Stochastic Loewner Evolution (SLE)

Linking universality, criticality and conformal invariance in complex systems

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Outline

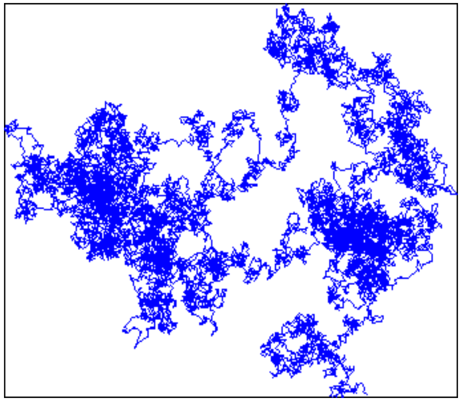
- Introduction
- Fractal shapes in 2d
- Fractal dimension
- Scaling limit
- Conformal transformations
- Loewner evolution
- Stochastic Loewner evolution
- Application to 2d turbulence
- Summary and conclusion

Introduction

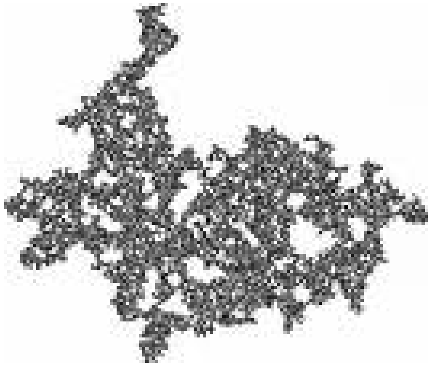
- SLE is a method to generate scale invariant (fractal) curves in 2d
- SLE operates in the complex plane and use conformal invariance
- SLE generates a family of fractal curves characterized by a parameter κ
- SLE is based on a method by Loewner for generating curves in 2d driven by a real function
- Schramm et al showed that driving the 2d curve by Brownian motion of strength κ the curve becomes random with fractal dimension $1+\kappa/8$
- SLE provides an extension of local field theory to extended fractal structures
- SLE represents an important step in probability theory and statistical mechanics
- SLE seems to have applications to turbulence and spin glasses

Some fractal shapes in 2d

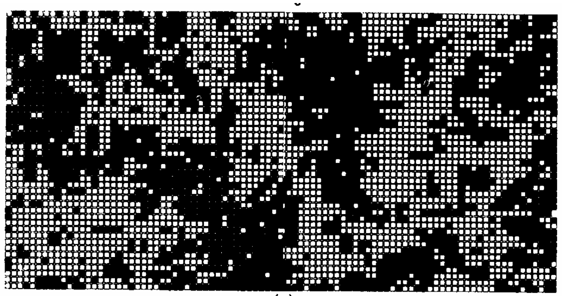
Random walk



Percolation cluster at the critical concentration

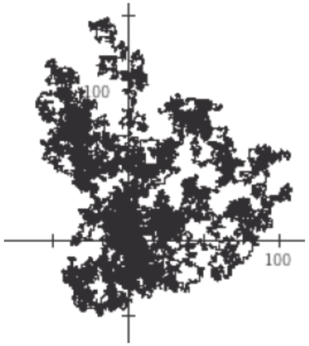


Ising cluster at the critical temperature



(a)

Self-avoiding random walk



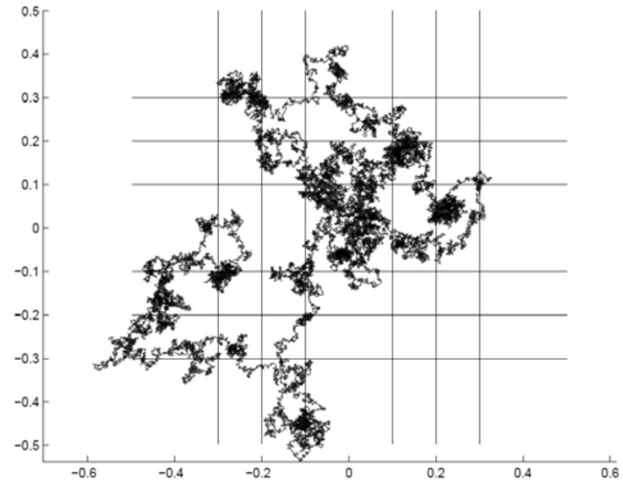
Cases to be discussed

- Random walk
- Self-avoiding random walk (SAW)
- Ising model
- Percolation

Random walk (RW)

- Basic random process - Wiener process
- Model for diffusion
- Wide applications in statistical physics and probability theory
- NN independent jumps on lattice (Markov property - no memory)
- Simple scaling properties

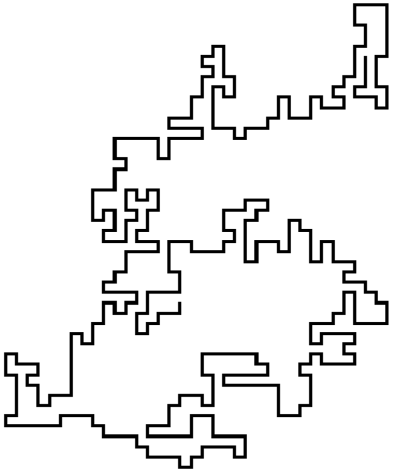
Long random walk



Self-avoiding random walk (SAW)

- Subtle random process
- NN independent jumps on lattice – BUT no self crossing - long range effects
- Model for dilute solution of polymers
- Subtle scaling properties

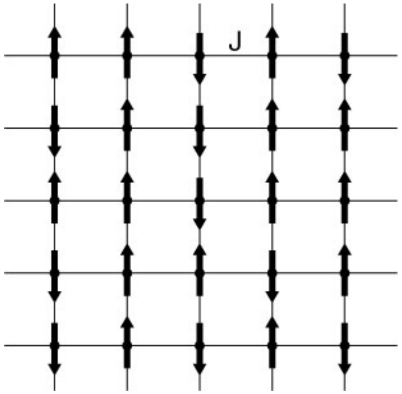
Self-avoiding RW



Ising model

- Simple model with a phase transition (critical point)
- Model for magnetism and many other systems in statistical mechanics
- Occupy site with a spin 1/2 degree of freedom and introduce NN interaction
- At critical temperature Ising model has a second order phase transition

2d Ising model on square lattice



Degree of freedom $\sigma_i = \pm 1$

Hamiltonian:

$$H(\{\sigma\}) = -J \sum_{<ij>} \sigma_i \sigma_j$$

J coupling , $<ij>$ NN sites

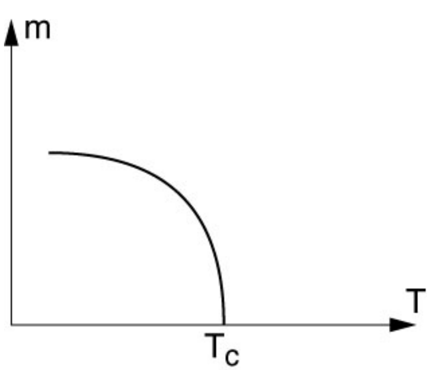
Ising model at critical point

Order parameter:

$$m = <\sigma_i> = \frac{1}{Z} \sum_{\{\sigma_k\}} \sigma_i \exp(-H/T)$$

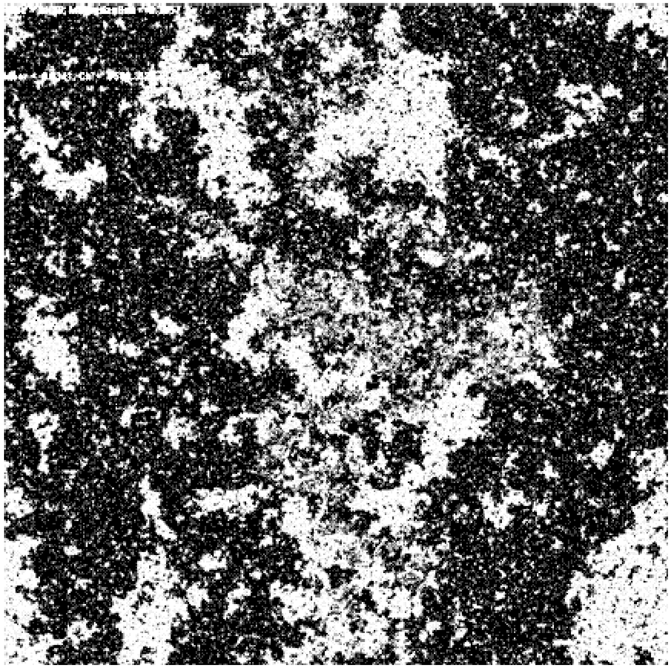
Partition function:

