \left[\frac{q_1^\nu (k_1 k_2) + k_2^\nu (k_1 k) - k_1^\nu (k_2 k)}{q_1^2 q_2^2} - \frac{p_1^\nu}{2 p_1 k} \left(\frac{k_1 k_2}{q_2^2} + \frac{1}{2} \right) \right. \\
& \left. + \frac{p_2^\nu}{2 p_2 k} \left(\frac{k_1 k_2}{q_1^2} + \frac{1}{2} \right) \right] \Bigg\},
\end{aligned}
$$

obtained, using KLT, from the amplitude with one gluon.

- Unlike ACV90 the four external particles are massive.

- Compute the contribution of the 3-particle discontinuity to the imaginary part of the elastic amplitude from the unitarity relation:

$$2\,\mathrm{Im}\,A_2^{(3p)} = \frac{1}{\left((2\pi)^{D-1}\right)^3}\int \frac{d^{D-1}k_1}{2E_{k_1}}\int \frac{d^{D-1}k_2}{2E_{k_2}}\int \frac{d^{D-1}k}{2E_k}$$
$$\times M^{\mu\nu}(p_1,p_2;k_1,k_2,k)M_{\mu\nu}(-k_1,-k,-k_2;p_3,p_4)$$
$$\times(2\pi)^D\delta^{(D)}(p_1+p_2+k_1+k_2+k)$$

- We compute it in the double-Regge limit

$$s \gg s_1, s_2 \to \infty\,, \quad \text{with}\ \frac{s_1 s_2}{s}\,,\ q_i^2 = -(p_i+k_i)^2\,,\ m_i^2\ \text{fixed}$$

where $s = -(p_1+p_2)^2$ and $s_i = -(k+k_i)^2$.

- In this limit the amplitude has no dilaton contribution as an intermediate state.

- In the double-Regge limit the amplitude becomes much simpler:

$$M^{\mu\nu}(p_1, p_2 \to k_2, k, k_1) = (8\pi G_N)^{3/2} \frac{(2p_1 p_2)^2}{2q_1^2 q_2^2}$$

$$\times \Bigg\{ \left[(q_1 - q_2)^\mu - \frac{s_1 s_2}{s} \left(-\frac{k_1^\mu}{k_1 k} + \frac{k_2^\mu}{k_2 k} \right) + \frac{k_2^\mu}{k_2 k} q_2^2 - \frac{k_1^\mu}{k_1 k} q_1^2 \right]$$

$$\times \left[(q_1 - q_2)^\nu - \frac{s_1 s_2}{s} \left(-\frac{k_1^\nu}{k_1 k} + \frac{k_2^\nu}{k_2 k} \right) + \frac{k_2^\nu}{k_2 k} q_2^2 - \frac{k_1^\nu}{k_1 k} q_1^2 \right]$$

$$-q_1^2 q_2^2 \left(-\frac{k_{1\mu}}{k_1 k} + \frac{k_{2\mu}}{k_2 k} \right) \left(\frac{k_{2\nu}}{k_2 k} - \frac{k_{1\nu}}{k_1 k} \right) \Bigg\}$$

where all momenta p_1, p_2, k_1, k_2, k are ingoing and

$$s = -(p_1 + p_2)^2 \ , \ \ s_1 = -(k_1 + k)^2 \ , \ \ s_2 = -(k_2 + k)^2$$
$$q_1 = -(k_1 + p_1) \ , \ \ q_2 = -(p_2 + k_2) \ ; \ \ k = q_1 + q_2$$

- We get the same expression as in the massless case.

- In this regime it is convenient to write the kinematic variables in terms of the $(D-2)$ space-like vectors orthogonal to the direction where the energetic states are boosted (taken to be x^{D-1}).
- By working in the Breit frame and taking light-cone variables for the time and longitudinal direction $(p_0 + p_{D-1}, \vec{p}, p_0 - p_{D-1})$, we have

$$p_1 = (\overline{m}_1 \mathrm{e}^{y_1}, -\frac{\vec{q}}{2}, \overline{m}_1 \mathrm{e}^{-y_1}) \; ; \; p_2 = (\overline{m}_2 \mathrm{e}^{y_2}, \frac{\vec{q}}{2}, \overline{m}_2 \mathrm{e}^{-y_2})$$

$$p_4 = (-\overline{m}_1 \mathrm{e}^{y_1}, -\frac{\vec{q}}{2}, -\overline{m}_1 \mathrm{e}^{-y_1}) \;\; ; \;\; p_3 = (-\overline{m}_2 \mathrm{e}^{y_2}, \frac{\vec{q}}{2}, -\overline{m}_2 \mathrm{e}^{-y_2})$$

  y_i are the rapidities of the ext. particles and $\overline{m}_{1,2}^2 = m_{1,2}^2 + \frac{\vec{q}^{\,2}}{4}$.
- The intermediate states with momentum k_1, k_2, k (all incoming) are

$$k_1 = (-\overline{m}_1' \mathrm{e}^{y_1'}, \frac{\vec{q}}{2} - \vec{q}_1, -\overline{m}_1' \mathrm{e}^{-y_1'}) \; ; \; k_2 = (-\overline{m}_2' \mathrm{e}^{y_2'}, -\frac{\vec{q}}{2} - \vec{q}_2, -\overline{m}_2' \mathrm{e}^{-y_2'})$$

$$k = (-|k| \mathrm{e}^{y}, \vec{k}, -|k| \mathrm{e}^{-y})$$

  where $(\overline{m}_1')^2 = m_1^2 + (\frac{\vec{q}}{2} - \vec{q}_1)^2$ and $(\overline{m}_2')^2 = m_2^2 + (\frac{\vec{q}}{2} + \vec{q}_2)^2$

- We can use the two non-transverse δ-functions to perform the integral over the rapidities $y'_{1,2}$ provided that

$$-\log \frac{\sqrt{s}}{m_{t,k}} \leq y_k \leq \log \frac{\sqrt{s}}{m_{t,k}} \Longrightarrow \int dy_k = 2 \log \frac{\sqrt{s}}{m_{t,k}} \sim \log s$$

- In the double Regge limit, one can approximate $q_i^2 \sim \vec{q}_i^{\,2}$ getting the result of ACV90

$$\text{Im}\, A_2^{(3p)} \simeq \frac{(16\pi G_N)^3 s^3 \log s}{2\pi} \int \frac{d^{D-2}\vec{q}_1}{(2\pi)^{D-2}} \int \frac{d^{D-2}\vec{q}_2}{(2\pi)^{D-2}} \frac{1}{(\vec{k}^2)^2}$$
$$\times \left[\frac{\left[(\vec{q}_1\vec{q}_4)(\vec{q}_2\vec{q}_3) + (\vec{q}_1\vec{q}_2)(\vec{q}_3\vec{q}_4) - (\vec{q}_1\vec{q}_3)(\vec{q}_2\vec{q}_4)\right]^2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{q}_3^{\,2}\vec{q}_4^{\,2}} \right.$$
$$\left. +1 - \frac{(\vec{q}_1\vec{q}_2)^2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}} - \frac{(\vec{q}_3\vec{q}_4)^2}{\vec{q}_3^{\,2}\vec{q}_4^{\,2}} \right]$$

where $k = q_1 + q_2, q_4 = q - q_1, q_3 = -q - q_2$.

