Some aspects of Classical-quantum dynamics

Jan Tuziemski, Stockholm University and NORDITA

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1. Realization of a classical system within quantum mechanics

2. A model of (consistent) classical quantum dynamics

Part 1

Koopman von Neumann mechanics

Koopman von Neumann mechanics

- Classical mechanics can be formulated in terms of Hilbert space
- In terms of states: one needs joint X and P eigenstates (they need to commute)
- In terms of dynamics: to have an analogue of Heisenberg equation of motion there should be operators not commuting with X and P

Koopman von Neumann mechanics

- Minimal realization involves 4 operators with non-trivial commutators:

$$[X_c, \lambda_{X_c}] = [P_c, \lambda_{P_c}] = i$$

- Liouville operator governing evolution of classical wavefunction

$$L_C = \frac{P_c}{m}\lambda_{X_c} - U'(X_c)\lambda_{P_c}$$

- Standard Liouville equation is recovered in XP representation, assuming that probability density is given by modulus squared of the wavefunction

Koopman von Neumann mechanics

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- Such a system can be realized within quantum mechanics

$$X_c = X_1 + X_2 \qquad \lambda_{P_c} = \frac{1}{2}(X_2 - X_1)$$

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- These are EPR operators

Koopman von Neumann mechanics

- Liouville operator (note the minus):

$$L_C(X_1, P_1, X_2, P_2) = \frac{1}{2}\left(\frac{P_1^2 - P_2^2}{m} - U'(X_1 + X_2)(X_1 - X_2)\right)$$

- Eg. for a harmonic oscillator – the second system has a negative mass

$$L_C(X_1, P_1, X_2, P_2) = \quad \frac{P_1^2 - P_2^2}{2} + \frac{X_1^2 - X_2^2}{2}$$

Koopman von Neumann mechanics

- One can realize classical systems within QM (free particle, particle in a linear potential, harmonic oscillator)

- To do so one needs:

    1) entanglement (EPR operators)

    2) (effective) negative mass

Koopman von Neumann mechanics

- One can realize classical systems within QM (free particle, particle in a linear potential, harmonic oscillator)

- This is useful for increasing measurements precision (e.g. a wavepacket does not spread)

Overcoming the Standard Quantum Limit in Gravitational Wave Detectors Using Spin Systems with a Negative Effective Mass

F. Ya. Khalili and E. S. Polzik
Phys. Rev. Lett. **121**, 031101 – Published 16 July 2018

Open problems

- Is there a similar trick for qubits (qudits)?

Part 2

A model of classical-quantum dynamics

Motivation

- Why it is interesting to consider composite classical-quantum systems ?
  - thermodynamics/chemistry: small molecules (quantum) interact with large reservoirs (classical)
  - Measurement theory: quantum systems interact with macroscopic devices (classical)
  - Gravity: quantum fields interact with gravitational field (effectively? classical)

Introduction

- Can one consistently couple classical and quantum systems?
  - Many formulations e.g. :
    - Koopman-von Neuman hybrid dynamics
    - Aleksandrov bracket $\frac{\partial \sigma}{\partial t} \stackrel{?}{=} -i[\mathbf{H}, \sigma] + \frac{1}{2}\Big(\{\mathbf{H}, \sigma\}_{PB} - \{\sigma, \mathbf{H}\}_{PB}\Big)$
    - Many others

Introduction

- Can one consistently couple classical and quantum systems?
  - Many formulations – many problems :
    - No-go theorems
    - Not completely positive dynamics (leading to “negative” probabilities)
    - Non-linear formulations (density matrix looses its statistical interpretation)

Introduction

- Classical-quantum dynamics introduced in

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North-Holland

PHYSICS LETTERS A

On the interaction between classical and quantum systems

Ph. Blanchard

Department of Physics and BiBoS, University of Bielefeld, Postfach 8640, W-4800 Bielefeld, Germany

and

A. Jadczyk

Institute of Theoretical Physics, University of Wrocław, Pl. Maxa Borna 9, PL-50 204 Wrocław, Poland

and further applied in different contexts.

Introduction

- Main features of classical quantum coupling:
  - Decoherence of quantum systems
  - “Collapse” of the wavefunction (quantum system jumps to a pure state that can be determined from classical dof)
  - Diffusion in the classical phase space

See: [A post-quantum theory of classical gravity?] J. Oppenheim
[Objective trajectories in hybrid classical-quantum dynamics] J. Oppenheim et al.