$$Z = \sum_{\{\sigma_k\}} \exp(-H/kT),\ T \text{ temperature}$$



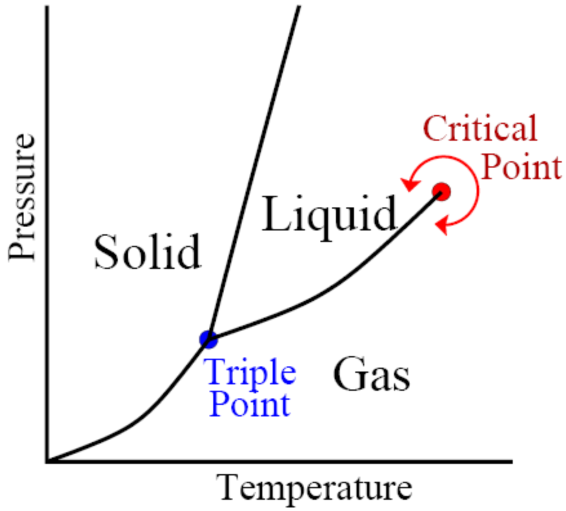
Ising model at the critical point

Domains of all sizes

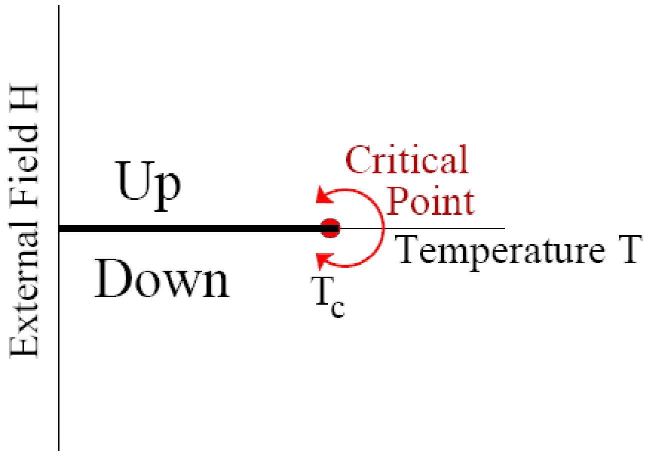


Phase diagrams for fluid and Ising

Fluid



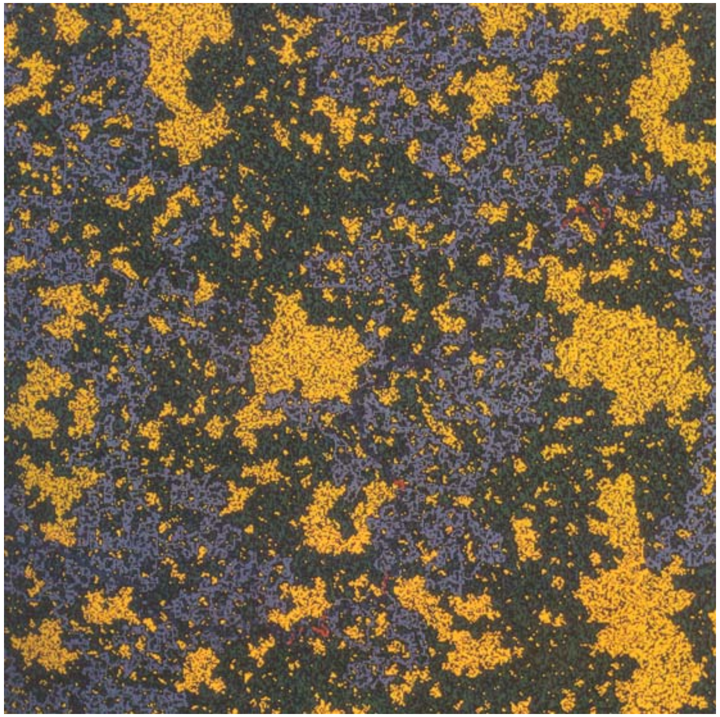
Ising model



Percolation

- Simple model with geometric phase transition
- Model for transport in porous medium
- Occupy site with probability p ($0<p<1$)
- No interaction
- At critical concentration p_c infinite cluster
- Cluster scaling properties

Critical spanning cluster (blue)



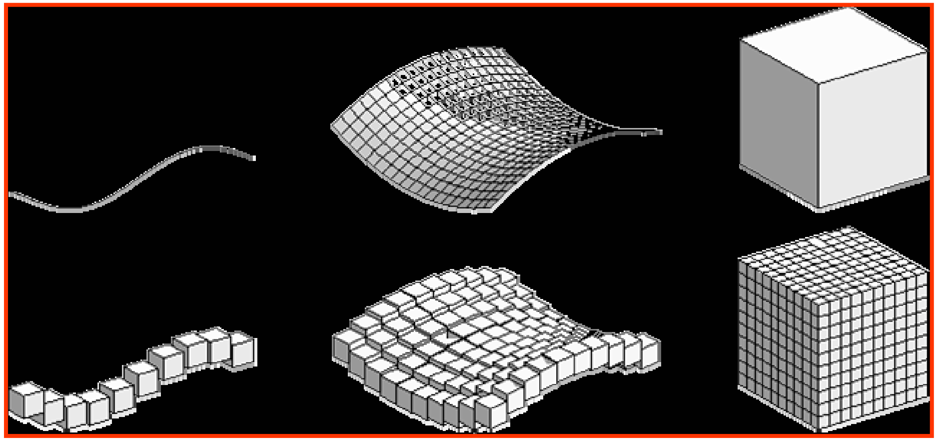
Fractal dimension

Mandelbrot and Nature
"Clouds are not spheres, mountains are not cones,
coastlines are not circles, and bark is not smooth,
nor does lightning travel in a straight line."
Mandelbrot, 1983.

New geometrical description of scale invariant objects
in natural sciences and mathematics
Fractals characterized by non integer dimension

Fractal dimension

Cover object with *N* boxes of size *a*
N(a) will depend on *a* as a power



$$N = a^{-D}$$

$$D = -\frac{\ln N}{\ln a}$$

D is the fractal dimension

Fractal dimension

$N(a)=a^{-D}$, D fractal dimension

$N(a=1) = 1$

$N(a=1/3) = 4$,

$N(a=1/9)=16$, ..

