# Exceptional Topology of Non-Hermitian Systems

**Rev. Mod. Phys. 93, 15005 (2021)**

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Credits to ***Flore Kunst*** *and* ***Jan Budich***
Elisabet Edvardsson, Marcus Stålhammar, Johan Carlström, Kang Yang, Fan Yang, Ipsita Mandal, Sid Morampudi, Daniel Varjas, Tsuneya Yoshida, Frank Wilczek,…

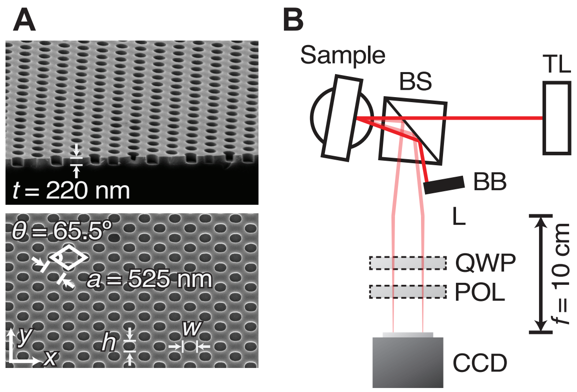
Quantum Connections 2022, Högberga

# Non-Hermitian, really?

- Complex “energies”, non-unitary time-evolution, …, Pandora’s box!?

## Relevance:

- Dissipative systems — experiments!
- Classical mechanical, electrical, robotic and optical metamaterials
- Photonic systems with gain and/or loss
- Open quantum systems
- Effective description of systems with finite lifetime states $\mathrm{Im}[E] \sim 1/\tau$
- Scattering problems, …




## New:

- Perspective of topological phases <—> uniquely NH phenomena

# A mathematical curiosity

- We are used to the stability of eigenvalues of Hermitian matrices:

$$\Delta E \leq |\epsilon|$$

*Largest eigenvalue of the perturbation*

- But for non-Hermitian $N \times N$ matrices we have

$$|\Delta E| \leq |c_N \epsilon^{1/N}|$$

$\mathcal{O}(1)$

Qualitatively new response for any $N \geq 2$

$\mathcal{O}(1)$ change for arbitrarily small $\epsilon$ as $N \to \infty$ !

# Minimal example:
# a two-level system

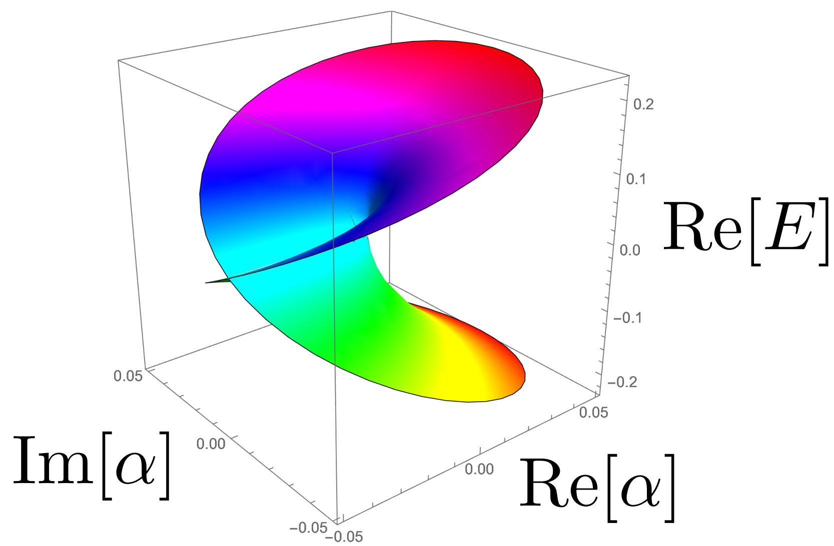
$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 1$$

- Take home: Exceptional degeneracies & Square roots

# A two-level system

$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

- Eigenvalues generally complex $E_\pm = \pm\sqrt{\alpha}$
  - Winding of $\alpha$ twice yield a winding of $E_\pm$ only once!




Note the branch point and branch cut

- Non-orthogonal eigenvectors $\Psi_{R,\pm} = \begin{pmatrix} \pm\sqrt{\alpha} \\ 1 \end{pmatrix}$
  - Left and right eigenvectors are different

$$\Psi_{L,\pm} = \begin{pmatrix} 1 & \pm\sqrt{\alpha} \end{pmatrix}$$

# An exceptional point $(\alpha = 0)$

$$H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

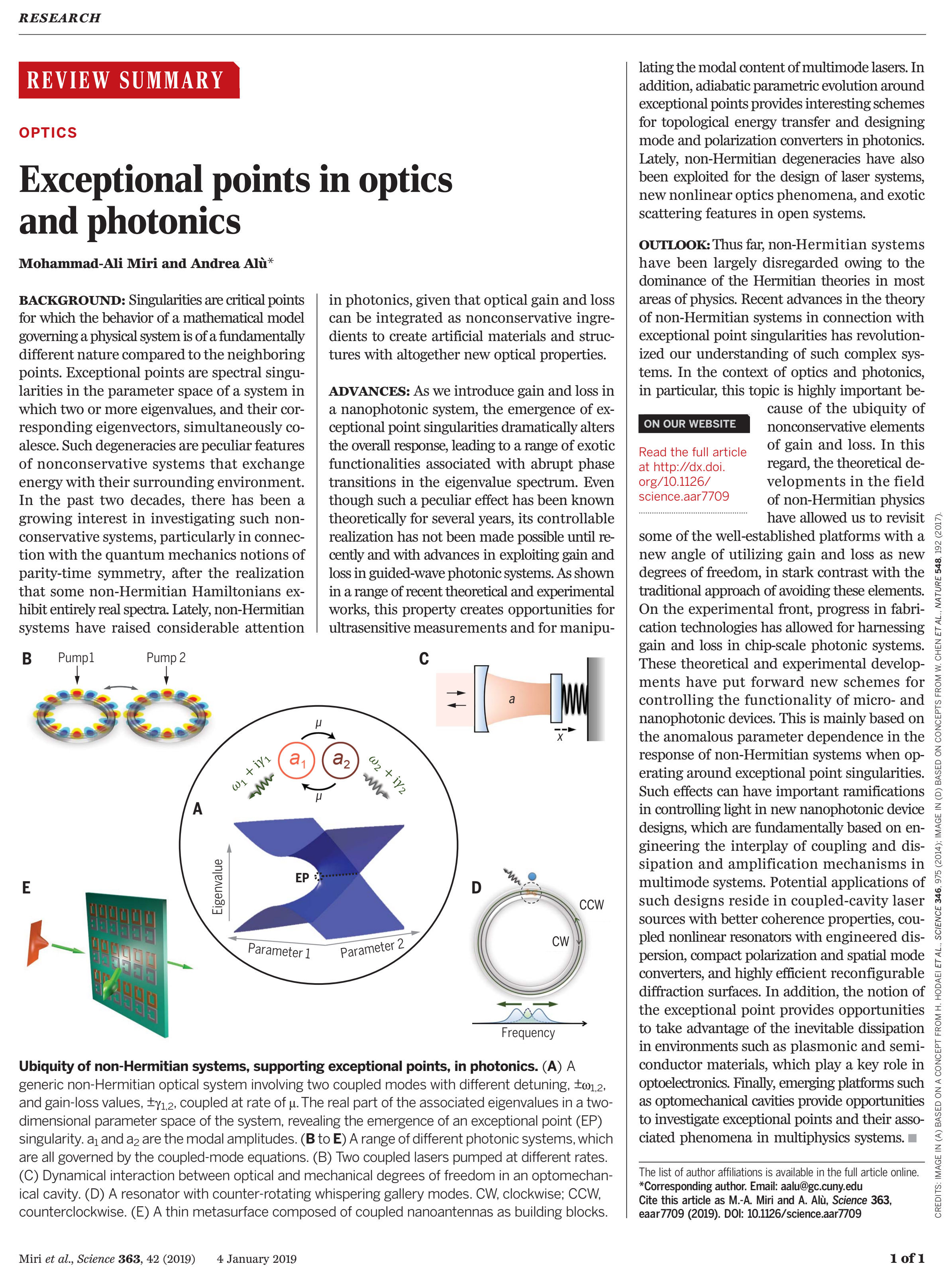
- Doubly degenerate eigenvalue $E_{\pm} = 0$
- But only one normalisable eigenvector! $\Psi_{R,\pm} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - The left eigenvector is the "opposite" $\Psi_{L,\pm} = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- "*Exceptional points*" (EPs) with singular behaviour
  - Diverging response $|\partial_\alpha E(\alpha)| \to \infty$
- When can we expect EPs to occur and what are their consequences?