- Then, we can proceed as in ACV90.
- The integral over y_k gives a *logs* term and the other integrals can be performed getting

$$\text{Im}\,\widetilde{A_2^{(3p)}}(s,b) \simeq \frac{1}{2s}\frac{(8G_N s)^3 \log s \Gamma^3(1-\epsilon)}{16(\pi b^2)^{1-3\epsilon}}\left[-\frac{1}{4\epsilon}+\frac{1}{2}+\mathcal{O}(\epsilon)\right]$$

- This result implies $p = 1$ and is nothing else than $2\,\text{Im}(\delta_2)$ and we can use it to obtain:

$$\text{Re}(2\delta_2) \simeq \frac{4G_N^3 s^2}{\hbar b^2}$$

as in ACV90.

The 2-loop amplitude in massive $\mathcal{N} = 8$ sugra

- By performing a KK reduction of 10-dimensional IIA supergravity to $M^4 \times T^6$ one obtains, at the massless level, $\mathcal{N} = 8$ sugra.
- One gets also massive states arising from KK states getting their mass from the component of the momentum in the six extra dimensions.
- Those states are scalar fields in $D = 4$ that can be taken with different masses if they acquire their mass from different components of the momentum in the six extra dimensions.
- We consider a $s - u$ symmetric case whose tree scattering amplitude is equal to

$$A_0(s, q^2) = \frac{8\pi G_N}{q^2} \frac{1}{2} \Big((s - m_1^2 - m_2^2)^2 + (u - m_1^2 - m_2^2)^2 - q^4 \Big)$$

- The advantage is that this theory with respect to GR contains a much simpler set of diagrams+no numerators.
- The disadvantage is that it contains also the exchange of the dilaton.

- We closely follow the approach of Parra-Martinez, Ruf and Zeng, 2005.04236 computing the various diagrams in the soft region by solving various diff. equations.
- With the important difference that also the boundary conditions are given in the soft region and not in the potential region.
- In the Regge limit at tree level we get

$$A_0 = \frac{32\pi G_N m_1^2 m_2^2 \sigma^2}{q^2} \; ; \; \sigma = -\frac{p_1 p_2}{m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2}$$

$$x = \sigma - \sqrt{\sigma^2 - 1} \; ; \; 4pE = 4 m_1 m_2 (\sigma^2 - 1) = \frac{2 m_1 m_2 (1 - x^2)}{x}$$

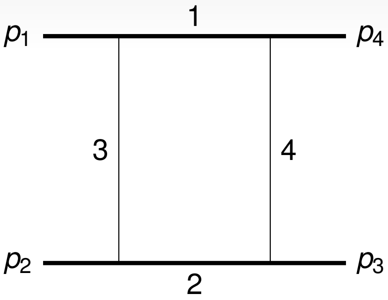
- From it we can get the leading eikonal:

$$2\delta_0(s, b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \mathrm{e}^{ibq} \frac{A_0(s, q^2)}{4pE} = G_N m_1 m_2 (\pi b^2)^{\epsilon} \Gamma(-\epsilon) \frac{2\sigma^2}{\sqrt{\sigma^2 - 1}}$$

- At one loop we get

$$A_1(s, q^2) = \frac{(8\pi G_N)^2}{2}$$
$$\times\left((s - m_1^2 - m_2^2)^4 + (u - m_1^2 - m_2^2)^4 - q^8\right)(I_{II} + I_{\widetilde{II}})$$

where we get contribution from the box diagram



and from the crossed box.

- We get

$$\frac{A_1(s,q^2)}{4pE} = \frac{(8\pi G_N)^2}{(4\pi)^2}\left(\frac{4\pi}{q^2}\right)^{\epsilon}\left\{\frac{2i\pi m_1^2 m_2^2}{q^2}\frac{\sigma^4}{\sigma^2-1}\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\right.$$

$$+\frac{4\sqrt{\pi}m_1 m_2(m_1+m_2)}{\sqrt{q^2}}\frac{\sigma^4}{(\sigma^2-1)^{\frac{3}{2}}}\frac{\Gamma(\epsilon+\frac{1}{2})\Gamma^2(\frac{1}{2}-\epsilon)}{\Gamma(-2\epsilon)} - \frac{\sigma^3}{(\sigma^2-1)^2}$$

$$\times\left[m_1 m_2\left[(1+2\epsilon)\left(\sigma^2\log x+\sigma\sqrt{\sigma^2-1}\right)+i\pi\left(2(\sigma^2-1)+\epsilon\sigma^2\right)\right]\right.$$

$$\left.\left.+i\frac{\pi\epsilon}{2}(m_1^2+m_2^2)\sigma\right]\frac{\Gamma^2(-\epsilon)\Gamma(1+\epsilon)}{\Gamma(-2\epsilon)}\right\}$$

- Go to impact parameter for the first, second and the last two lines

- From the term in the first line we get

$$i\tilde{A}_1^{(1)}(s,b) = \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \frac{iA_1^{(1)}(s,t=-\vec{q}^{\,2})}{4Ep} \, \mathrm{e}^{\mathrm{i}\vec{b}\vec{q}}$$

$$= \frac{1}{2}\left(\frac{2\mathrm{i}m_1 m_2 G_N (\pi b^2)^{\epsilon} \sigma^2 \Gamma(-\epsilon)}{\hbar^2\sqrt{\sigma^2-1}}\right)^2 = \frac{1}{2}(2\mathrm{i}\delta_0)^2$$

that reproduces the quadratic term of the expansion of the leading eikonal.

- From the term in the second line we get

$$i\tilde{A}_1^{(2)} \equiv \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \frac{iA_1^{(2)}(s, t=-\vec{q}^{\,2})}{4Ep} e^{i\vec{b}\vec{q}} = 2i\delta_1$$

$$= \frac{4(\pi b^2)^{2\epsilon} m_1 m_2 (m_1+m_2) G_N^2}{\sqrt{\pi}\sqrt{b^2}} \left[\frac{1+x^2}{x} \left(\frac{1+x^2}{1-x^2} \right)^3 \right]$$

$$\times \frac{\Gamma(\frac{1}{2}-2\epsilon)\Gamma^2(\frac{1}{2}-\epsilon)}{\Gamma(-2\epsilon)}$$

$$= \frac{8i(\pi b^2)^{2\epsilon} G^2 m_1 m_2 (m_1+m_2)}{\hbar\sqrt{\pi}\sqrt{b^2}} \frac{\sigma^4}{(\sigma^2-1)^{\frac{3}{2}}} \frac{\Gamma(\frac{1}{2}-2\epsilon)\Gamma^2(\frac{1}{2}-\epsilon)}{\Gamma(-2\epsilon)}$$

that vanishes for $\epsilon = 0$ in agreement with Caron.Huot, Zahraee, 1810.04694.