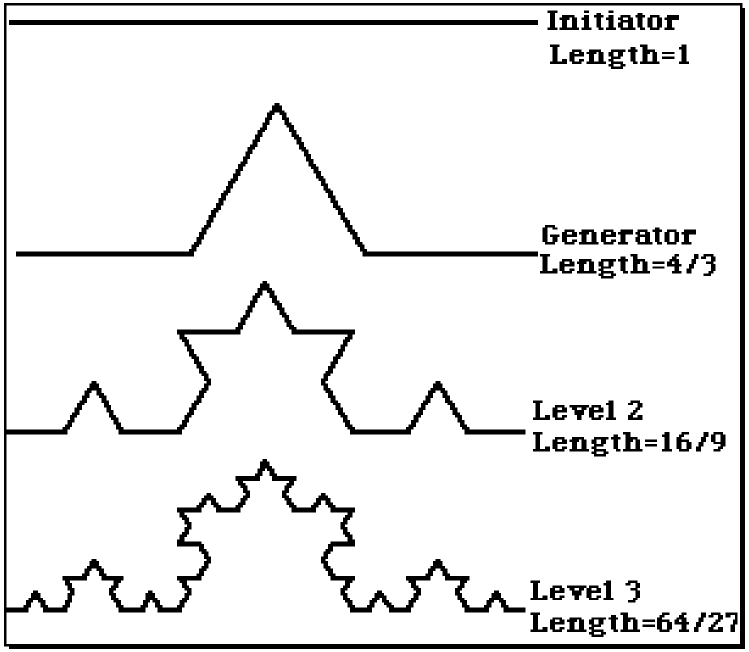
$N(a=(1/3)^n)=4^n$

$D_{Koch} = \log 4/\log 3 \sim 1.26$

$1< D_{Koch} <2$

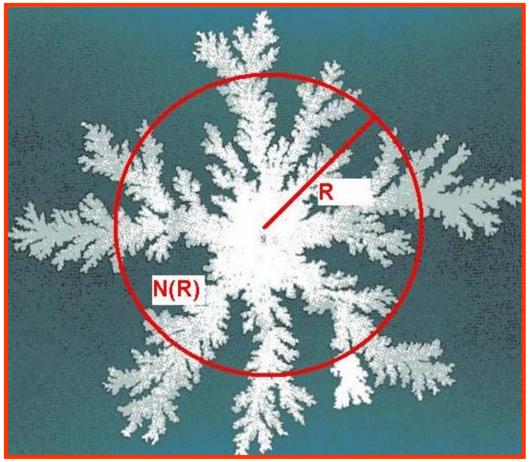
Koch kurve is self-similar

The Koch curve



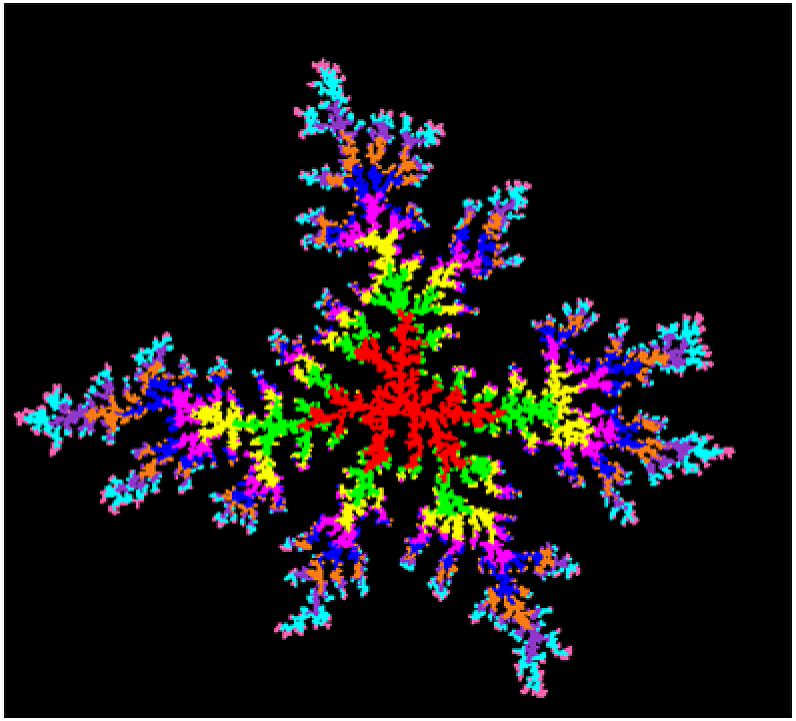
Diffusion limited aggregation (DLA)

Fractal dimension
D ≈ 1.70



$$N(R) \approx R^D$$

DLA cluster

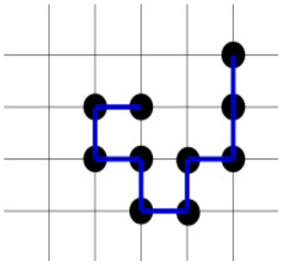


Scaling limit

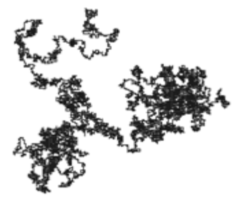
- Lattice shrinks (continuum limit)
- Site variables becomes local fields
- Lattice models become field theories
- Lattice models at the critical point become conformal field theories
- Issue: How to understand scaling in the continuum limit

Scaling limit of random walk

Random walk

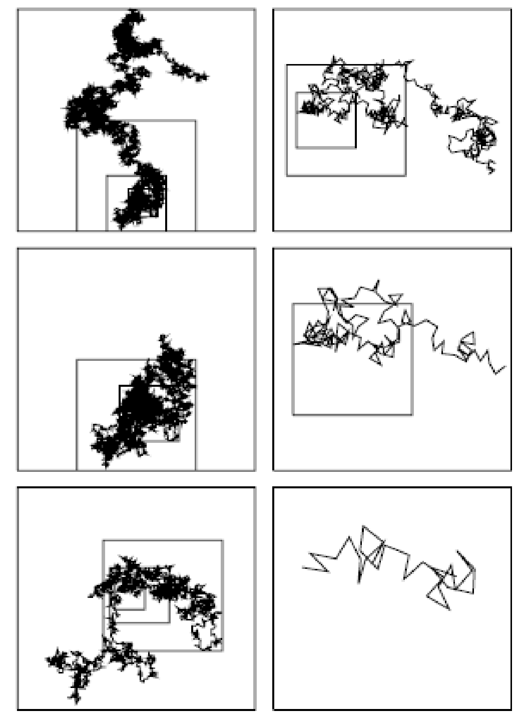


Brownian motion (BM)



<- R ->

Self similarity



- BM is a continuous curve
- BM is not differentiable
- BM is scale invariant
- BM is at a critical point
- BM has fractal dimension D=2
- BM is plane-filling

Size:

$R \propto \sqrt{N} \propto \sqrt{t} = t^{\upsilon}$

scaling exponent υ=1/2

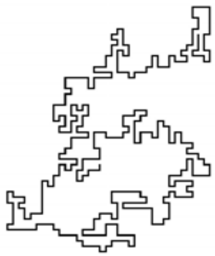
steps:

$N \propto R^2 = R^D$

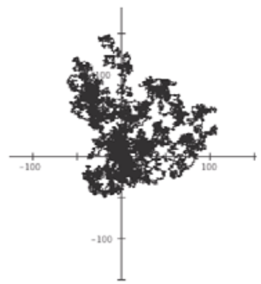
fractal dimension D=2

Scaling limit of SAW

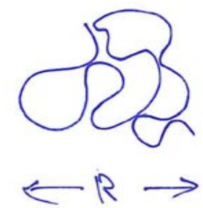
SAW



SAW



Mean field theory for dilute polymers and SAW (Flory)



Size of polymer:

$R \propto N^{\upsilon}$

scaling exponent $\upsilon = \dfrac{3}{2+\mathrm{d}}$

$N \propto R^{1/\upsilon} = R^{D}$

fractal dimension $\mathrm{D} = \dfrac{2+\mathrm{d}}{3}$

In d=2, $\mathrm{D} = \dfrac{4}{3}$

- SAW is a continuous non crossing curve
- SAW is not differentiable
- SAW is scale invariant
- SAW is at a critical point
- SAW has fractal dimension D=4/3
- SAW is not plane-filling

Scaling limit of Ising model

- In scaling limit the Ising model described by Landau functional
- Renormalization group (RG) methods yield exponents
- The local field and its correlations are the central objects
- RG method does not describe critical domains etc
- RG method does not give the GEOMETRY at criticality

Site variable $\rightarrow$ local field

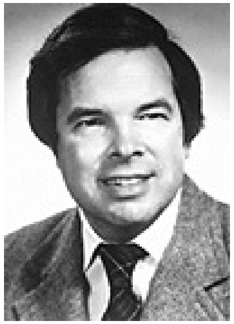
$$\sigma_i \rightarrow \Phi(r)$$

Hamiltonian $\rightarrow$ Landau functional

$$H = -J\sum_{<ij>}\sigma_i\sigma_j \rightarrow F = \int\left[(\nabla\Phi)^2 + R\Phi^2 + U\Phi^4\right]d^2r$$

Partition function $\rightarrow$ path integral

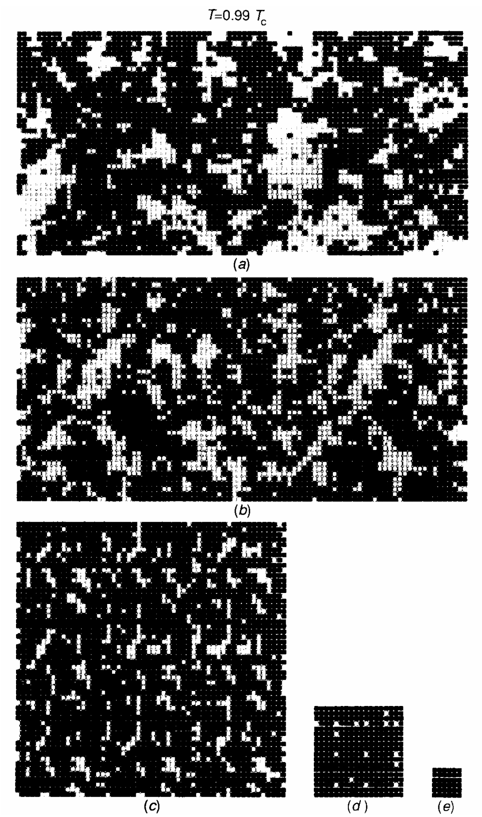
$$Z = \sum_{\{\sigma_i\}}\exp(-H/kT) \rightarrow \int\prod_r D\Phi\exp(-F)$$