# Relevant for physics?

- Yes, EPs have a long history especially in optics/photonics
  - I remember hearing about them at a talk by Michael Berry when I was undergraduate…

Review:
Miri et al., Science 363, 42 (2019)

- New perspective…





REVIEW SUMMARY

OPTICS

## Exceptional points in optics and photonics

Mohammad-Ali Miri and Andrea Alù*

**BACKGROUND:** Singularities are critical points for which the behavior of a mathematical model governing a physical system is of a fundamentally different nature compared to the neighboring points. Exceptional points are spectral singularities in the parameter space of a system in which two or more eigenvalues, and their corresponding eigenvectors, simultaneously coalesce. Such degeneracies are peculiar features of nonconservative systems that exchange energy with their surrounding environment. In the past two decades, there has been a growing interest in investigating such nonconservative systems, particularly in connection with the quantum mechanics notions of parity-time symmetry, after the realization that some non-Hermitian Hamiltonians exhibit entirely real spectra. Lately, non-Hermitian systems have raised considerable attention in photonics, given that optical gain and loss can be integrated as nonconservative ingredients to create artificial materials and structures with altogether new optical properties.

**ADVANCES:** As we introduce gain and loss in a nanophotonic system, the emergence of exceptional point singularities dramatically alters the overall response, leading to a range of exotic functionalities associated with abrupt phase transitions in the eigenvalue spectrum. Even though such a peculiar effect has been known theoretically for several years, its controllable realization has not been made possible until recently and with advances in exploiting gain and loss in guided-wave photonic systems. As shown in a range of recent theoretical and experimental works, this property creates opportunities for ultrasensitive measurements and for manipulating the modal content of multimode lasers. In addition, adiabatic parametric evolution around exceptional points provides interesting schemes for topological energy transfer and designing mode and polarization converters in photonics. Lately, non-Hermitian degeneracies have also been exploited for the design of laser systems, new nonlinear optics phenomena, and exotic scattering features in open systems.

**OUTLOOK:** Thus far, non-Hermitian systems have been largely disregarded owing to the dominance of the Hermitian theories in most areas of physics. Recent advances in the theory of non-Hermitian systems in connection with exceptional point singularities has revolutionized our understanding of such complex systems. In the context of optics and photonics, in particular, this topic is highly important because of the ubiquity of nonconservative elements of gain and loss. In this regard, the theoretical developments in the field of non-Hermitian physics have allowed us to revisit some of the well-established platforms with a new angle of utilizing gain and loss as new degrees of freedom, in stark contrast with the traditional approach of avoiding these elements. On the experimental front, progress in fabrication technologies has allowed for harnessing gain and loss in chip-scale photonic systems. These theoretical and experimental developments have put forward new schemes for controlling the functionality of micro- and nanophotonic devices. This is mainly based on the anomalous parameter dependence in the response of non-Hermitian systems when operating around exceptional point singularities. Such effects can have important ramifications in controlling light in new nanophotonic device designs, which are fundamentally based on engineering the interplay of coupling and dissipation and amplification mechanisms in multimode systems. Potential applications of such designs reside in coupled-cavity laser sources with better coherence properties, coupled nonlinear resonators with engineered dispersion, compact polarization and spatial mode converters, and highly efficient reconfigurable diffraction surfaces. In addition, the notion of the exceptional point provides opportunities to take advantage of the inevitable dissipation in environments such as plasmonic and semiconductor materials, which play a key role in optoelectronics. Finally, emerging platforms such as optomechanical cavities provide opportunities to investigate exceptional points and their associated phenomena in multiphysics systems. ■

ON OUR WEBSITE

Read the full article at http://dx.doi.org/10.1126/science.aar7709



**Ubiquity of non-Hermitian systems, supporting exceptional points, in photonics.** (**A**) A generic non-Hermitian optical system involving two coupled modes with different detuning, $\pm\omega_{1,2}$, and gain-loss values, $\pm\gamma_{1,2}$, coupled at rate of $\mu$. The real part of the associated eigenvalues in a two-dimensional parameter space of the system, revealing the emergence of an exceptional point (EP) singularity. $a_1$ and $a_2$ are the modal amplitudes. (**B** to **E**) A range of different photonic systems, which are all governed by the coupled-mode equations. (B) Two coupled lasers pumped at different rates. (C) Dynamical interaction between optical and mechanical degrees of freedom in an optomechanical cavity. (D) A resonator with counter-rotating whispering gallery modes. CW, clockwise; CCW, counterclockwise. (E) A thin metasurface composed of coupled nanoantennas as building blocks.

The list of author affiliations is available in the full article online.
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Cite this article as M.-A. Miri and A. Alù, *Science* 363, eaar7709 (2019). DOI: 10.1126/science.aar7709

# Today:

## Minimal examples:

1: $H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 1$

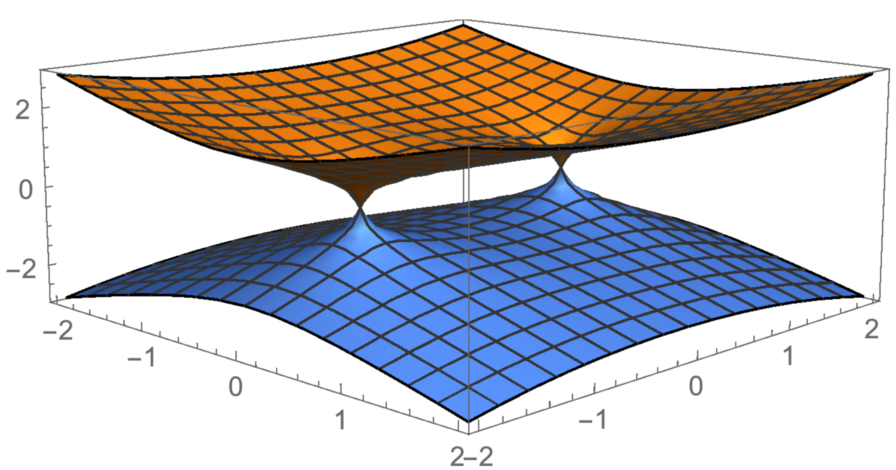
$N = 2$

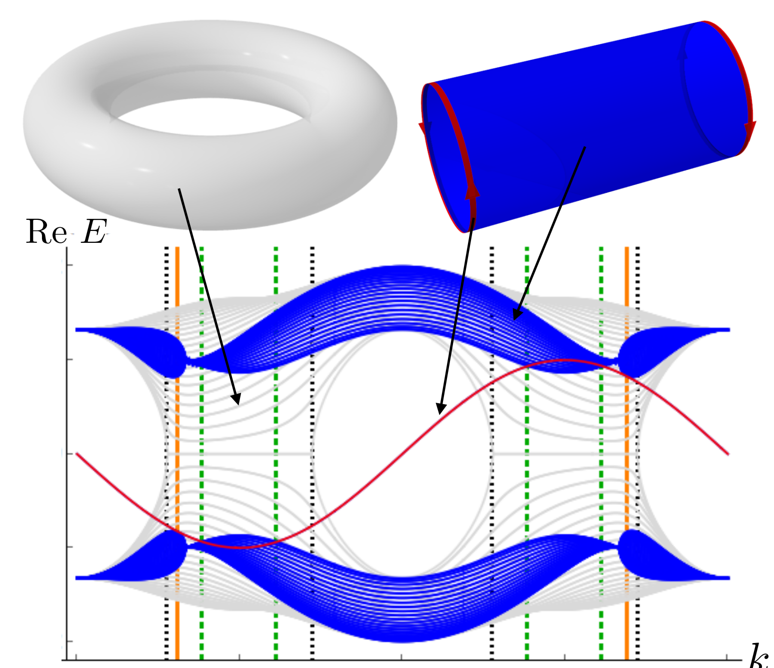
2: $H = \sum_i \left( J_L c_i^\dagger c_{i+1} + J_R c_{i+1}^\dagger c_i \right)$