From the last two lines we extract a quantum term

$$2Re\Delta_1 = \frac{4m_1 m_2 G_N^2(\pi b^2)^{2\epsilon}}{\pi b^2} \left(\frac{1+x^2}{1-x^2}\right)^3 \left[\frac{1+x^2}{x} + \frac{(1+x^2)^2}{x(1-x^2)}\right.$$

$$\left.\times \log x\right] (1+2\epsilon)\Gamma^2(1-\epsilon)$$

$$= \frac{8G_N^2 m_1 m_2 (\pi b^2)^{2\epsilon}}{\pi b^2} \frac{\sigma^4 \left(\sigma \log x + \sqrt{\sigma^2 - 1}\right)}{(\sigma^2-1)^2} (1+2\epsilon)\Gamma^2(1-\epsilon)$$

that in the UR ($s >> m_1^2, m_2^2$) behaves as

$$2Re\Delta_1 \Longrightarrow \frac{4G_N^2(\pi b^2)^{2\epsilon}}{\pi b^2} s(-\log \frac{s}{m_1 m_2} + 1)(1+2\epsilon)\Gamma^2(1-\epsilon)$$

and

$$2Im\Delta_1 = \frac{4(\pi b^2)^{2\epsilon} G_N^2}{b^2} \left(\frac{1+x^2}{1-x^2} \right)^3$$

$$\times \left[\epsilon (m_1^2 + m_2^2) \frac{1+x^2}{1-x^2} + 2m_1 m_2 \frac{1-x^2}{x} + \epsilon m_1 m_2 \frac{(1+x^2)^2}{x(1-x^2)} \right] \Gamma^2(1-\epsilon)$$

$$= \frac{8G^2(\pi b^2)^{2\epsilon}\Gamma^2(1-\epsilon)}{b^2} \frac{\sigma^3}{(\sigma^2-1)^2} \left[\frac{\epsilon}{2}(m_1^2+m_2^2)\sigma + 2m_1 m_2(\sigma^2-1) \right.$$

$$\left. + \epsilon m_1 m_2 \sigma^2 \right] \Longrightarrow \frac{4G^2(\pi b^2)^{2\epsilon}(\epsilon+2)s\Gamma^2(1-\epsilon)}{b^2}$$

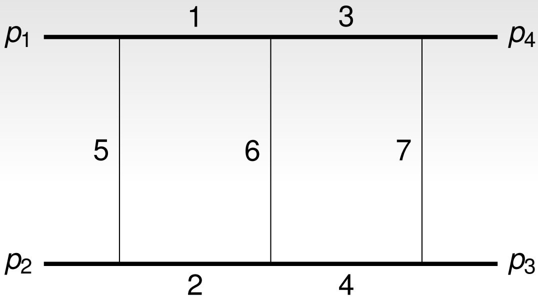
- At two loops we get

$$A_2(s,q^2) = \frac{(8\pi G_N)^3}{2}\left((s-m_1^2-m_2^2)^4 + (u-m_1^2-m_2^2)^4 - t^4\right)$$

$$\times\Bigg[(s-m_1^2-m_2^2)^2\,(I_{III}+I_{IX}+I_{XI})$$

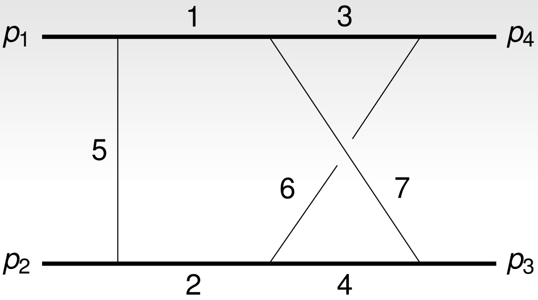
$$+(u-m_1^2-m_2^2)^2\,\left(I_{\widetilde{III}}+I_{\widetilde{IX}}+I_{\widetilde{XI}}\right)+t^2(I_H+I_{\overline{H}})\Bigg]\;\;;\;\; q^2=-t\,.$$

- in terms of the double box diagram



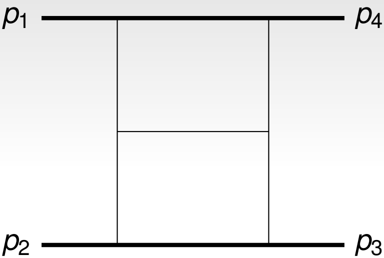
of its crossed version

- of the non-planar double box



of its crossed version

- and of the H diagram



and its crossed version

- In the UR limit one gets:

$$A_2(s,q^2) \simeq \frac{(8\pi G_N)^3 s^3}{(4\pi)^4}\left(\frac{4\pi e^{-\gamma_E}}{q^2}\right)^{2\epsilon}\left\{-\frac{2\pi^2 s}{\epsilon^2 q^2}\left(1+\frac{2t}{s}\right)-\frac{4\pi i}{\epsilon^2}\right.$$
$$\left.+\frac{1}{\epsilon}\left[4\pi^2\left(1+\frac{2i}{\pi}\log\frac{s}{m_1 m_2}\right)-8\pi i-i\frac{\pi^3}{3}\right]\right\}+\mathcal{O}(\epsilon^0)$$

- Going to impact parameter we get

$$\tilde{A}_2(s,b) \simeq \left\{\frac{G_N^3 s^3(\pi b^2)^{3\epsilon}\Gamma^3(1-\epsilon)}{6\hbar^3\epsilon^3}-\frac{8G_N^3(i-\pi)s^2(\pi b^2)^{3\epsilon}\Gamma^3(1-\epsilon)}{\epsilon\pi b^2\hbar}\right.$$
$$\left.+\frac{2G_N^3 s^2\Gamma^3(1-\epsilon)}{(\pi b^2)^{1-3\epsilon}\hbar}\left[4\pi+8i\log\frac{s}{m_1 m_2}-8i-i\frac{\pi^2}{3}\right]+\mathcal{O}(\epsilon)\right\}$$

- In order to compute the new contribution to the eikonal we must first subtract the contribution of the lower eikonal δ_0 and Δ_1 that are equal to

$$2\delta_0 = \frac{G_N s \Gamma(1-\epsilon)(\pi b^2)^\epsilon}{-\epsilon \hbar}$$

$$2\,\text{Im}\,\Delta_1 \simeq \frac{8 G_N^2 s (\pi b^2)^{2\epsilon} \Gamma^2(1-\epsilon)}{b^2}\left(1+\frac{\epsilon}{2}\right)$$

- Using them we can write the real part of the amplitude as follows:

$$\text{Re}\,\tilde{A}_2(s,b) \simeq -\frac{i}{6}(2i\delta_0)^3 - \text{Im}(2\Delta_1)2\delta_0$$
$$+\frac{4 G_N^3 s^2 (\pi b^2)^{3\epsilon}\Gamma^3(1-\epsilon)}{b^2 \hbar} + \mathcal{O}(\epsilon)$$

recovering the universal value for $\text{Re}\,2\delta_2$.

- Up to now all results are in the UR limit ($s >> m_i^2$).

- In the general case (with arbitrary masses) from the two-loop amplitude we can extract the sub-sub-leading eikonal through the relations:

$$Re(2\delta_2) = Re\tilde{A}_2 - \frac{1}{6}(2i\delta_0)^3 + 2\delta_0 2Im2\Delta_1$$

where $\tilde{A}_2(s, b)$ is the Fourier transform in impact parameter space of the two-loop amplitude divided by the factor $4pE$.

- We get

$$Re(2\delta_2) = \frac{16m_1^2m_2^2G_N^3\sigma^6}{b^2(\sigma^2-1)^2} - \frac{16m_1^2m_2^2\sigma^4G_N^3}{b^2(\sigma^2-1)}\cosh^{-1}(\sigma)$$
$$\times\left[1 - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right] \;;\; \cosh^{-1}(\sigma) = \log\left(\sigma + \sqrt{\sigma^2-1}\right)$$

- The corresponding 3PM contribution to the scattering angle as a function of the angular momentum $J = pb$ reads

$$\chi_{3\text{PM}} = -\frac{16 m_1^3 m_2^3 \sigma^6 G_N^3}{3J^3 \left(\sigma^2 - 1\right)^{3/2}} + \frac{32 m_1^4 m_2^4 \sigma^6 G_N^3}{J^3 \left(\sigma^2 - 1\right) s}$$

$$-\frac{64 m_1^4 m_2^4 G_N^3 \sigma^4}{J^3 s} \left(1 - \frac{\sigma\left(\sigma^2 - 2\right)}{\left(\sigma^2 - 1\right)^{3/2}}\right) \operatorname{arcsinh} \sqrt{\frac{\sigma - 1}{2}}$$

that is finite at high energy! ; $\sigma = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2} = -\frac{p_1 p_2}{m_1 m_2}$

- Blue terms are those obtained with boundary conditions in the potential region.
- Red terms are the additional terms obtained with boundary conditions in the soft region.
- In the PN expansion the sum of the terms in red are of the order 1.5 PN, while the terms in blue are of the order 0 PN and 2 PN.