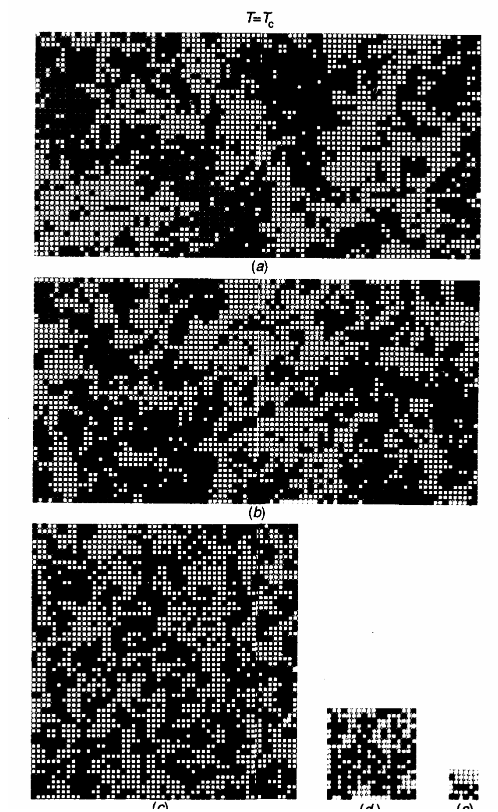
Renormalization group
Ken Wilson
Nobel prize 1982

Ising self similarity at T_c

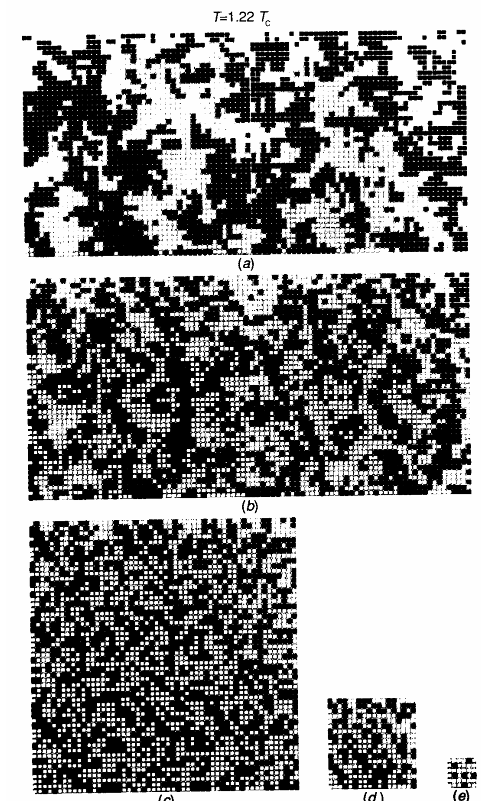
T=0.99 T_c



T=T_c

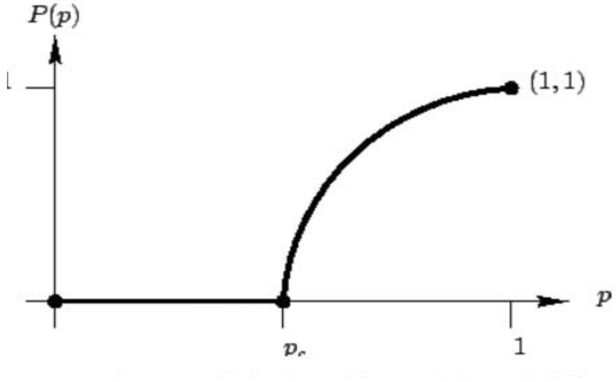


T=1.22 T_c

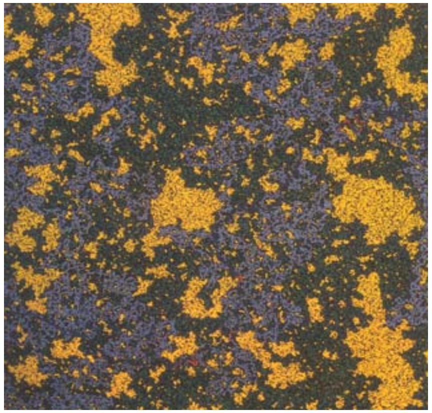


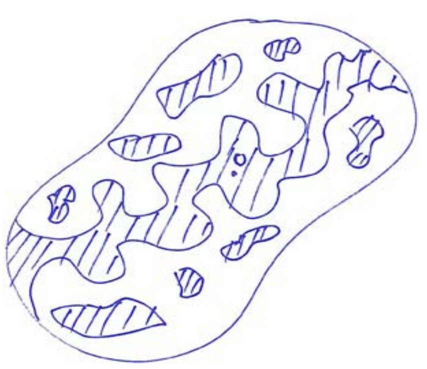
Scaling limit of percolation

- Percolation probability P(p)
- P(p) probability that origin is included in infinite cluster
- Geometrical phase transitions at critical concentration $p=p_c$
- At p_c clusters of all sizes



Infinite cluster (blue)



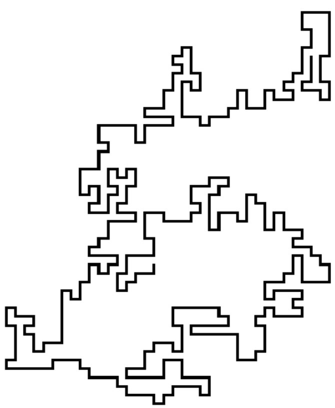


Critical domain walls and curves

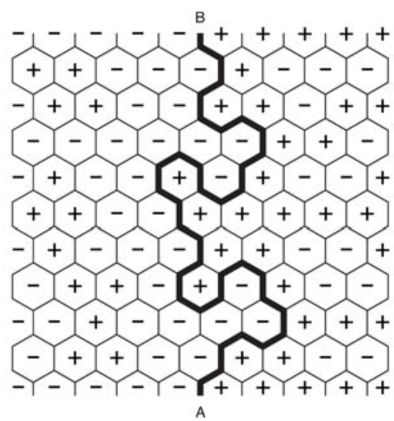
Enforce domain wall by choosing appropriate boundary conditions

By construction domain walls are non crossing

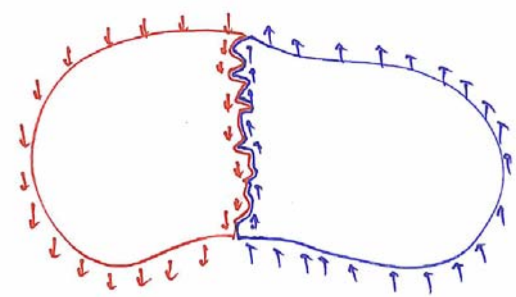
SAW random path



Percolation cluster boundary at critical concentration



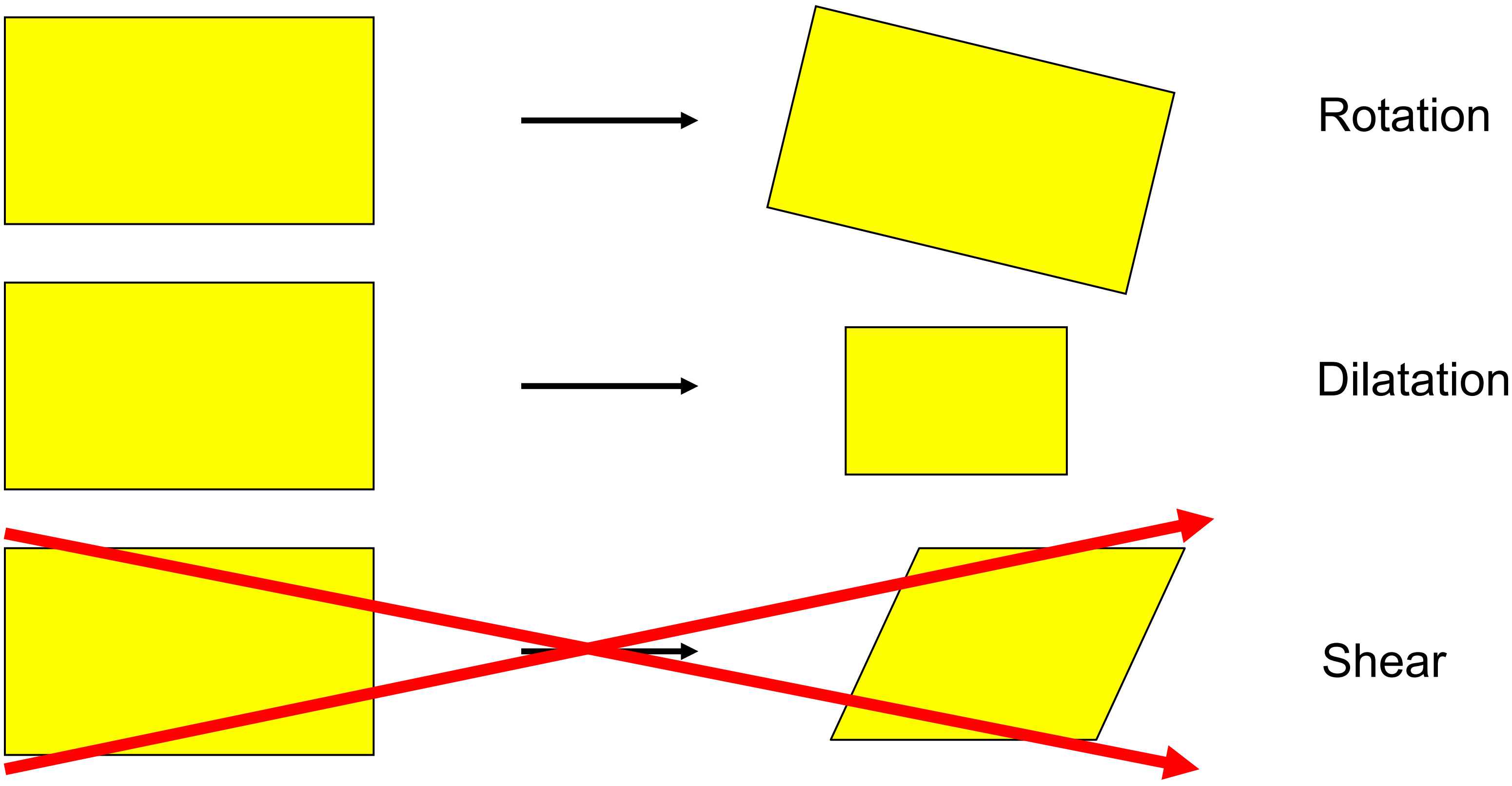
Ising domain wall at critical temperature