$\text{large } N$

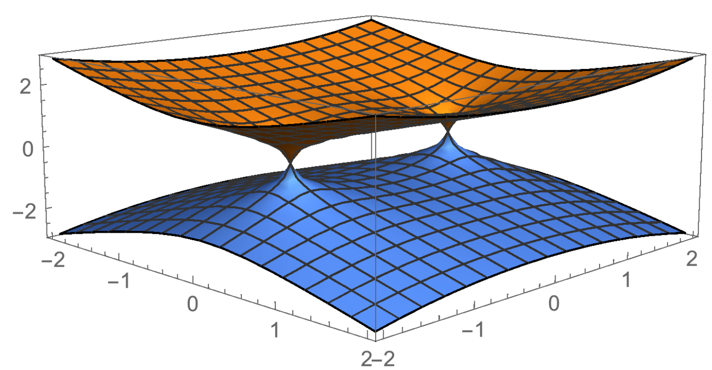
## Focus:

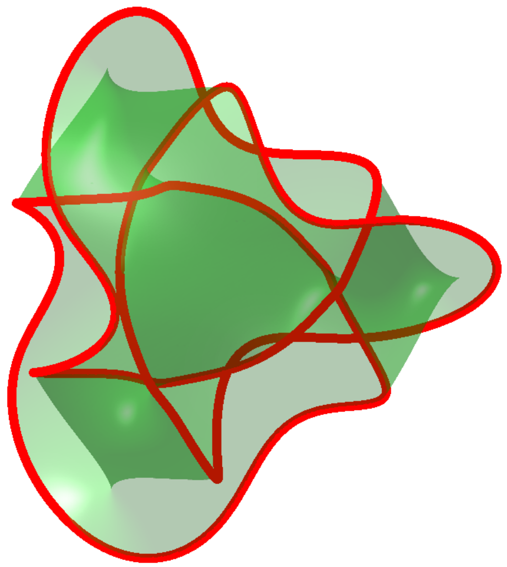
- Exceptional nodal phases
- Anomalous bulk-boundary correspondence



# Exceptional nodal phases



- Take home: Abundant & conceptually rich

# A step back:
# Band crossings in Hermitian systems

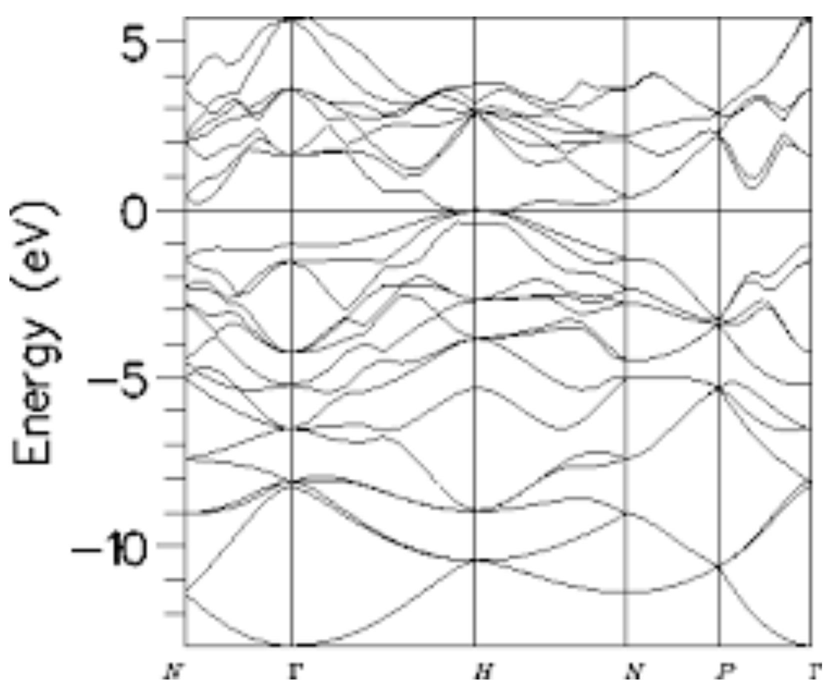
- When can we expect two energy bands to cross at a single point?

von Neumann & Wigner 1929

$$H(\mathbf{k}) = \begin{pmatrix} d_3(\mathbf{k}) + d_0(\mathbf{k}) & d_1(\mathbf{k}) - i d_2(\mathbf{k}) \\ d_1(\mathbf{k}) + i d_2(\mathbf{k}) & -d_3(\mathbf{k}) + d_0(\mathbf{k}) \end{pmatrix}$$

$$= \mathbf{d}(\mathbf{k}) \cdot \sigma + d_0(\mathbf{k})$$

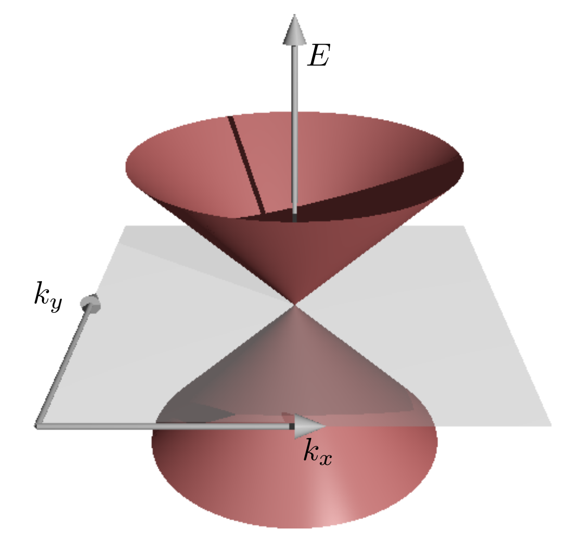
$$E(\mathbf{k}) = \pm\sqrt{d_1^2(\mathbf{k}) + d_2^2(\mathbf{k}) + d_3^2(\mathbf{k})} + d_0(\mathbf{k})$$




- 3 real constraints
- Fine-tuning in 2d
- Stable and generic in 3d!