Solving differential equations (one-loop case)

- In the elastic scattering of two particles

$$p_1 + p_2 + p_3 + p_4 = 0 \;\; ; \;\; -p_1^2 = m_1^2 = p_4^2 \;\; ; \;\; p_2^2 = m_2^2 = p_3^2$$

it is convenient to introduce the new variables:

$$p_1 = -\bar{p}_1 + q/2\,, \quad p_4 = \bar{p}_1 + q/2$$
$$p_2 = -\bar{p}_2 - q/2\,, \quad p_3 = \bar{p}_2 - q/2$$

and

$$u_1^\mu = \frac{\bar{p}_1^\mu}{\bar{m}_1}, \quad u_2^\mu = \frac{\bar{p}_2^\mu}{\bar{m}_2},$$

with

$$\bar{m}_1^2 = -\bar{p}_1^2 = m_1^2 + \frac{q^2}{4}, \quad \bar{m}_2^2 = -\bar{p}_2^2 = m_2^2 + \frac{q^2}{4}\,.$$

- They are useful because $\bar{p}_i$ is orthogonal to q

$$p_1^2 - p_4^2 = -2\,\bar{p}_1 \cdot q = 0$$
$$p_2^2 - p_3^2 = 2\,\bar{p}_2 \cdot q = 0$$

- Introduce the quantity

$$y = -u_1 \cdot u_2 \sim \sigma - \frac{2m_1 m_2 + \sigma(m_1^2 + m_2^2)}{8m_1^2 m_2^2} q^2 + \mathcal{O}\left(q^4\right)$$

that tends to 1 in the static limit and to ∞ for large velocities.

- Introduce also

$$x_y = y - \sqrt{y^2 - 1} \ \ ; \ \ x = \sigma - \sqrt{\sigma^2 - 1} \ \ ; \ \ \sigma = -\frac{p_1 p_2}{m_1 m_2}$$

- We want to compute the box diagram

$$I_{\rm II} = \int_\ell \frac{1}{[\bar{m}_1 \rho_1 + (\ell^2 - q \cdot \ell)][\bar{m}_2 \rho_2 + (\ell^2 - q \cdot \ell)]\rho_3 \rho_4}$$

where

$$\rho_1 = 2u_1 \cdot \ell - i0, \quad \rho_2 = -2u_2 \cdot \ell - i0, \quad \rho_3 = \ell^2 - i0, \quad \rho_4 = (\ell - q)^2 - i0$$

- To get the non-analytic dependence on q^2 that gives the classical contribution we expand the integrand in the soft region characterised by

$$\ell \sim q << m_{1,2}$$

- Expand the integrand in the soft region and IBP reduce it to get

$$I_{\rm II} = \frac{1}{\epsilon^2 \bar{m}_1 \bar{m}_2 \sqrt{y^2-1}} \frac{1}{q^2} f_3 + \frac{(\bar{m}_1+\bar{m}_2)}{\bar{m}_1^2 \bar{m}_2^2 (y-1)} \frac{1}{q} f_2 - \frac{(1+2\epsilon)\left(2\bar{m}_2\bar{m}_1 y + \bar{m}_1^2 + \bar{m}_2^2\right)}{8\epsilon^2 \bar{m}_1^3 \bar{m}_2^3 \left(y^2-1\right)^{3/2}} f_3 - \frac{(1+2\epsilon)\left[\left(\bar{m}_1^2+\bar{m}_2^2\right) y + 2\bar{m}_1\bar{m}_2\right]}{8\epsilon \bar{m}_1^3 \bar{m}_2^3 \left(y^2-1\right)} f_1$$

in terms of the three master integrals

$$f_1 = \epsilon\, q^2 G_{0,0,2,1}\,, \quad f_2 = -\epsilon\, q\, G_{1,0,1,1}\,, \quad f_3 = \epsilon^2 \sqrt{y^2-1}\, q^2 G_{1,1,1,1}$$

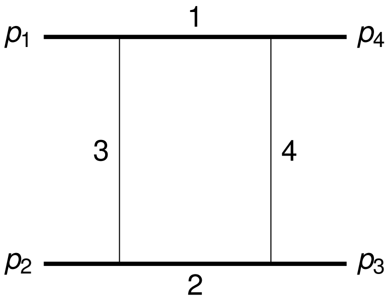
- The relevant integrals at one loop are

$$G_{i_1,i_2,i_3,i_4} = \int \frac{\mathrm{d}^D\ell \, e^{\gamma_E \epsilon}}{\mathrm{i}\pi^{D/2}} \frac{1}{\rho_1^{i_1}\rho_2^{i_2}\rho_3^{i_3}\rho_4^{i_4}}$$

where i_k are integers and

$$\rho_1 = 2u_1\cdot\ell - i0, \quad \rho_2 = -2u_2\cdot\ell - i0, \quad \rho_3 = \ell^2 - i0, \quad \rho_4 = (\ell - q)^2 - i0$$

-



- They are determined by the differential equation:

$$\frac{\mathrm{d}\vec{f}}{\mathrm{d}\log x_y} = \epsilon A \vec{f}\,, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Their solutions

$$f_1 = c_1\,, \qquad f_2 = c_2\,, \qquad f_3 = \epsilon\, c_1 \log x_y + c_3$$

are uniquely determined by fixing the integration constants c_1, c_2, c_3 using the boundary conditions at $x_y = 1$.

$$I_{\rm II} = -\frac{e^{\gamma_E \epsilon}\Gamma(1-\epsilon)^2\Gamma(\epsilon+1)\left(\log x + i\pi\right)}{m_1 m_2 \sqrt{\sigma^2-1}\,\epsilon\Gamma(1-2\epsilon)(q^2)^{1+\epsilon}}$$

$$+ \frac{e^{\gamma_E \epsilon}\sqrt{\pi}\,(m_1+m_2)\,\Gamma\left(\frac{1}{2}-\epsilon\right)^2\Gamma\left(\epsilon+\frac{1}{2}\right)}{4m_1^2 m_2^2(\sigma-1)\Gamma(-2\epsilon)(q^2)^{\frac{1}{2}+\epsilon}}$$

$$+ \frac{e^{\gamma_E \epsilon}\Gamma(1-\epsilon)^2\Gamma(\epsilon+1)\left[\sqrt{\sigma^2-1}\left(m_1^2\sigma + m_2^2\sigma + 2m_1 m_2\right) + s(\log x e^{i\pi})\right]}{4m_1^3 m_2^3(\sigma^2-1)^{3/2}\Gamma(1-2\epsilon)(q^2)^{\epsilon}}$$

Conclusions

- Analyticity and crossing symmetry allow to show that $\mathrm{Re}(2\delta_2)$ and the deflection angle are not diverging at high energy.
- Explicit evaluation of $\mathrm{Im}(2\delta_2)$ allows to fix $\mathrm{Re}(2\delta_2)$ in agreement with ACV90.
- Explicit evaluation of the loop integrals in the soft region in massive sugra shows a universal behaviour at high energy.
- The interpretation of the previous result is that the deflection angle computed by Bern et al contains only the conservative part and does not take into account the effect of the radiation-reaction.
- When one takes into account the effect of the radiation reaction then one gets extra terms that eliminate the problem with the divergence in the ultra-relativistic limit.
- Recently Damour in 2010.01641 has added a term coming from radiation to the conservative part of Bern et al and again found a finite behaviour at high energy: his result is valid in GR.