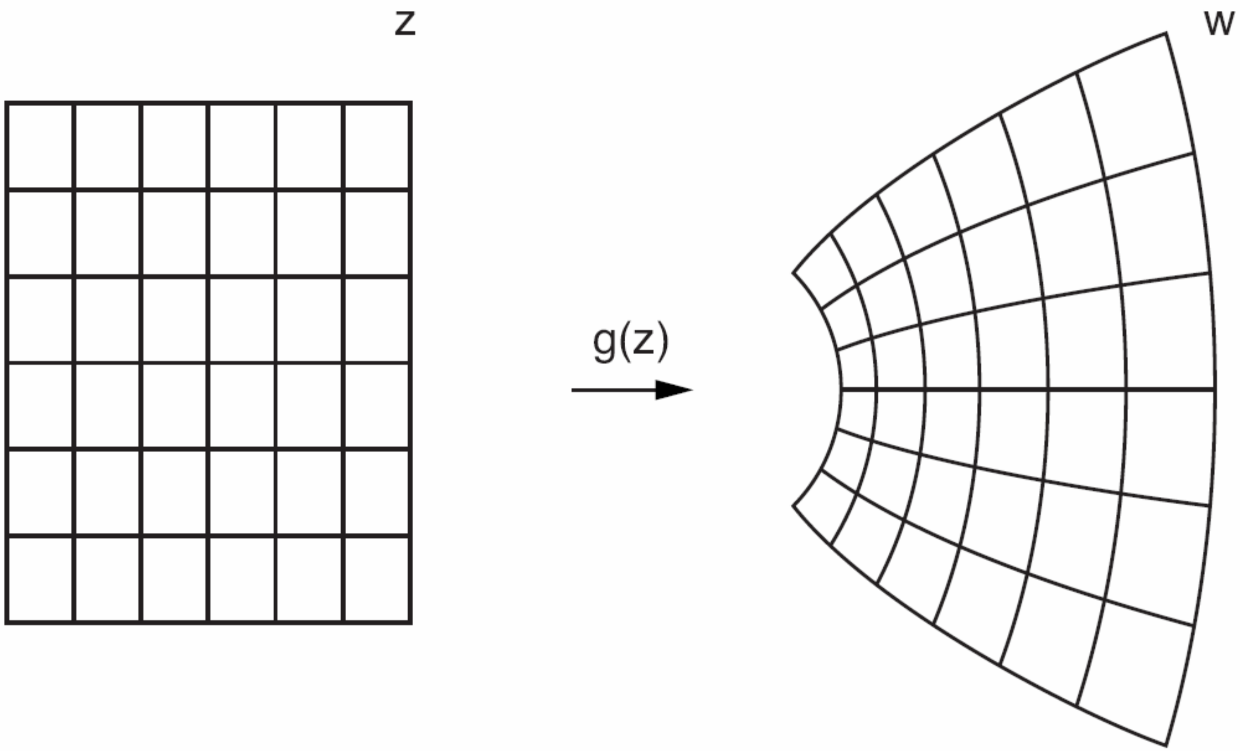
The issue

- Investigate critical behavior directly in the continuum limit
- Focus on critical domains and curves
- Assume *conformal invariance* at the critical point

Conformal transformations

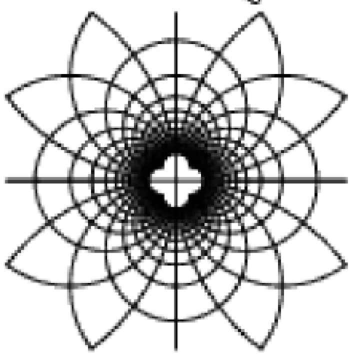


Conformal transformation

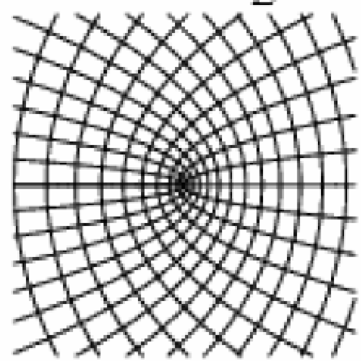


Examples (transformation of grid)

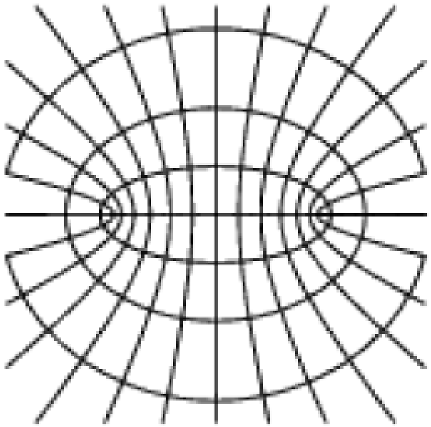
$$f(z) = \frac{1}{z}$$



$$f(z) = \frac{z^2}{2}$$



$$f(z) = \cos z$$



Conformal transformations

Riemann's mapping theorem

Riemann's theorem:
A shape without holes in plane z can be mapped to the unit disk in plane w by means of an analytic function:

$$w = g(z)$$

Any shape can be mapped to any other shape

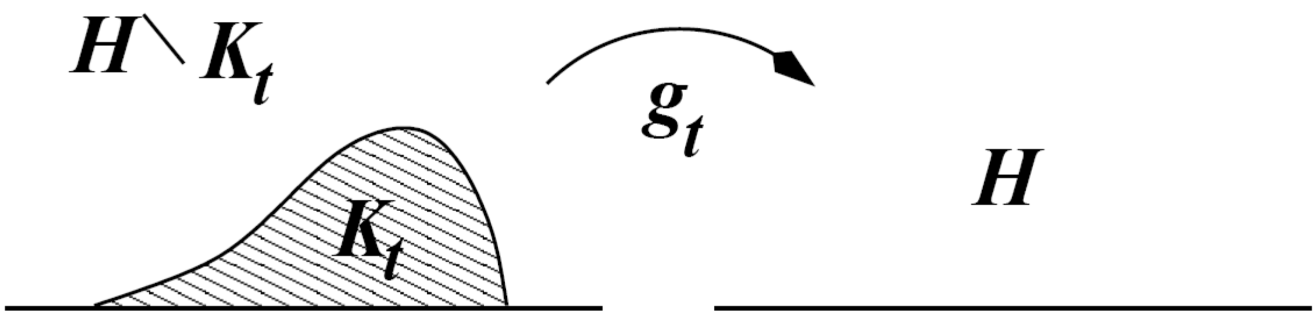
In 2D *geometry* is the same as *complex analysis* !

