Koopman - von Neumann Made Tricky

The Koopman-von Neumann formalism is a way of re-casting classical mechanics into the same form framework as we have come to know and love in quantum theory, with wave functions, probabilities, Schrödinger equations … .

It is a special case of a more general trick, that embeds any system governed by first-order differential equations, plus normal quantum variables it acts upon, into a quantum evolution.

Consider a dynamical system of c-number variables:

$$\frac{dy^j}{dt} = f^j(y)\,.$$

Elevate the y^j to operators in Hilbert space, and introduce conjugate variables ρ_k with

$$[y^j, \rho_k] = i\delta^j_k\,.$$

Define the Hamiltonian

$$H \equiv \frac{1}{2}(\rho_j f^j(y) + f^j(y)\rho_j) \, .$$

Then evolution of the y^j according to the standard quantum rule

$$\dot{y}^j = i[H, y^j]$$

reproduces our original dynamical system.

Thus, at the price of introducing the auxiliary variables ρ_k and a rather unusual Hamiltonian, we have embedded our classical dynamical system of y^k into a conventional quantum system.

We can take over the concepts of superposition, transformation theory, operators, probability interpretation, …

Wave functions $\psi(y,t) \;=\; \Pi_j\, \delta(y^j - s^j(t))$, where $s^j(t)$ solve the classical dynamical system, are solutions of this quantum system.

Harmonic Oscillator

Classical Within Quantum Double

$$\dot{x}_1 = p_2$$

$$\dot{p}_2 = -x_1$$

$$\begin{aligned} H &= p_1 p_2 + x_1 x_2 \\ &\to p_1 \tilde{x}_2 - x_1 \tilde{p}_2 \end{aligned}$$

$$\propto (p_1 + \tilde{x}_2)^2 - (p_1 - \tilde{x}_2)^2 - (x_1 + \tilde{p}_2)^2 + (x_1 - \tilde{p}_2)^2$$

Adventure 1: Bringing in Quantum Jitter

We can expand our Hamiltonian to include ordinary quantum variables, according to

$$H_1(y, \rho, Q) \;=\; \frac{1}{2}\{\rho_j, f^j(y)\} + H_0(y, Q)$$

where the Q are q-numbers that commute with the y and ρ (but not necessarily with one another).

Along this path we preserve the dynamical equations for the y^j, and keep the ρ_k out of the equations for other variables.

Along this road the y^j represent a classical system that feels no back-reaction.

We wander off that road at our peril, but maybe also to our profit.

One possibility is to bring a touch of quantum mechanics into our classical system.

We can do this by replacing

$$H_c \;\equiv\; \frac{1}{2}(\rho_j f^j(y) + f^j(y)\rho_j) + g(y)$$

with

$$H_q \;\equiv\; H_c + \epsilon H_c^2 + h(y)\,.$$

It is easy to insure $H_q \geq 0$.

This kind of construction is also suggested by the phase space path integral. In that context, the ρ_k were Lagrange multipliers. We relax them to let the y^j jiggle.

Alternatively:

Perturbation theory in ϵp^2 or $\epsilon p A p$.

Adventure 2: Back-Reaction

We can also consider relaxing the form

$$H_1(y,\rho,Q) \;=\; \frac{1}{2}\{\rho_j, f^j(y)\} + H_0(y,Q),$$

as a way of bringing in back-reaction effects.

In can be the case that half of the y^j are positions and the other half momenta, and our dynamical system is of Hamiltonian form, so that we're starting with a respectable conservative symplectic dynamical system, but that is not necessary -

- and it isn't obvious that our construction acquires nice special features in that case. Indeed, the quantum H seems to be a rather artificial function of the underlying classical Hamiltonian.

(Maybe this deserves a closer look.)