- According to the last Damour's paper the 3PM contribution to the deflection angle is equal to

$$\chi_3 = \chi_3^{Schw} - \frac{2\nu p}{h^2(\sigma,\nu)} \bar{C}^{Tot}(\sigma) \ ; \ \chi_3^{Schw} = \frac{64\sigma^6 - 120\sigma^4 + 60\sigma^2 - 5}{3(\sigma^2-1)^{3/2}}$$

where $\bar{C}^{Tot} = \bar{C}^{cons} + \bar{C}^{rad}$

- with

$$\bar{C}^{cons}(\sigma) = \frac{2}{3}\sigma(14\sigma^2+25) + \frac{4(4\sigma^4 - 12\sigma^2 - 3)}{\sqrt{\sigma^2-1}} \sinh^{-1}\sqrt{\frac{\sigma-1}{2}}$$

- and

$$\bar{C}^{rad} = -\frac{(2\sigma^2-1)^2}{2\sqrt{\sigma^2-1}} \left(-\frac{8}{3} + \frac{1}{v^2} + 2\frac{3v^2-1}{v^3} \sinh^{-1}\sqrt{\frac{\sigma-1}{2}} \right)$$

where

$$v = \frac{\sqrt{\sigma^2-1}}{\sigma}$$

Outlook

- Extend our results from massive $\mathcal{N} = 8$ sugra to GR.
- Compute radiation emitted in the process.
- Compute EOB Hamiltonian and the Potential at two-loop order in $\mathcal{N} = 8$ massive sugra: check if the potential is real.
- Go to higher orders: 4PM?

Additional explanatory slides

- The scattering angle in the elastic scattering of two particles is given in perturbation theory by

$$\sin\frac{\theta}{2} = \frac{|q|}{2p}$$

q is the momentum transfer $q^2 = -t = (p_1 + p_4)^2$ and p is the absolute value of the momentum in the center of mass frame. $2p = \sqrt{s}$ in the massless case.

- More precisely

$$q^2 = -t = (p_1 + p_4)^2 = 2p^2(1 - \cos\theta)$$

where in the c.o.m. frame we use

$$p_1 = (E_1, \vec{p}) \;\; ; \;\; p_4 = (-E_1, -\vec{p}') \;\; ; \;\; |\vec{p}| \equiv p = |\vec{p}'|$$

- After exponentiation we can transform back to momentum space getting

$$iA_L(s, \vec{Q}) = 2s \int d^{D-2}\vec{b}_e \left[\mathrm{e}^{\frac{i}{\hbar}\vec{b}_e\vec{Q}+2\mathrm{i}\delta_0(s,\vec{b}_e)} - 1 \right]$$

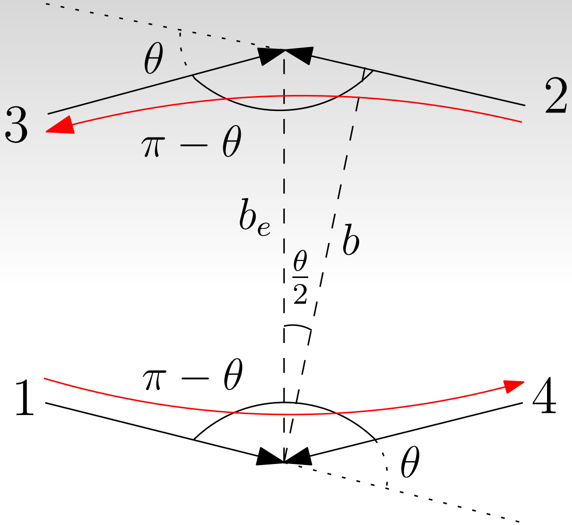
and here we get the momentum $\vec{Q}$ as the one that describes the full momentum exchanged in the eikonal process where many gravitons are exchanged, to distinguish it from q that is the momentum exchanged in the perturbative amplitude.

- For large b we can approximate the integral with the saddle point equation:

$$\vec{Q} = -\hbar \frac{\partial(2\delta_0)}{\partial b_e} \frac{\vec{b}_e}{b_e} = 2Gs \frac{\Gamma(\frac{D}{2}-1)}{\pi^{\frac{D}{2}-2} b_e^{D-3}} \frac{\vec{b}_e}{b_e}$$

- and after the resummation we get

$$\sin\frac{\theta}{2} = \frac{|Q|}{2p} = -\frac{\hbar}{2p} \frac{\partial(2\delta_0)}{\partial b_e}$$


θ
2
3
π − θ
b_e
b
θ/2
1
π − θ
4
θ

Why $D-2$ and not D

- In the Breit or brick wall frame one has

$$p_1 = (E_1, -\frac{\vec{q}}{2}, \bar{p}_1) \; ; \; p_4 = (-E_1, -\frac{\vec{q}}{2}, -\bar{p}_1)$$
$$p_2 = (E_2, \frac{\vec{q}}{2}, -\bar{p}_2) \; ; \; p_3 = (-E_2, \frac{\vec{q}}{2}, \bar{p}_2) \, .$$

The exchanged momentum $-q \equiv p_1 + p_4$ lies in a $(D-2)$ trans. space and the incoming particle bounces back along the $(D-1)$ dir..

- In e center of mass frame one has

$$p_1 = (E_1, 0, p) \quad ; \quad p_4 = (-E_1, q, -\sqrt{E_1^2 - q^2 - m_1^2})$$
$$p_2 = (E_2, 0, -p) \quad ; \quad p_3 = (-E_2, -q, \sqrt{E_2^2 - q^2 - m_2^2})$$
$$q = p_1 + p_4 = (0, q, p - \sqrt{p^2 - q^2}) \sim (0, q, \frac{q^2}{p} + \dots) \sim (0, q, 0)$$

$\frac{q^2}{p}$ negligible with respect to q at high energy and in the class. lim.

DIMENSION OF THE ELASTIC AMPLITUDE AND FACTORS OF $\hbar$

- The elastic amplitude has dimension $L^2 E^2 L^{-2\epsilon}$.
- G_N has dimension $L^{1-2\epsilon} E^{-1}$.

THE PROBE LIMIT

- In the probe limit m_1 becomes very large and there is no back reaction of the other particle on the very massive one.
- The scattering can be described by the motion of the probe particle in the metric generated by the very massive one.
- In the rest frame of the massive particle we get

$$s = -(p_1 + p_2)^2 = (m_1 + E_2)^2 - \vec{p}_2^{\,2} = m_1^2 + 2m_1 E_2 + m_2^2$$
$$\Longrightarrow \sigma = \frac{s - m_1^2 - m_2^2}{2m_1 m_2} = \frac{E_2}{m_2}$$

THE PN EXPANSION

In the Post-Newtonian expansion the counting is as follows:

$$\epsilon \sim v^2 \sim \frac{G_N m}{r} << 1 \;\; ; \;\; (v^2)^n G_N^m \sim \epsilon^{n+m} \Longrightarrow (n+m)PN$$

where

$$\sigma - 1 \sim v^2 \;\; ; \;\; \cosh^{-1}(\sigma) \sim (\sigma - 1)^{1/2} = v$$