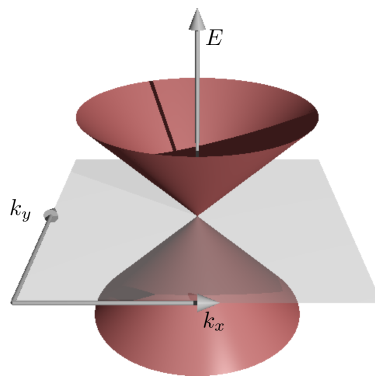
- Simplest case — the Weyl Hamiltonian

$$H = v\mathbf{k} \cdot \sigma$$

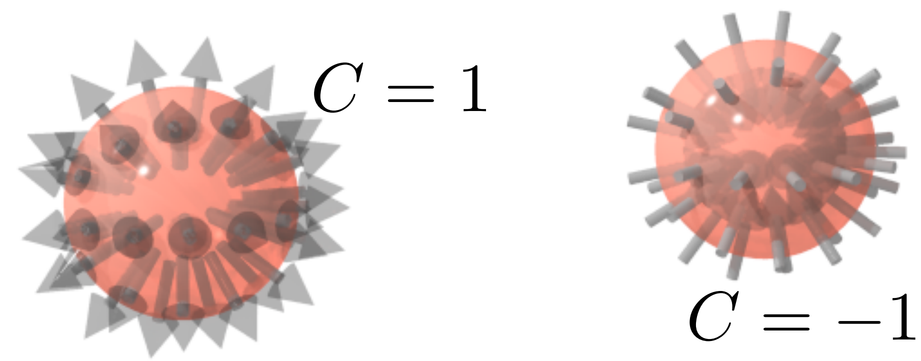


# Weyl semimetals

- Band touching points generic and protected by Chern numbers



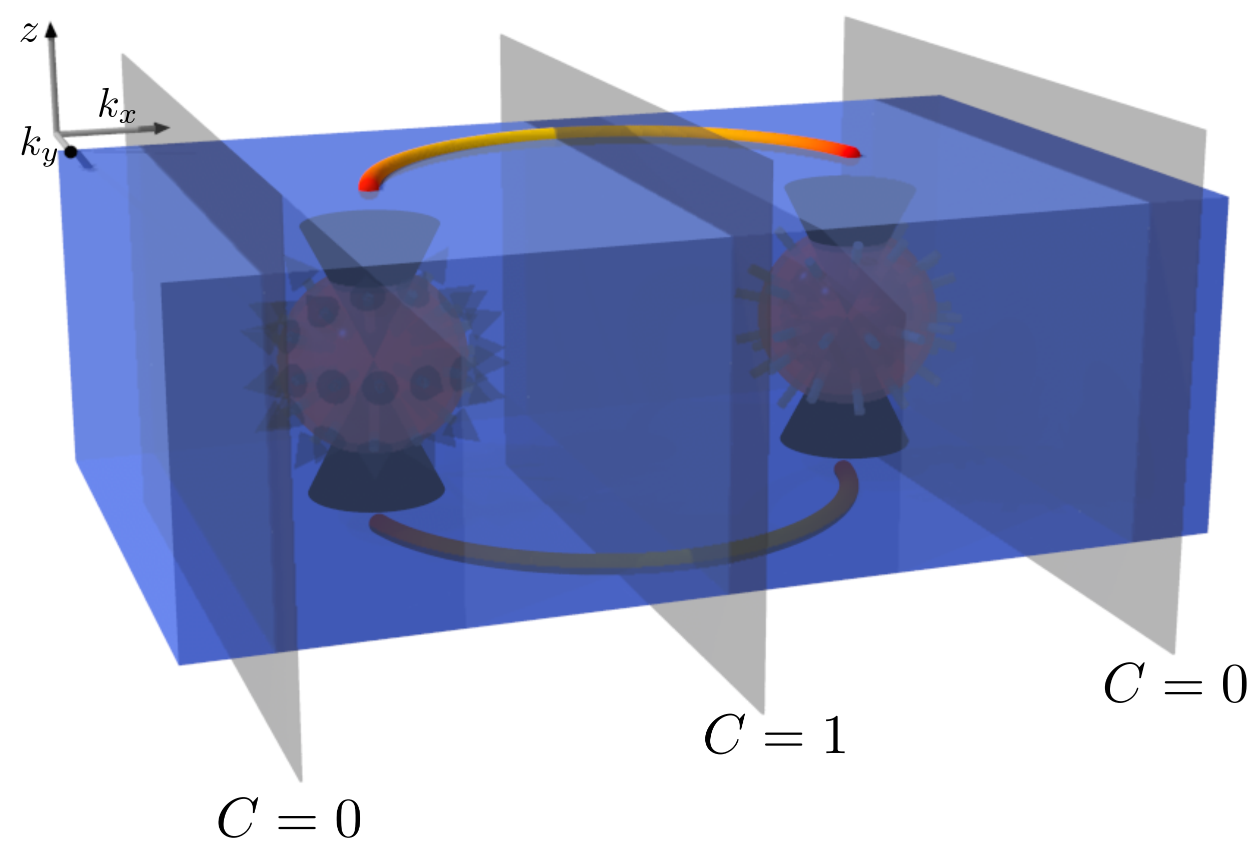

$$H = v\mathbf{k} \cdot \sigma$$




$$C = \mathrm{sign}(v)$$

- Implies novel “Fermi arc” surface states

X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011)



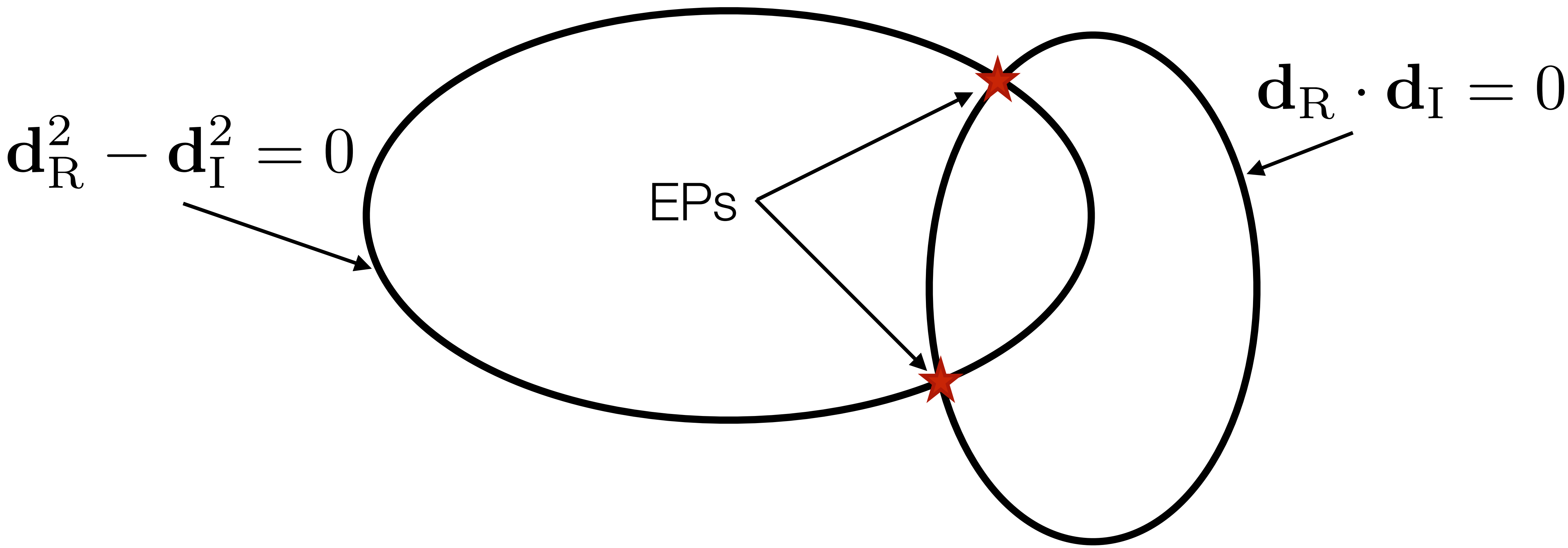
# Non-Hermitian band-crossings

$$H(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma \quad \text{now with} \quad \mathbf{d}(\mathbf{k}) = \mathbf{d}_\mathrm{R}(\mathbf{k}) + i\mathbf{d}_\mathrm{I}(\mathbf{k})$$

$$E(\mathbf{k}) = \pm\sqrt{\mathbf{d}_\mathrm{R}(\mathbf{k})^2 - \mathbf{d}_\mathrm{I}(\mathbf{k})^2 + 2i\mathbf{d}_\mathrm{R}(\mathbf{k}) \cdot \mathbf{d}_\mathrm{I}(\mathbf{k})}$$

- Generic band crossings from tuning only ***two real parameters***!
  - Look at $E^2(\mathbf{k})$ in 2d