Loewner evolution (LE)

- Loewner evolution designed to generate curves in 2d in the continuum limit
- Idea: Define gradual conformal transformation g(z)
- Change shapes by changing conformal transformation
- Parametrize transformation by ”time variable” t
- Assume identity transformation at infinity
- Map shape to half plane (reference plane)

Mapping shape by Riemann's theorem

- Half plane: H
- Growing shape: K_t
- Complement to shape: $H \backslash K_t$
- Complement $H \backslash K_t$ mapped to H by analytic function g_t
- Real axis plus boundary of K_t mapped to real axis



Boundary and initial conditions

The identity map at infinity in H

$$g_t(z) = z + \frac{2t}{z} \text{ for large } z$$

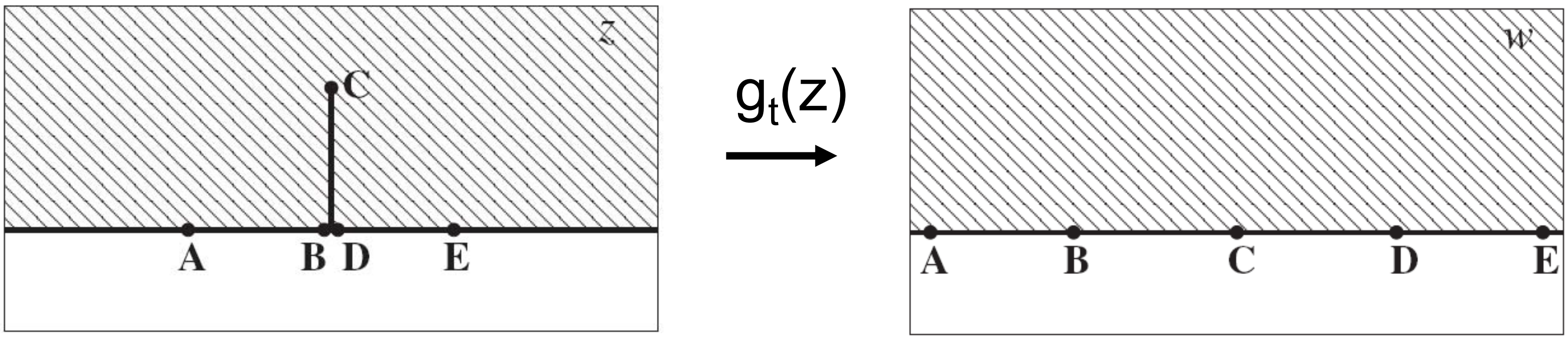
The identity map at the initial time t=0

$$g_{t=0}(z) = z$$

The growing stick

$$w = g_t(z) = \sqrt{z^2 + 4t}$$

$$z = f_t(w) = \sqrt{w^2 - 4t}, \text{ inverse map}$$



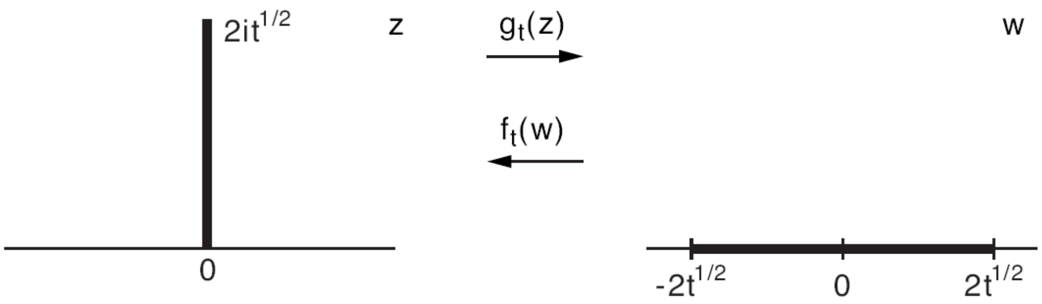
C at $(0, 2i\sqrt{t})$ in z plane

B and D at $(\mp 2\sqrt{t}, 0)$ in w plane

Equation of motion for growing stick

Growing stick map:

$$w = g_t(z) = \sqrt{z^2 + 4t}$$

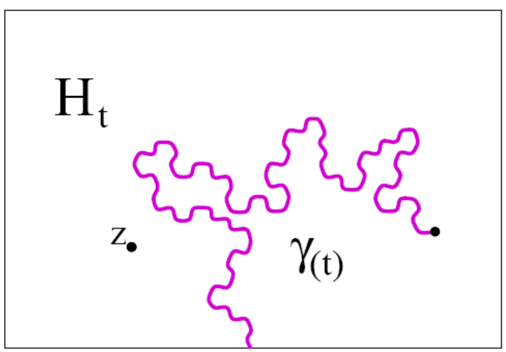


Equation of motion:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z)}$$

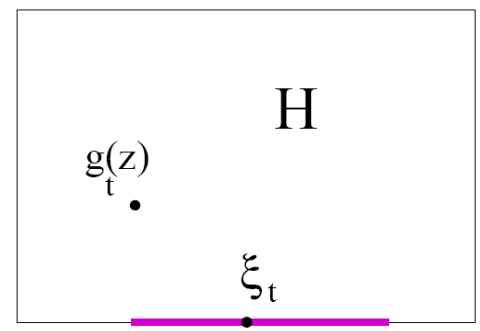
The Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}$$



Riemann's theorem

$$\xrightarrow{g_t(z)}$$

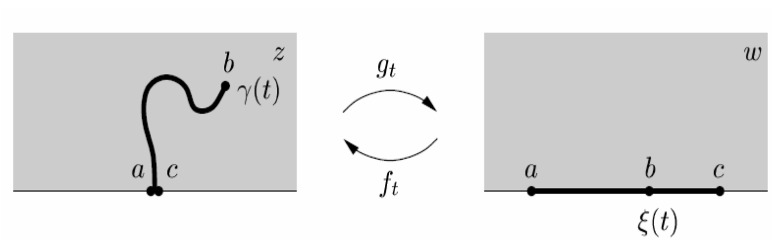


Curve $\gamma_{[0,t]}$ coded by the real function ξ_t

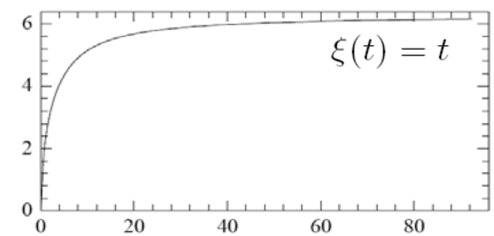
ξ_t = image of the tip of the curve: $g_t(\gamma_t) = \xi_t$.

Loewner evolution - examples

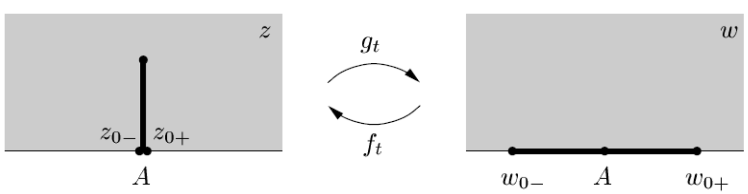
$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \qquad g_{t=0}(z) = z$$



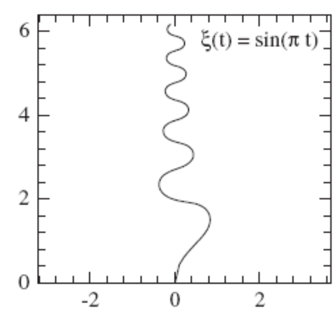
Linear forcing



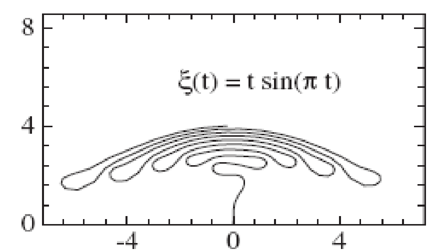
Constant forcing



Periodic forcing



Growing periodic



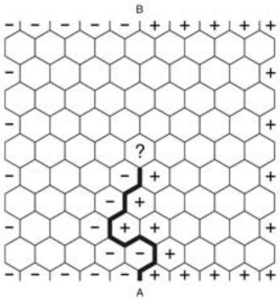
Loewner evolution - summary

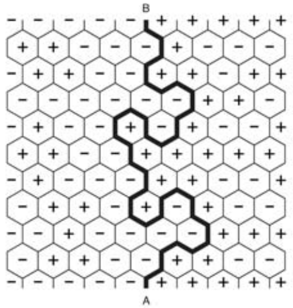
$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}$$

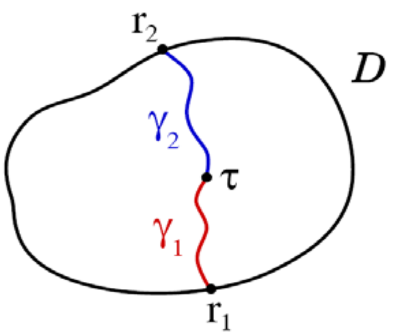
- Time evolution of map $g_t(z)$ from z to w
- Curve γ(t) in 2d driven by real function ξ(t)
- Tip of curve $z_c(t)$ given by $\xi(t)= g_t(z_c(t))$
- ξ smooth - curve γ non-intersecting
- ξ periodic - curve γ self-similar
- ξ singular - curve γ self-intersecting at finite time

Stochastic Loewner evolution (SLE)

- Schramm Loewner Evolution
- Idea: Grow domain wall step by step like a random walk (Markov process)
- Implement Markov property in the continuum limit