- The change in length ΔL is given by:

$$\Delta L = hL$$

- $h \sim 10^{-21}$ is the strength of the gravitational waves.
- $L = 4 \cdot 10^3 m$ is the length of the two arms of the interferometer.
- Then we get

$$\Delta L = 10^{-16} cm = 10^{-3} proton$$

Waveform from GR

- In GR one can describe the merging or the scattering of two black holes by the following Lagrangian:

$$L = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R - \sum_{i=1}^{2} m_i \int d\tau_i \sqrt{-g_{\mu\nu}(x)\dot{x}_i^{\mu}\dot{x}_i^{\nu}}$$

  where $x(\tau)$ is a function of the world-line parameter τ and $\dot{x} = \frac{dx}{d\tau}$ and $2\kappa^2 = 16\pi G_N$ (G_N is the Newton constant).
- The previous Lagrangian describes the two black holes as point-particles.
- One can add additional terms containing the Riemann tensor.
- The lowest ones are

$$c_E \int d\tau E_{\mu\nu}E^{\mu\nu} + c_B \int d\tau B_{\mu\nu}B^{\mu\nu} + \ldots$$

  where

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta}\dot{x}^{\alpha}\dot{x}^{\beta} \;\; ; \;\; B_{\mu\nu} = \epsilon_{\mu\alpha\beta\rho}\dot{x}^{\rho}R^{\alpha\beta}{}_{\sigma\nu}\dot{x}^{\sigma}$$

- Because of these additional terms the particle does not move anymore along a geodesic and this implies stretching by tidal forces as in the case of extended objects.
- In order to find the motion of the particles and the waveform of the gravitational field one must derive the equations of motion from the previous Lagrangian and solve them.
- This is an extremely difficult problem.
- One can expand for small velocities and perform the so-called post-Newtonian (PN) expansion.
- But when the two black holes start to merge this approximation is not valid anymore and one must solve the problem numerically: numerical GR.
- A simple case is when one particle has a mass much bigger then the other.
- In this case one can compute the motion of the light particle in the gravitational field generated by the heavy particle: geodesic motion.

- In this seminar I will consider the scattering of two point-particle (rather than their merging) and concentrate myself on the calculation of the deflection angle.
- It can be shown that the two regimes are related.
- Compute the four-point scattering amplitude involving two scalar particles with mass m_1 and m_2 in a perturbative expansion in the Newton constant G_N: Post-Minkowskian expansion (PM).
- Extract from it classical quantities as the deflection angle and the Hamiltonian.
- We will neglect the spin.
- Our system is described by the following Lagrangian:

$$S = \int d^D x \sqrt{|G|} \left\{ \frac{R}{2\kappa_D^2} - \frac{1}{2} \sum_{i=1}^{2} \left[\partial_\mu \Phi_i \partial_\nu \Phi_i G^{\mu\nu} + m_i^2 \Phi_i^2 \right] \right\}$$

- We work in D space-time dimensions to regularise infrared divergences: dimensional regularisation.
- In perturbation theory we expand the metric $g_{\mu\nu}$ around flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \ ; \ \kappa = \sqrt{8\pi G_N}$$

- One has to choose a gauge: a very convenient gauge is the de Donder gauge: $\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = 0$.
- The expansion of the Einstein-Hilbert Lagrangian is non-polynomial in $h_{\mu\nu}$ and a lot of terms are generated.
- Very soon it becomes too complicated to proceed this way and new techniques have been used to go to higher loops.
- Use double copy or BCJ relations from gluon amplitudes that are much easier to construct.
- Construct the unitarity cuts relevant for classical physics that provide the integrand.
- Then you have to perform the integrals.

General mass case in D dimensions

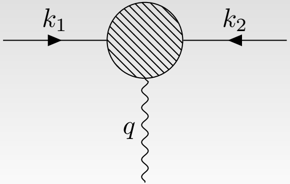
- Consider the scattering of two particles with mass m_1 and m_2 at 1PM and 2PM.
- In order to extract the classical contributions we take the limit:

$$16\pi G_N = 2\kappa^2 \to 0 \,, \quad s \gg q^2 = |t| \,, \quad \text{with} \;\; G_N M^* \;\; \text{fixed}$$

  where M^* is the largest mass scale in the process.
- Therefore we need to compute the amplitude in the Regge limit where $|t| << s$
- In this case the gravitons exchanged in the process are almost on shell and we can construct the amplitude by glueing together almost on-shell amplitudes.
- For 1PM we can use the three-point vertex involving a graviton and two massive scalars given by:

$$A^{\mu\nu}(k_1, k_2, q) = -i\kappa_D \left(k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - (k_1 k_2 - m^2)\eta_{\mu\nu} \right)$$

- For small q we can neglect the term proportional to $\eta^{\mu\nu}$ and use

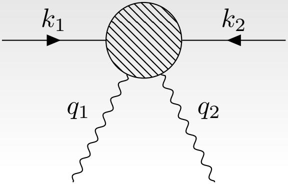


$$= A_3^{\mu\nu}(k_1, k_2, q) = -i\kappa_D \left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu\right)$$

- For 2PM we can use the four-point amplitude involving two gravitons and two massive scalars:

$$\hat{A}_4^{\alpha\beta;\rho\sigma}(k_1, k_2, q_1, q_2) = \frac{2\kappa_D^2(k_2 q_1)(k_1 q_1)}{(q_1 q_2)}$$
$$\times \left[\frac{k_2^\rho k_1^\alpha}{k_2 q_1} + \frac{k_2^\alpha k_1^\rho}{k_1 q_1} + \eta^{\rho\alpha}\right] \left[\frac{k_2^\sigma k_1^\beta}{k_2 q_1} + \frac{k_2^\beta k_1^\alpha}{k_1 q_1} + \eta^{\sigma\beta}\right].$$

- For our purposes it will be convenient to use the following expression



$$= A_4^{\alpha\beta;\rho\sigma}(k_1, k_2, q_1, q_2)$$

$$= \frac{-2\kappa_D^2[(k_1q_1) + (q_2q_1)](k_1q_1)}{(q_1q_2)}\left(\frac{(k_1+q_2)^\rho k_1^\alpha}{(k_1q_1)+(q_2q_1)} - \frac{(k_1+q_1)^\alpha k_1^\rho}{k_1q_1} + \eta^{\rho\alpha}\right.$$
$$\times\left(\frac{(k_1+q_2)^\sigma k_1^\beta}{(k_1q_1)+(q_1q_2)} - \frac{(k_1+q_1)^\beta k_1^\alpha}{k_1q_1} + \eta^{\sigma\beta}\right)$$

- The two previous amplitudes are equivalent on shell.
- The second one is transverse in a slightly more general sense.

- It vanishes when the polarization of the graviton takes the form $\epsilon^{\mu\nu} = q^\mu \zeta^\nu + q^\nu \zeta^\mu$ using only momentum conservation to rewrite products between momenta such as $k_i k_j$.
- Without the need of using it to rewrite the products between momenta and the arbitrary vectors ζ^μ.
- In other words, it is transverse without any restriction on the polarization ζ^μ.
- The 1PM and 2PM contributions are then obtained by glueing the gravitons with a de Donder propagator:

$$[G(q)]^{\mu\nu;\rho\sigma} = \frac{-i}{2q^2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right) .$$

- The S-matrix is given by

$$S = 1 + iT = 1 + (2\pi\hbar)^D \delta^{(D)}(P_f - P_i)A$$

- A_{fi} is the scattering amplitude from an initial state i to a final state f:

$$A_{fi} = \langle k_1, k_2 \dots k_{N_f} f | A | k_1{}', k_2{}' \dots k_{N_i}{}' \rangle$$

- Taking into account that the creation and annihilation operators satisfy the following covariant commutation relation:

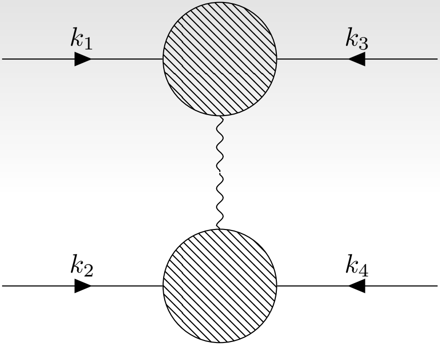
$$[a(k), a^\dagger(p)] = 2E_p (2\pi\hbar)^{D-1} \delta^{(D-1)}(\vec{k} - \vec{p})$$

we can derive that the amplitude A_{fi}, for $N_i = N_f = 2$ has dimension of $L^2 E^2 L^{-2\epsilon}$ where $2\epsilon = 4 - D$.