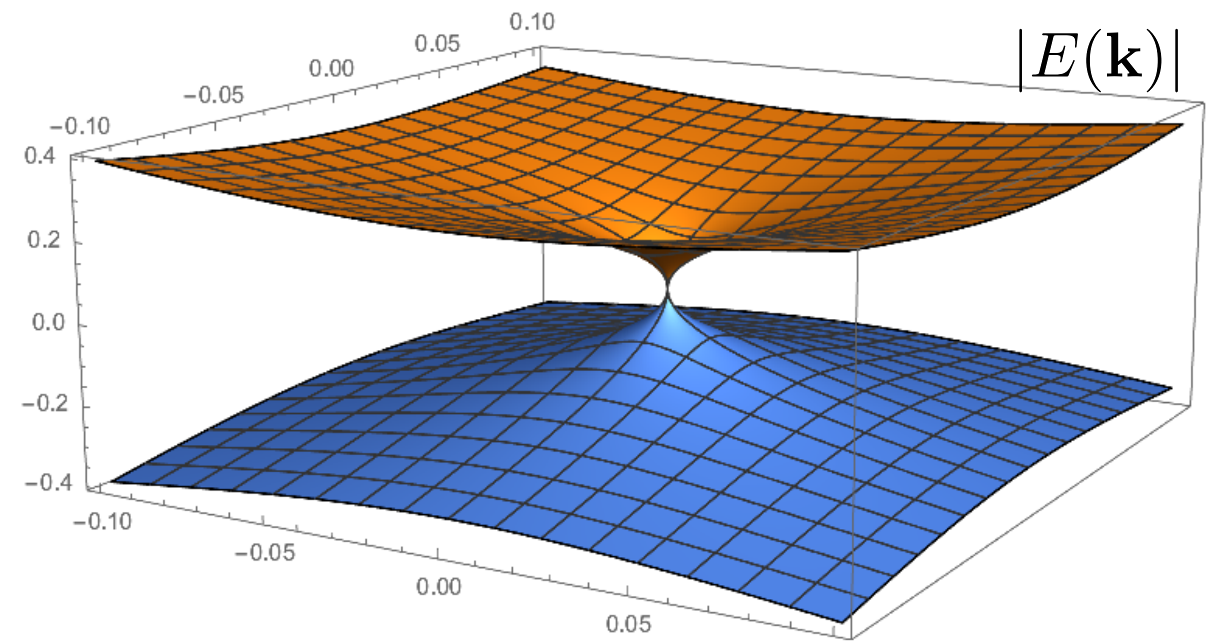
(Pancharatnam, Berry, …)



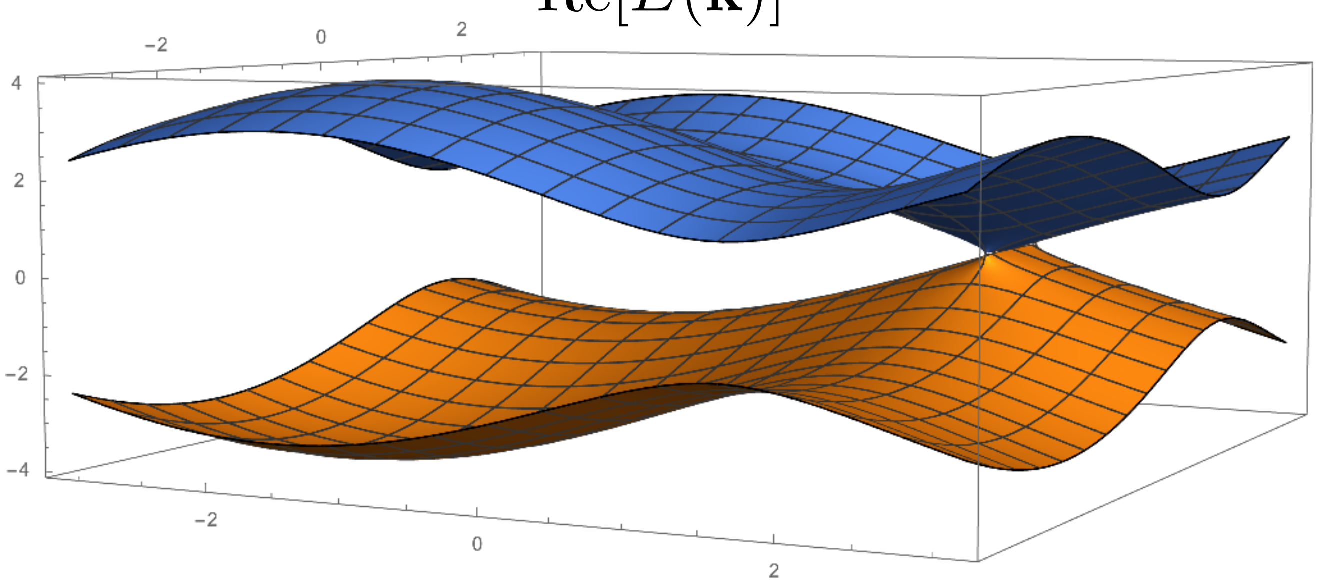

- EPs come in pairs and are generic in 2d, hence much more abundant than in the Hermitian case!

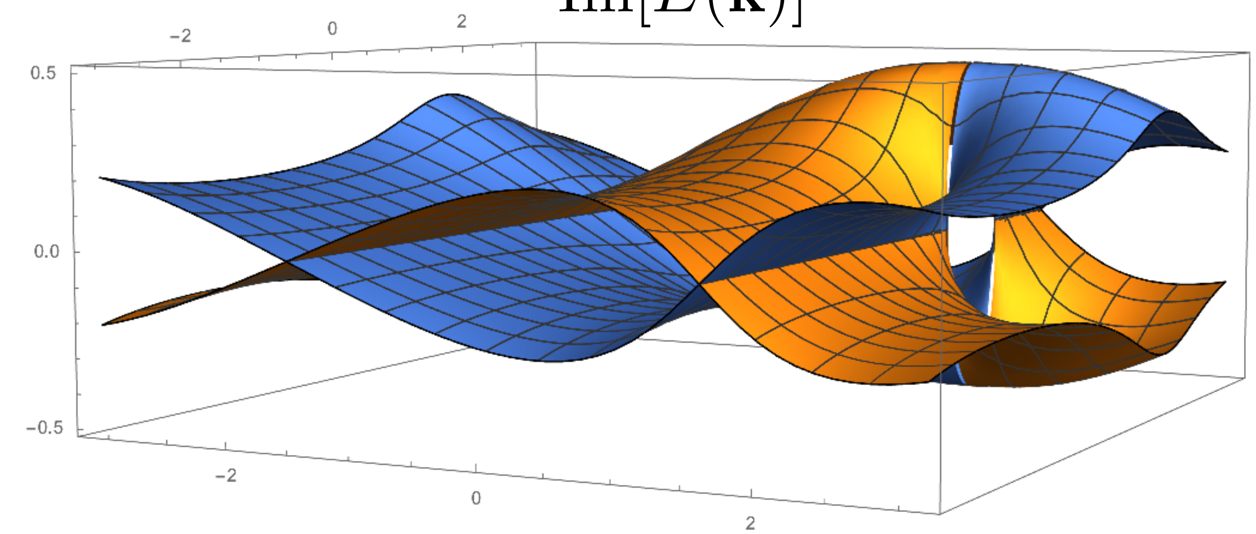
# Spectral features

- EPs are non-analytical, “square roots of Weyl points”




- $E(\mathbf{k})$ is different than what one naively infers from $E^2(\mathbf{k})$!