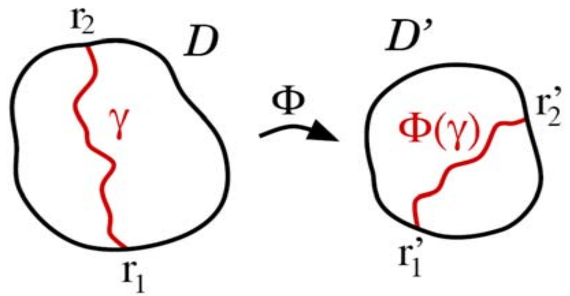


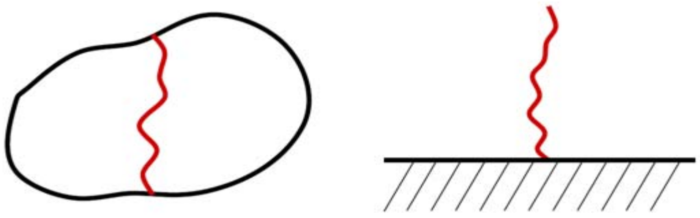
P is probability density (measure) on curve γ

$$P(\gamma_2 \,|\, \gamma_1; D, r_1, r_2) = P(\gamma_2; D \setminus \gamma_1, \tau, r_2)$$

Stochastic Loewner evolution (SLE)

- Implement conformal invariance to ensure scaling
- Probability distribution invariant under conformal transformation
- Conformal transformation to upper half plane as reference plane





Transformation of the probability density

$$(\Phi * P)(\gamma; D, r_1, r_2) = P(\Phi(\gamma); D', r_1', r_2')$$

Stochastic Loewner evolution (SLE)

Loewner equation (1923):

$$\boxed{\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}}$$

Schramm (1999):

$$\left.\begin{array}{c}\text{Markov property} \\ \text{conformal invariance}\end{array}\right\} \Longrightarrow \boxed{a_t = \sqrt{\kappa}\, B_t}$$

B_t: Brownian motion, $\langle (B_t - B_s)^2 \rangle = |t-s|$

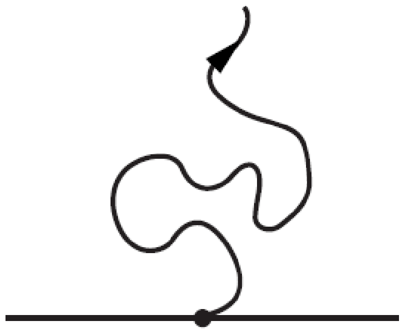
Parameter κ, SLE_κ

SLE - summary

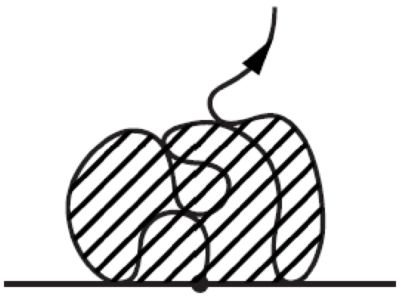
$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}$$

- Drive curve $\gamma(t)$ by random function $\xi(t)$
- Curve is fractal if $\xi(t)$ is a 1d Brownian motion
- $< [\xi(t) - \xi(s)]^2 >= \kappa\, |t-s|$
- κ is the SLE parameter, notation SLE_κ
- Fractal dimension $D=1+\kappa/8$
- $0< \kappa <4$: curve non-intersecting $(1<D<3/2)$
- $4< \kappa <8$: curve intersecting $(3/2<D<2)$
- $\kappa >8$: curve space-filling $(D=2)$

Phases of SLE$_\kappa$



$\kappa \leq 4$



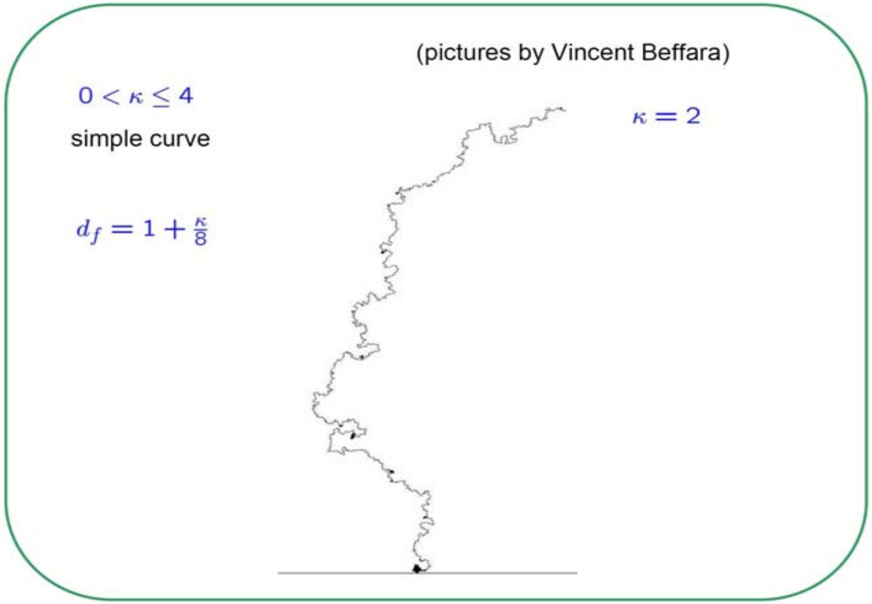
$4 < \kappa < 8$

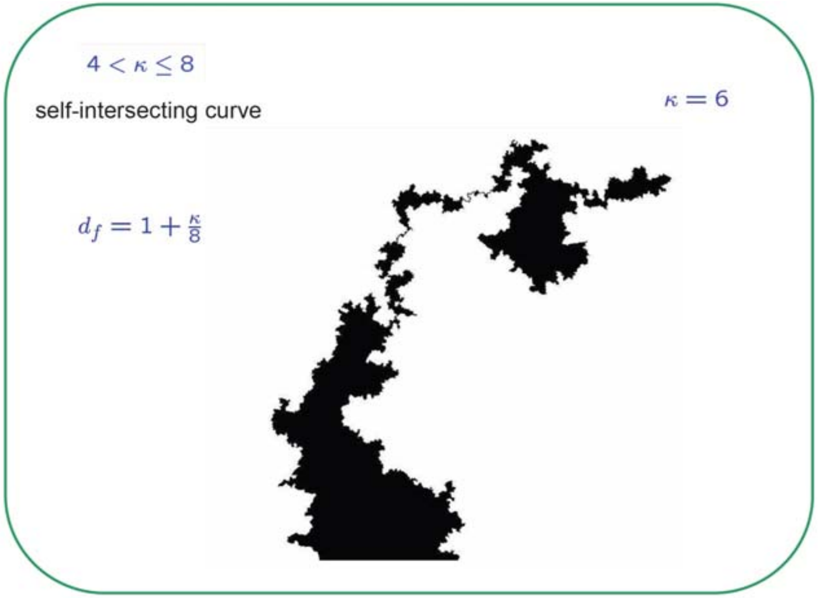
$\kappa \geq 8$

Phases of SLE$_\kappa$

Non-intersecting fractal curve
Fractal dimension D=5/4

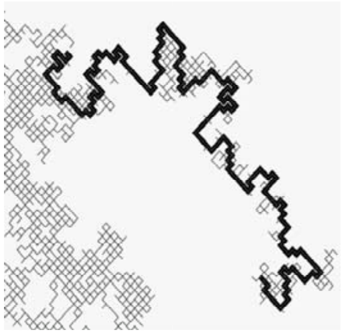
Self-intersecting fractal curve
Fractal dimension D=7/4



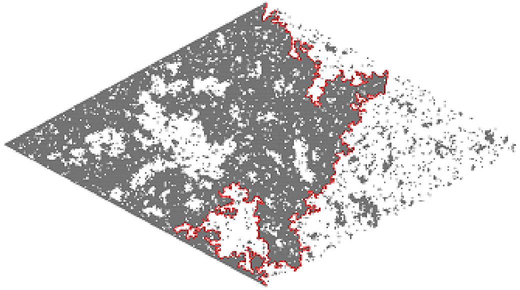


Some results of SLE$_\kappa$

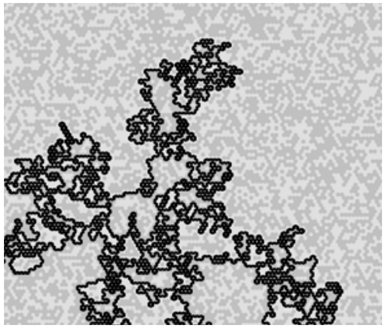
Case discussed by Schramm: Loop erased random walk
$\kappa = 2$, D=5/4