- For the 1PM we have to compute the following quantity

$$i\mathcal{A}_0 = [G(k_1+k_3)]_{\mu_1\nu_1;\mu_2\nu_2}\, A_3^{\mu_1\nu_1}(k_1, k_3, -k_1-k_3) A_3^{\mu_2\nu_2}(k_2, k_4, k_1+k_3)$$

represented by the following diagram:



getting

$$i\mathcal{A}_0 = \frac{2i\kappa_D^2}{q^2}\left(\frac{1}{2}(s - m_1^2 - m_2^2)^2 - \frac{2}{D-2} m_1^2 m_2^2\right) = \frac{2i\kappa_D^2 \gamma(s)}{\frac{q^2}{\hbar^2}\hbar}$$

- $q \equiv k_1 + k_3$ is the momentum exchanged between the two massive scalars and

$$\gamma(s) = 2(k_1 k_2)^2 - \frac{2}{D-2} m_1^2 m_2^2 = \frac{1}{2}(s - m_1^2 - m_2^2)^2 - \frac{2}{D-2} m_1^2 m_2^2$$

- We can go to impact parameter space by using:

$$\tilde{\mathcal{A}}(s,b) = \frac{1}{4Ep} \int \frac{\mathrm{d}^{D-2}\mathbf{q}}{(2\pi\hbar)^{D-2}} e^{i\frac{\mathbf{q}}{\hbar}\mathbf{b}} \mathcal{A}(s,q^2)$$

where $E = \sqrt{s} = E_1 + E_2$ and $p = |\vec{k_1}| = |\vec{k_2}|$ is the absolute value of the space-like momentum in the center of mass frame of the two scattering particles.

- At high energy q has only non-zero components along the $D-2$ directions orthogonal to the energy and the direction of motion.
- We get

$$i\tilde{\mathcal{A}}_0(s,b) \equiv 2i\delta_0 = \frac{i\kappa_D^2 \gamma(s)}{2Ep} \frac{1}{4\pi^{\frac{D-2}{2}}} \Gamma\left(\frac{D}{2} - 2\right) \frac{1}{\hbar\, \mathbf{b}^{D-4}}$$

that is dimensionless.

- In terms of the more convenient variable:

$$\sigma = -\frac{k_1 k_2}{m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2} \;\; ; \;\; 4pE = 4 m_1 m_2 \sqrt{\sigma^2 - 1}$$

the leading eikonal in the massive case is equal to

$$i\tilde{\mathcal{A}}_0(\sigma, b) \equiv 2i\delta_0 = -\frac{G_N m_1 m_2 \Gamma(1-\epsilon)(\pi b^2)^\epsilon (2\sigma^2 - \frac{2}{D-2})}{\epsilon \hbar \sqrt{\sigma^2 - 1}}$$

where $\epsilon = \frac{4-D}{2}$.

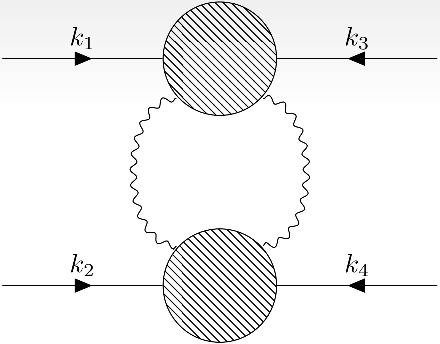
- It is the Fourier transform of the following tree-level amplitude

$$A_0(\sigma, q^2) = \frac{16\pi G_N m_1^2 m_2^2 (2\sigma^2 - \frac{2}{D-2})}{q^2}$$

- For 2PM we need to compute

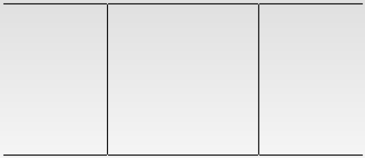
$$iA_2 = \int \frac{d^D k}{(2\pi)^D} [G(k)]_{\alpha_1\beta_1;\alpha_2\beta_2} \, [G(k+q)]_{\rho_1\sigma_1;\rho_2\sigma_2} \times A_4^{\alpha_1\beta_1;\rho_1\sigma_1}(k_1, k_3, k, -k-q) A_4^{\alpha_2\beta_2;\rho_2\sigma_2}(k_2, k_4, -k, k+q)$$

corresponding to the following diagram:

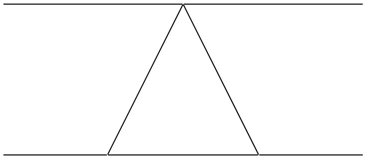


- The amplitude contains four propagators (two massive and two massless) and a numerator.

- The contribution to the classical limit is given by the two following topologies



corresponding to box diagram and



corresponding to triangular diagrams.

- They are extracted from the original amplitude.

- The box gives the following contribution to the amplitude:

$$i\mathcal{A}_2 = 4\kappa_D^4(\gamma^2(s)\mathcal{I}_4(s,t) + \gamma^2(u)\mathcal{I}_4(u,t))$$

where we have included the crossed diagram and

$$\mathcal{I}_4(s,t) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{(q+k)^2} \frac{1}{(k_1+k)^2+m_1^2} \frac{1}{(k_2-k)^2+m_2^2} \, ,$$

$$\mathcal{I}_4(u,t) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{1}{(q+k)^2} \frac{1}{(k_3+k)^2+m_1^2} \frac{1}{(k_2-k)^2+m_2^2}$$

- In order to extract the classical contributions we take the following limit:

$$16\pi G_N = 2\kappa^2 \to 0 \, , \quad s \gg q^2 = |t| \, , \quad \text{with } \; G_N M^* \; \text{ fixed}$$

where M^* is the largest mass scale in the process.

- The leading term is equal to

$$iA_2^{(1)}(s,q^2) = -\frac{4\pi^{\frac{D+2}{2}}\kappa_D^4\gamma^2(s)}{2(2\pi)^D\sqrt{(k_1k_2)^2 - m_1^2m_2^2}}\frac{\Gamma^2(\frac{D}{2}-2)\Gamma(3-\frac{D}{2})}{\hbar^2\Gamma(D-4)}(\frac{q^2}{\hbar^2})^{\frac{D-6}{2}}$$

- In impact parameter space it becomes:

$$i\tilde{A}_2^{(1)}(s,b) = -\frac{\kappa_D^4\gamma^2(s)}{(Ep)^2}\frac{1}{2^7\pi^{D-2}}\Gamma^2\left(\frac{D}{2}-2\right)\frac{1}{\hbar^2\mathbf{b}^{2D-8}} = \frac{1}{2}(i\tilde{A}_1)^2$$

  that is the first sign of an exponentiation.
- The subleading contribution is:

$$iA_2^{(2)}(s,q^2) = \frac{i2\kappa_D^4\gamma^2(s)\sqrt{\pi}(m_1+m_2)}{(4\pi)^{\frac{D}{2}}\left((k_1k_2)^2 - m_1^2m_2^2\right)}\frac{\Gamma\left(\frac{5-D}{2}\right)\Gamma^2\left(\frac{D-3}{2}\right)}{\hbar\Gamma(D-4)}(\frac{q^2}{\hbar^2})^{\frac{D-5}{2}}$$

- It vanishes for $D=4$.