Some details

- Object of study: hybrid density $\varrho(z)$

- $z = q, p$ - phase space variables

- $\mathrm{prob}(z) = \mathrm{Tr}\left[\varrho(z)\right]$ - distribution over z; $\rho = \int_{\mathcal{M}} \mathrm{d}z \, \varrho(z)$ - a valid quantum state.

Some details

- Object of study: hybrid density $\varrho(z)$

- e.g. for a qubit

$$\varrho(q,p) = \begin{pmatrix} u_0(q,p) & c(q,p) \\ c^\star(q,p) & u_1(q,p) \end{pmatrix}$$

Some details

- Evolution equation (GKLS/Lindblad)

Commutator

$$\frac{\partial\,\varrho(z,t)}{\partial t} = -\,i\left[H(z),\varrho(z,t)\right] \;\; + \int \mathrm{d}\Delta \sum_{\alpha,\beta} W^{\alpha\beta}\left(z|z-\Delta\right)\, L_\alpha\, \varrho(z-\Delta,t)\, L_\beta^\dagger - \frac{1}{2}\sum_{\alpha,\beta} W^{\alpha\beta}\left(z\right)\left\{L_\beta^\dagger L_\alpha, \varrho(z,t)\right\}_+$$

- $W^{\alpha\beta}\left(z|z-\Delta\right)$ - Describes change in phase space and jump of the quantum system

- $W^{\alpha\beta}\left(z\right) = \int \mathrm{d}\Delta\, W^{\alpha\beta}\left(z+\Delta|z\right) \quad \forall\, \alpha, \beta.$

Some details

- How to include "free" classical evolution?

$$\int \mathrm{d}\Delta \sum_{\alpha} W^{\alpha\beta}\left(z|z-\Delta\right)\, L_\alpha\, \varrho(z-\Delta,t)\, L_\beta^\dagger = \sum_{\alpha\beta} \frac{1}{\tau_{\alpha\beta}} e^{\tau_{\alpha\beta}\left\{h^{\alpha\beta}(z),\,\cdot\right\}}\, L_\alpha\, \varrho(z,t)\, L_\beta^\dagger,$$

- where $H(z) = \sum h^{\alpha\beta}(z) L_\beta^\dagger L_\alpha$

Poisson Bracket

Some details

- How to include “free” classical evolution?

Expansion

$$\frac{\partial\,\varrho(z,t)}{\partial t} = -\,i\,[H(z),\varrho(z,t)] \quad + \boxed{\int \mathrm{d}\Delta \sum_{\alpha,\beta} W^{\alpha\beta}\,(z|z-\Delta)\;L_\alpha\,\varrho(z-\Delta,t)\,L_\beta^\dagger} - \frac{1}{2}\sum_{\alpha,\beta} W^{\alpha\beta}\,(z)\,\left\{L_\beta^\dagger L_\alpha,\varrho(z,t)\right\}_+$$