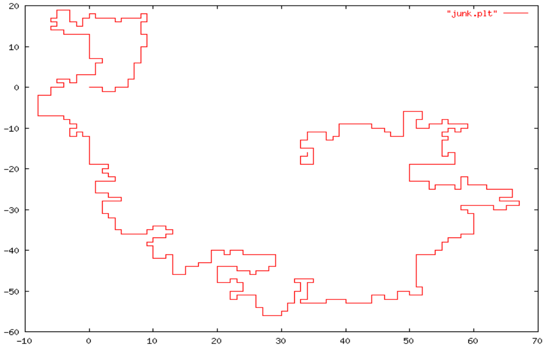
Ising domain wall random walk
$\kappa = 3$, D=11/8



Percolation domain
$\kappa = 6$, D=7/4



Self avoiding random walk
$\kappa = 8/3$, D=4/3



Application to 2d turbulence

- 2d turbulence governed by Navier Stokes equation with random forcing
- Energy and enstrophy (vorticity) injected at scale L
- Direct enstrophy cascade (from large to small scales)
- Inverse energy cascade (from small to large scales)

Navier Stokes equation

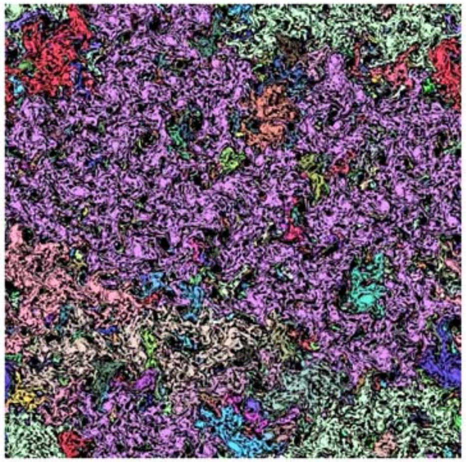
$$\frac{\partial \mathrm{v}}{\partial t} + (\mathrm{v} \cdot \nabla)\mathrm{v} = -\frac{\nabla p}{\rho} + \upsilon \Delta \mathrm{v} + \mathrm{f}$$

$$\text{Energy} = \frac{1}{2}\int \mathrm{v}^2 d^2 r$$

$$\text{Enstrophy} = \frac{1}{2}\int (\nabla \times \mathrm{v})^2 d^2 r$$

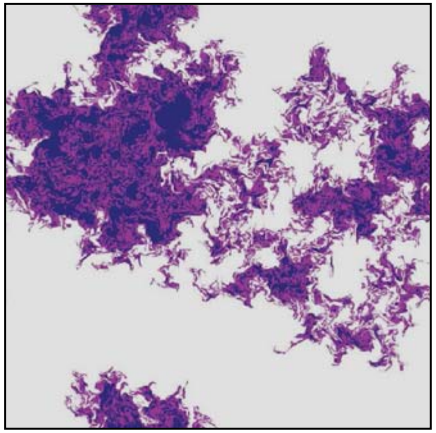
$$\text{Vorticity} = \nabla \times \mathrm{v}$$

Colored clusters of vorticity of given sign

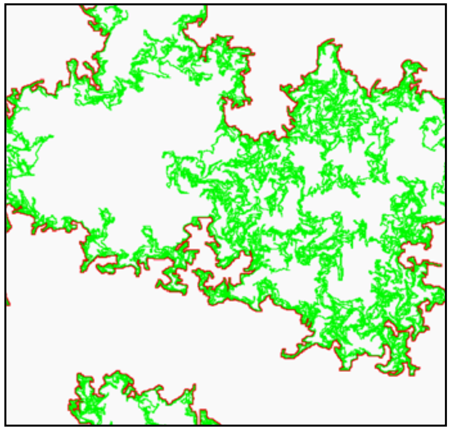


2d turbulence: vorticity clusters

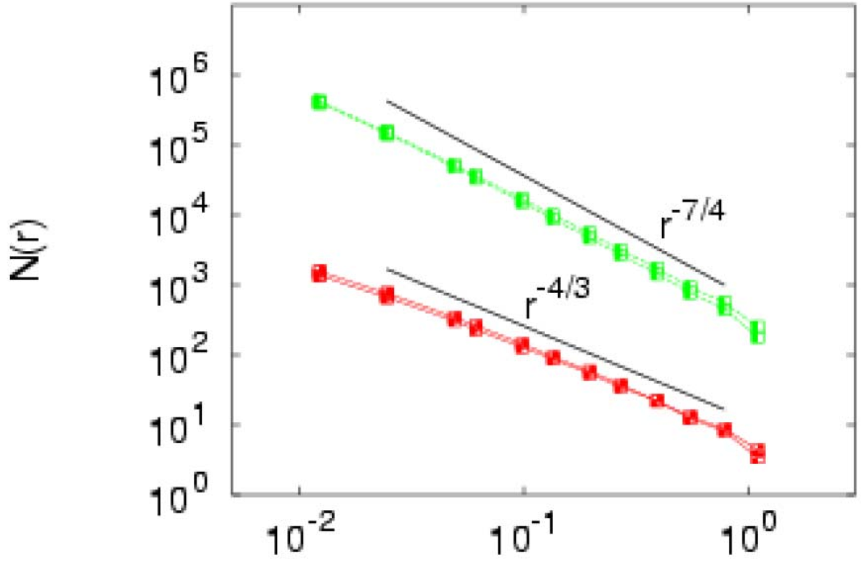
A large macroscopic filled vorticity cluster



Frontier of a vorticity cluster



Vorticity clusters and SLE

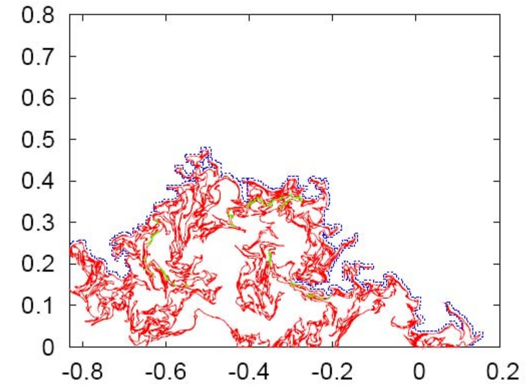


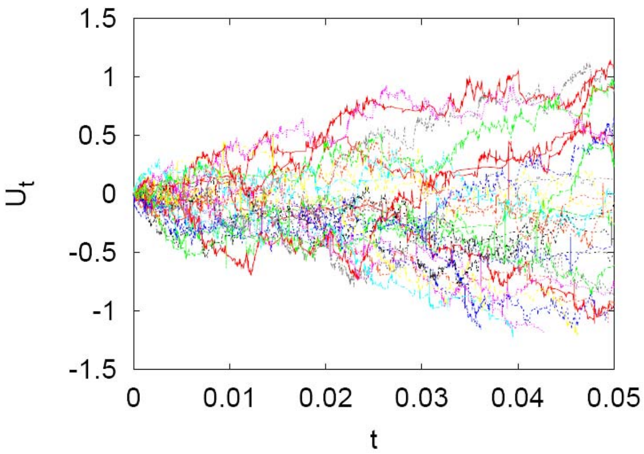
Fractal dimension vorticity cluster frontier: D=7/4
Corresponds to κ=6, percolation

Fractal dimension of external perimeter: D=4/3
Corresponds to κ=8/3,
Self-avoiding random walk (polymer)

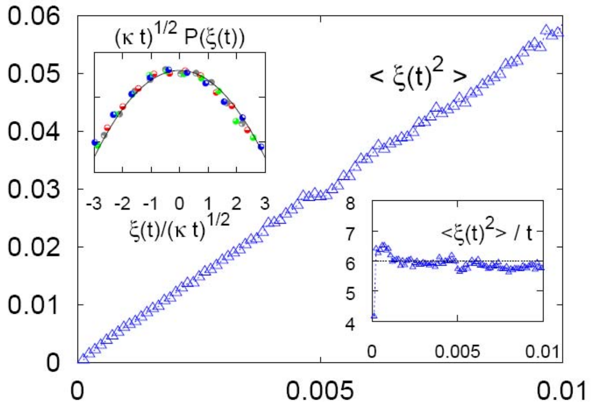
SLE in 2d turbulence

- Extract contour samples from simulation of turbulent flows
- Code them into conformal maps
- Reconstruct Loewner driving source
- Analyse statistics





Reconstructed driving sources



Distribution of driving sources

Summary and conclusion

- SLE is only 6-7 years old – but moving fast
- SLE represents *qualitative progress* in 2d critical phenomena in the scaling limit
- SLE is mainly driven by mathematicians – but the theoretical physicists are catching up
- SLE demonstrates the power of analysis when it applies
- SLE provides geometrical understanding of conformal field theory
- The SLE parameter κ delimits universality classes in 2d
- SLE ideas have already been applied to *2d turbulence* and *spin glasses*
- There is surely more to come

Thank you for your attention