- In impact parameter space it becomes:

$$i\tilde{\mathcal{A}}_2^{(2)}(s,b) = \frac{i\kappa_D^4\gamma^2(s)(m_1+m_2)}{Ep((k_1k_2)^2 - m_1^2m_2^2)} \frac{1}{64\pi^{D-\frac{3}{2}}} \frac{\Gamma\left(\frac{2D-7}{2}\right)\Gamma^2\left(\frac{D-3}{2}\right)}{\hbar\Gamma(D-4)} \frac{1}{\mathbf{b}^{2D-7}}$$

- We did not expect any big contribution from the subsubleading term, but actually we found a large logarithmic term.
- In impact parameter space we get:

$$i\tilde{\mathcal{A}}_2^{(3)}(s,b)$$
$$= \frac{\kappa_D^4\gamma^2(s)}{Ep}\frac{i}{128\pi^{D-1}}\Gamma^2\left(\frac{D-2}{2}\right)\frac{1}{(\mathbf{b}^2)^{D-3}}\left[\frac{4(5-D)}{(k_1k_2)^2-m_1^2m_2^2}\right.$$
$$\left.\times\left(1+\frac{2k_1k_2\,\mathrm{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{(k_1k_2)^2-m_1^2m_2^2}}\right)+i\frac{\pi(D-4)(k_1+k_2)^2}{[(k_1k_2)^2-m_1^2m_2^2]^{3/2}}\right]$$
$$+\frac{\kappa_D^4\psi(s)}{Ep}\frac{i}{8\pi^{D-1}}\frac{\mathrm{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{(k_1k_2)^2-m_1^2m_2^2}}\Gamma^2\left(\frac{D-2}{2}\right)\frac{1}{(\mathbf{b}^2)^{D-3}}$$

- No $\hbar$ in the denominator $\Longrightarrow$ it is a quantum contribution.
- By using $\text{arcsinh}\, y = \log(y + \sqrt{y^2+1})$ one can see that the terms with arcsinh are log-divergent at large energies.
- The same arcsinh-function also appears in the recent 3PM result, violating perturbative unitarity in the high-energy limit: $\frac{s}{m_i^2} \to \infty$ [Z. Bern et al arXiv:1901.04424]
- Finally, one gets also a classical contribution from the triangular diagrams.
- In impact parameter space one gets:

$$i\tilde{\mathcal{A}}_2^{(2)} = i \frac{\kappa_D^4}{64\pi^{D-\frac{3}{2}}\, Ep} \frac{\Gamma\left(\frac{2D-7}{2}\right)\Gamma^2\left(\frac{D-3}{2}\right)}{\Gamma(D-3)} \frac{m_1+m_2}{\hbar\, \mathbf{b}^{2D-7}} \left\{ (s-m_1^2-m_2^2)^2 - \frac{4m_1^2m_2^2}{(D-2)^2} - \frac{(D-3)\left((s-m_1^2-m_2^2)^2 - 4m_1^2m_2^2\right)}{4(D-2)^2} \right\}$$

The eikonal and the deflection angle

- $$2\delta_0 = -\frac{G_N m_1 m_2 \Gamma(1-\epsilon)(\pi b^2)^{\epsilon}(2\sigma^2 - \frac{2}{D-2})}{\epsilon \hbar \sqrt{\sigma^2 - 1}} \quad ; \; \epsilon = \frac{4-D}{2}$$
- The next to the leading eikonal is given by

$$2\delta_1(s, m_i, \mathbf{b}) = \frac{(8\pi G_N)^2 (m_1 + m_2)}{\hbar E p \pi^{D-\frac{3}{2}}} \frac{\Gamma(\frac{2D-7}{2})\Gamma^2(\frac{D-3}{2})}{16\, \mathbf{b}^{2D-7}}$$

$$\times \left\{ \frac{\gamma^2(s)}{\Gamma(D-4)\left[(s-m_1^2-m_2^2)^2 - 4m_1^2 m_2^2\right]} + \frac{1}{4\Gamma(D-3)} \right.$$

$$\times \left[(s-m_1^2-m_2^2)^2 - \frac{4m_1^2 m_2^2}{(D-2)^2} \right.$$

$$\left. \left. - \frac{(D-3)\left((s-m_1^2-m_2^2)^2 - 4m_1^2 m_2^2\right)}{4(D-2)^2} \right] \right\}$$

- The second line comes from the box diagrams and vanishes for $D = 4$.

- The rest comes from the triangular diagrams and coincides with known results for $D = 4$.
- In $D = 4$ it can be written in a compact form:

$$2\delta_1 = \frac{3\pi G_N^2 m_1 m_2 (m_1 + m_2)}{4\hbar\sqrt{\sigma^2 - 1}\, b}(5\sigma^2 - 1)$$

- The deflection angle is given by

$$\chi = -\frac{\hbar}{p}\frac{\partial}{\partial \mathbf{b}}(2\delta_0 + 2\delta_1) + \ldots$$

- The leading deflection angle is equal to

$$\theta_{1PM} = -\frac{\hbar}{p}\frac{\partial}{\partial \mathbf{b}} 2\delta_0 = \frac{2G_N m_1 m_2 \sqrt{s}(2\sigma^2 - 1)}{b(\sigma^2 - 1)} = \frac{G m_1 m_2}{J}\frac{2(2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}}$$

where $J = pb$.

- The subleading contribution is equal to

$$\theta_{2PM} = -\frac{\hbar}{p}\frac{\partial}{\partial \mathbf{b}} 2\delta_1 = \frac{3\pi G_N^2 m_1 m_2 (m_1 + m_2)(5\sigma^2 - 1)}{4\sqrt{\sigma^2 - 1} b^2 p}$$

$$= \left(\frac{G_N m_1 m_2}{J}\right)^2 \frac{m_1 + m_2}{\sqrt{s}} \frac{3\pi(5\sigma^2 - 1)}{4}$$

- It vanishes for $m_1 = m_2 = 0$ in agreement with ACV90.

- The subleading contribution to the deflection angle is equal

$$\theta^{(2)} = \frac{(8\pi G_N)^2(m_1+m_2)}{Ep^2\pi^{D-\frac{3}{2}}}\frac{2\Gamma(\frac{2D-5}{2})\Gamma^2(\frac{D-3}{2})}{16\,\mathbf{b}^{2D-6}}$$
$$\times\Bigg\{\frac{\gamma^2(s)}{\Gamma(D-4)\left[(s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right]}+\frac{1}{4\Gamma(D-3)}$$
$$\times\Bigg[(s-m_1^2-m_2^2)^2-\frac{4m_1^2m_2^2}{(D-2)^2}$$
$$-\frac{(D-3)\left((s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right)}{4(D-2)^2}\Bigg]\Bigg\}$$

- It vanishes for $m_1 = m_2 = 0$ in agreement with ACV90.
- Both the leading and subleading angles agree with what is obtained from alternative calculations (mostly for $D = 4$ where the box diagram does not contribute).
- For instance, in the probe limit (one mass much larger than the other and of the energy involved).