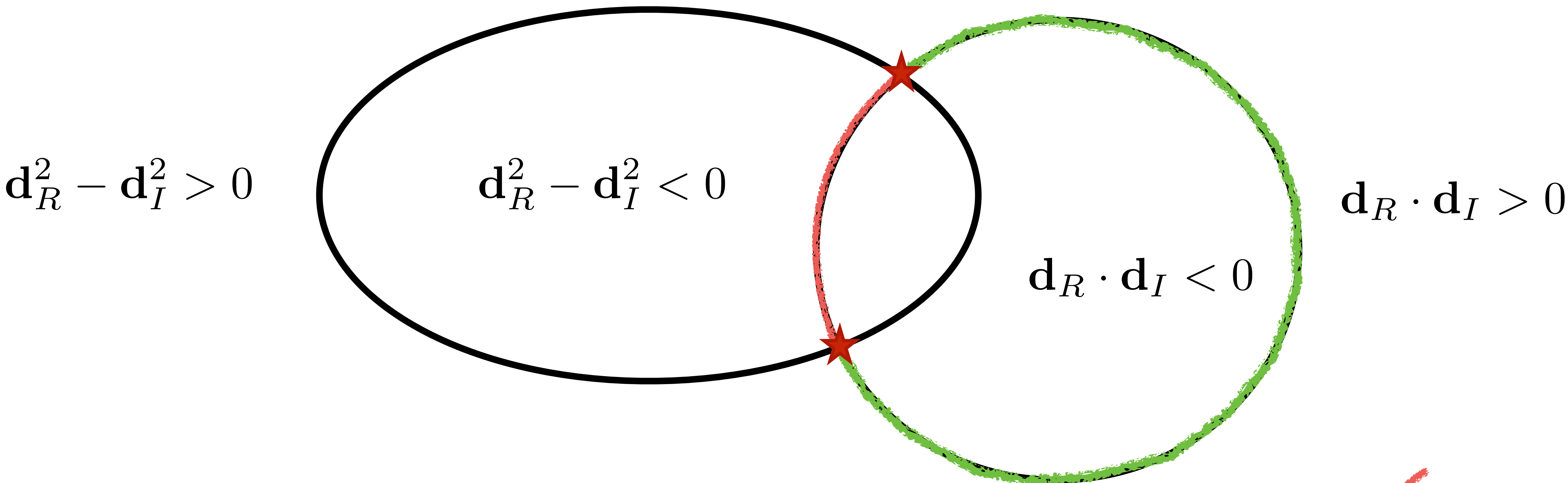




- 2d bulk Fermi arcs!

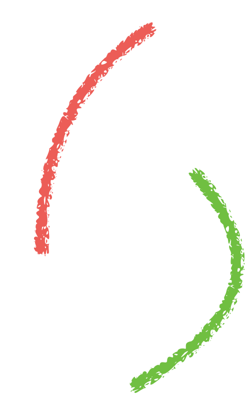
V. Kozii and L. Fu, arXiv:1708.05841

# Let's have a closer look: arcs

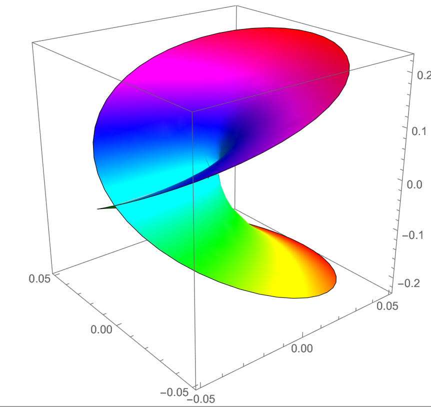
$$E(\mathbf{k}) = \pm\sqrt{\mathbf{d}_\mathrm{R}(\mathbf{k})^2 - \mathbf{d}_\mathrm{I}(\mathbf{k})^2 + 2i\mathbf{d}_\mathrm{R}(\mathbf{k})\cdot\mathbf{d}_\mathrm{I}(\mathbf{k})}$$



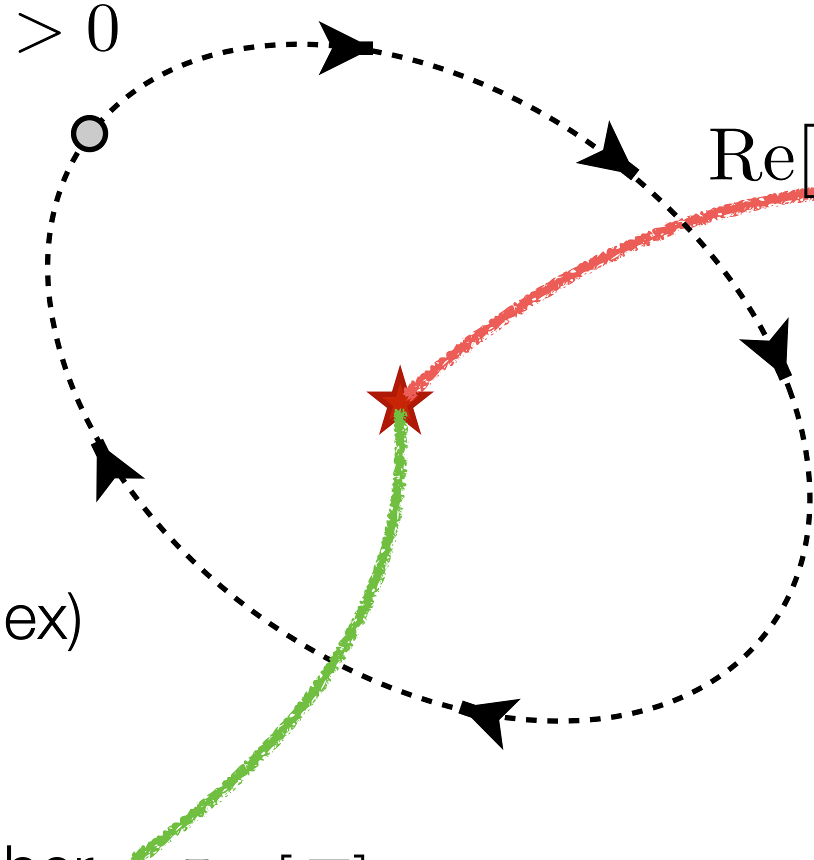

- Fermi arcs $\mathrm{Re}[E] = 0$ when $\mathbf{d}_\mathrm{R}\cdot\mathbf{d}_\mathrm{I} = 0$ and $\mathbf{d}_R^2 - \mathbf{d}_I^2 < 0$
- i-Fermi arcs $\mathrm{Im}[E] = 0$ when $\mathbf{d}_\mathrm{R}\cdot\mathbf{d}_\mathrm{I} = 0$ and $\mathbf{d}_R^2 - \mathbf{d}_I^2 > 0$
- Irremovable degeneracies; generic (d-1)-dimensional open nodal surfaces/arcs



# “Braiding”



- Move around an exceptional point, track an eigenstate




  - $4\pi$ periodicity!
  - Unique, no (complex) band closing
  - “Half” winding number

$$\Psi_{\pm} \propto \begin{pmatrix} \pm r e^{i\theta/2} \\ 1 \end{pmatrix}$$

- We end up in the other eigenstate after one closed orbit!

- Cf. our simple example and non-Abelian braiding
  - But no obvious analogue of adiabatic transport…

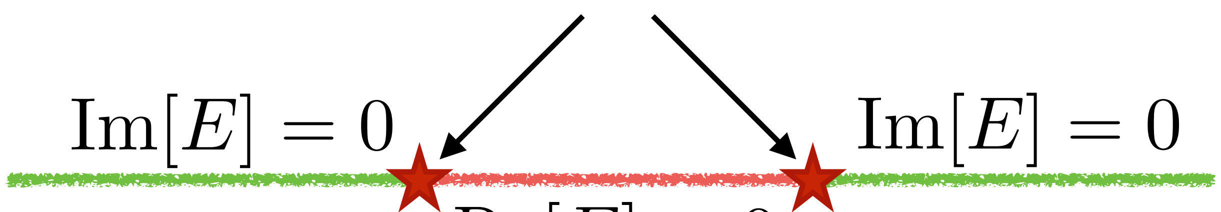
# Splitting Weyl/Dirac points

- Minimal 2d model

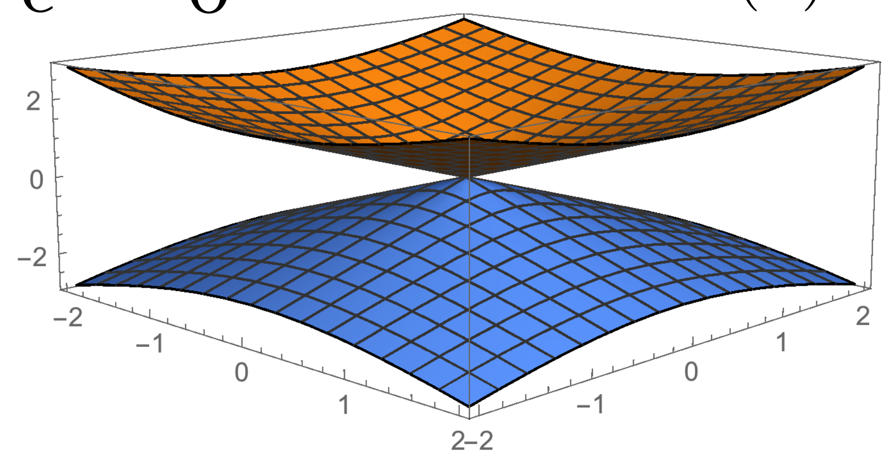
$$H = k_x\sigma_x + k_y\sigma_y + i\epsilon\sigma_x$$