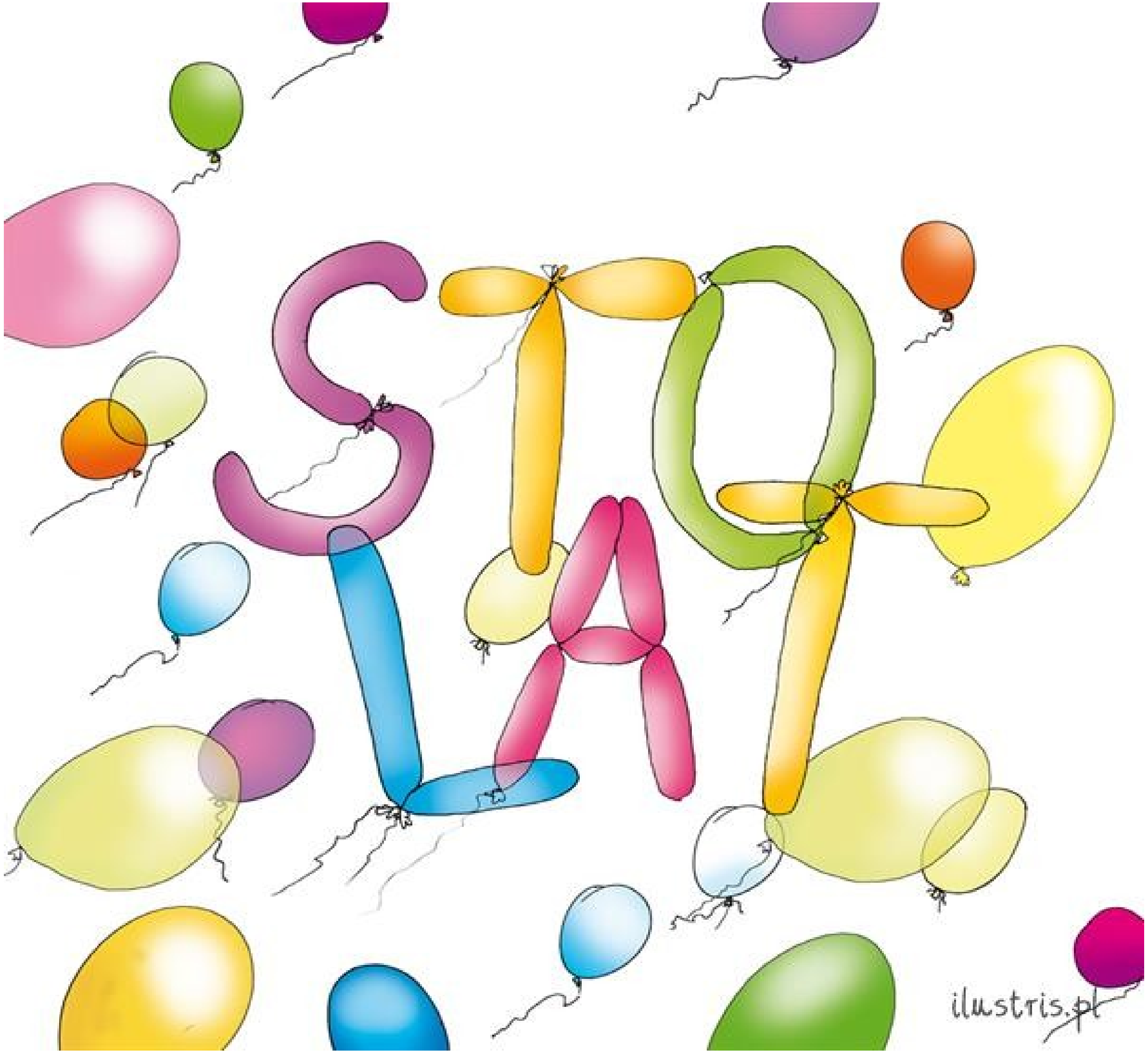
- First order: Poisson bracket
- Second order: diffusion
- ...

Some details

- Gravity (main features of the framework):
  - Decomposition based on ADM Hamiltonian (Lindblad operators → field operators)
  - Constraints implemented on the level of equations of motion → constrains possible realizations (see also “The constraints of post-quantum classical gravity” J. Oppenheim et al. arXiv:2011.15112)

Open problems

- Characterization of decoherence/diffusion interplay
- Weak-field limit of the framework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Some details

- Evolution equation (expansion):

$$\frac{\partial\,\varrho(z,t)}{\partial t} = -\,i\,[H(z),\varrho(z,t)] + \sum_\alpha \frac{1}{\tau_{\alpha\beta}}\left(L_\alpha\,\varrho(z,t)\,L_\beta^\dagger - \frac{1}{2}\left\{L_\beta^\dagger L_\alpha,\varrho(z,t)\right\}_+\right)$$
$$+\sum\{h^{\alpha\beta}(z),L_\alpha\varrho(z)L_\beta^\dagger\}+\cdots$$

$$H(z)=\sum h^{\alpha\beta}(z)L_\beta^\dagger L_\alpha$$

$$L_{\alpha=0}=\mathbb{I} \qquad H_C(z):=h^{00}(z)\,\mathbb{I}$$

Some details

- Next term (diffusion)

$$\mathcal{D}(\varrho) = \tau D^{\alpha\beta}_{p,ij}(z) L_\alpha \frac{\partial^2 \varrho}{\partial p_i \partial p_j} L^\dagger_\beta + \tau D^{\alpha\beta}_{qp,ij}(z) L_\alpha \frac{\partial^2 \varrho}{\partial q_i \partial p_j} L^\dagger_\beta + \tau D^{\alpha\beta}_{q,ij}(z) L_\alpha \frac{\partial^2 \varrho}{\partial q_i \partial q_j} L^\dagger_\beta$$