$$\Rightarrow \quad E = \pm\sqrt{k_x^2 + k_y^2 - \epsilon^2 + 2i\epsilon k_x}$$

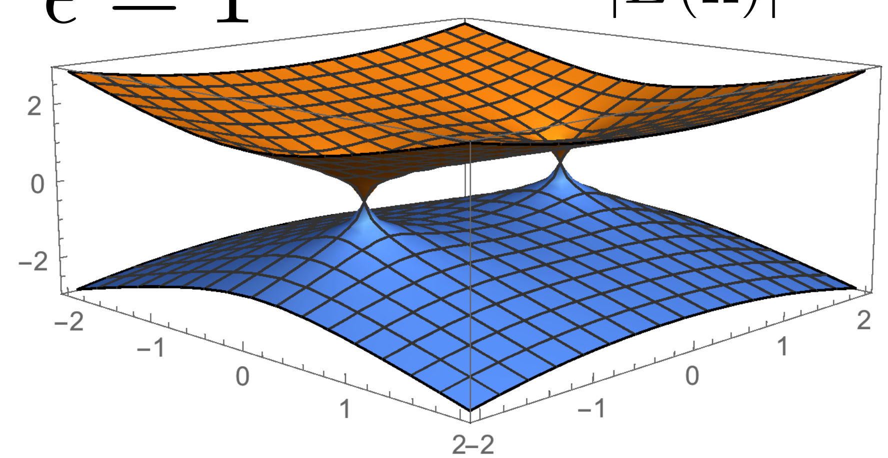
- EPs at $k_x = 0, k_y = \pm\epsilon$



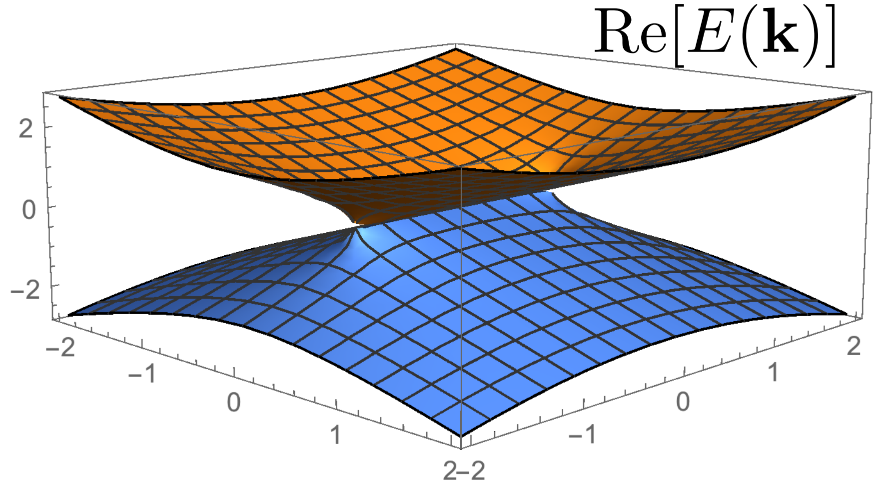

$\epsilon = 0$ $E(\mathbf{k})$



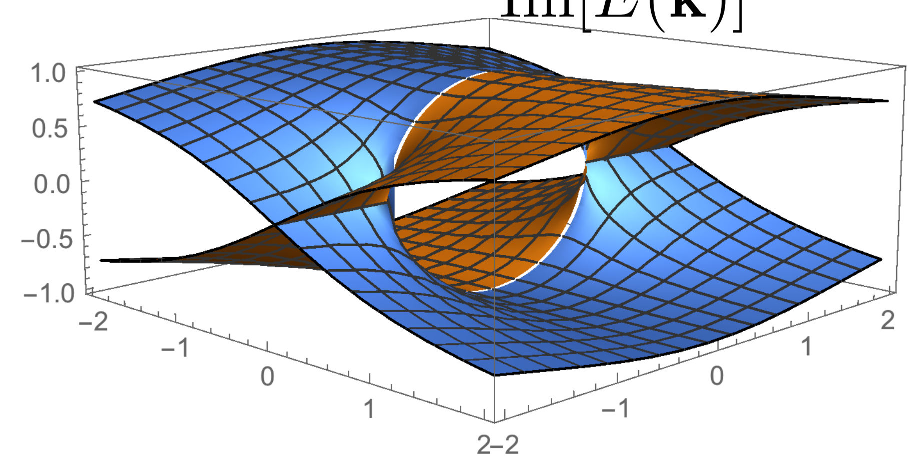
$\epsilon = 1$ $|E(\mathbf{k})|$



$\mathrm{Re}[E(\mathbf{k})]$



$\mathrm{Im}[E(\mathbf{k})]$

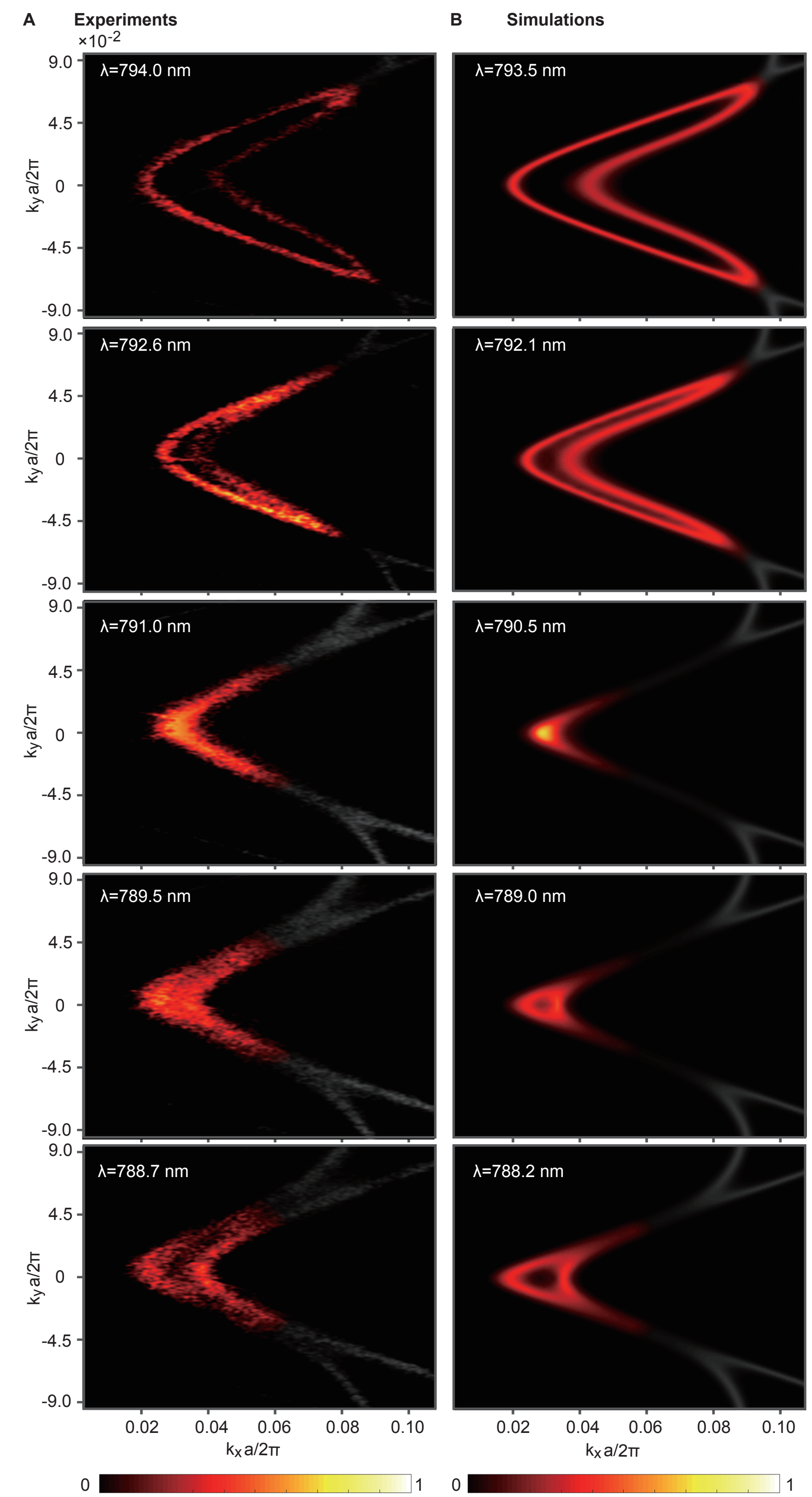


# Experimental observation of 2d bulk Fermi arcs

- Fermi arcs observed in photonic crystal slabs with losses

H.Zhou, et. al. Science p. eaap9859 (2018)

- These experiments directly measure the *spectral density of states*.




- Light scattering, iso-frequency contours vs. theoretical band structure

# Quantum materials (more next time)

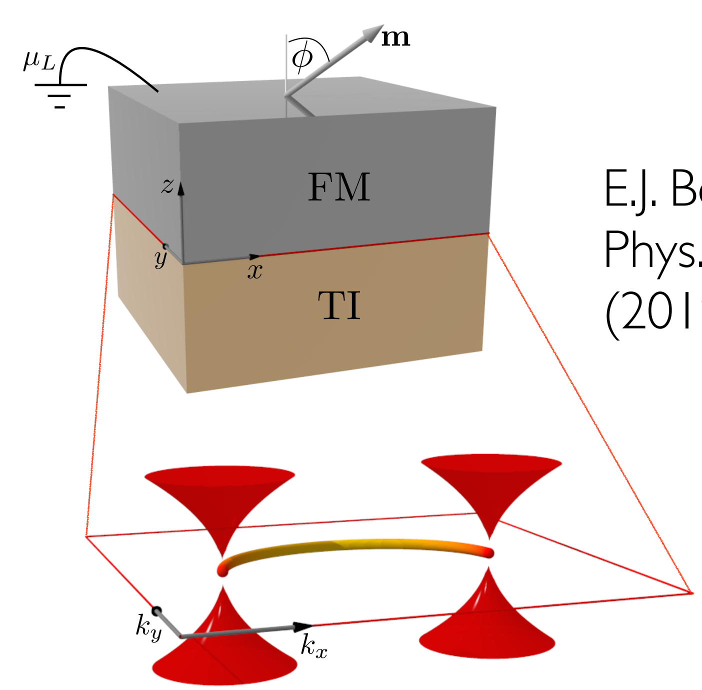
- Self-energies of interacting and disordered systems — intrinsic dissipation

V. Kozii and L. Fu, arXiv:1708.05841 and many later works

- Coupling to environments — example: 3d topological insulator coupled to a ferromagnetic lead

$$H_{\mathrm{NH}} = H + \Sigma_L^r(\omega = 0)$$

- Symmetry protected state promoted to a generic topological phase!
  - Sufficiently generic coupling needed




E.J. Bergholtz and J.C. Budich, Phys. Rev. Research 1, 012003 (2019)

- Similar ideas can be applied e.g. to Kitaev spin liquids

K. Yang, S.C. Morampudi and E.J. Bergholtz Phys. Rev. Lett. 126, 077201 (2021)

K. Yang, D. Varjas, E.J. Bergholtz, S. Morampudi, and F. Wilczek arXiv:2202.03445

# Symmetries in non-Hermitian systems

- Specifically non-Hermitian symmetries (43 or 38 sym. classes)

D. Bernard and A. LeClair, arXiv:cond-mat/0110649 (2001)

- Example $H = qH^\dagger q^{-1}, \quad q^\dagger q^{-1} = qq^\dagger = \mathbb{I}.$ "Pseudo-hermiticity"

  - For 2-band models, pick $q = \sigma_x$

$$d_x, d_0 \in \mathbb{R}, \qquad d_y, d_z \in i\mathbb{R}.$$

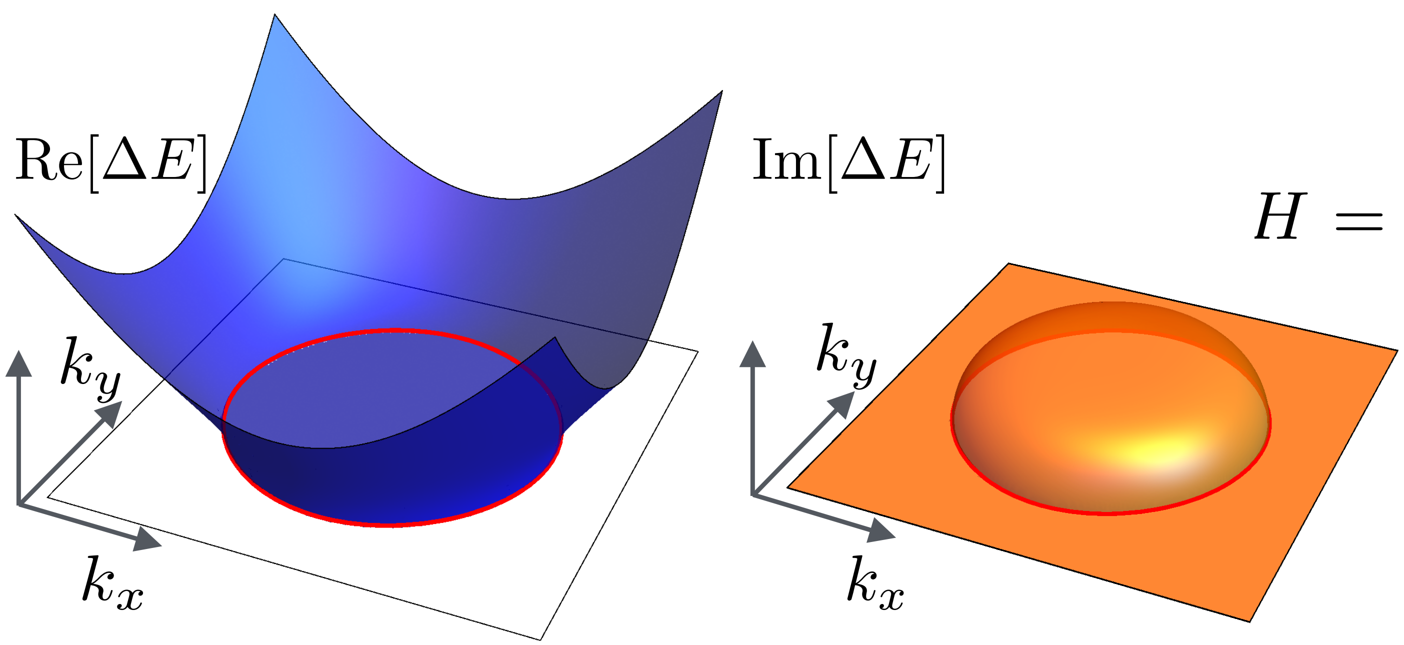
  - Generally $\mathbf{d}_R \cdot \mathbf{d}_I = 0 \Longrightarrow E_\pm = \pm\sqrt{d_R^2 - d_I^2}, \qquad (d_0 = 0)$

Purely real or imaginary!

- $\mathcal{PT}$ symmetric systems, popular in optics, work analogously. Homework!

# Symmetry protected nodal non-Hermitian phases

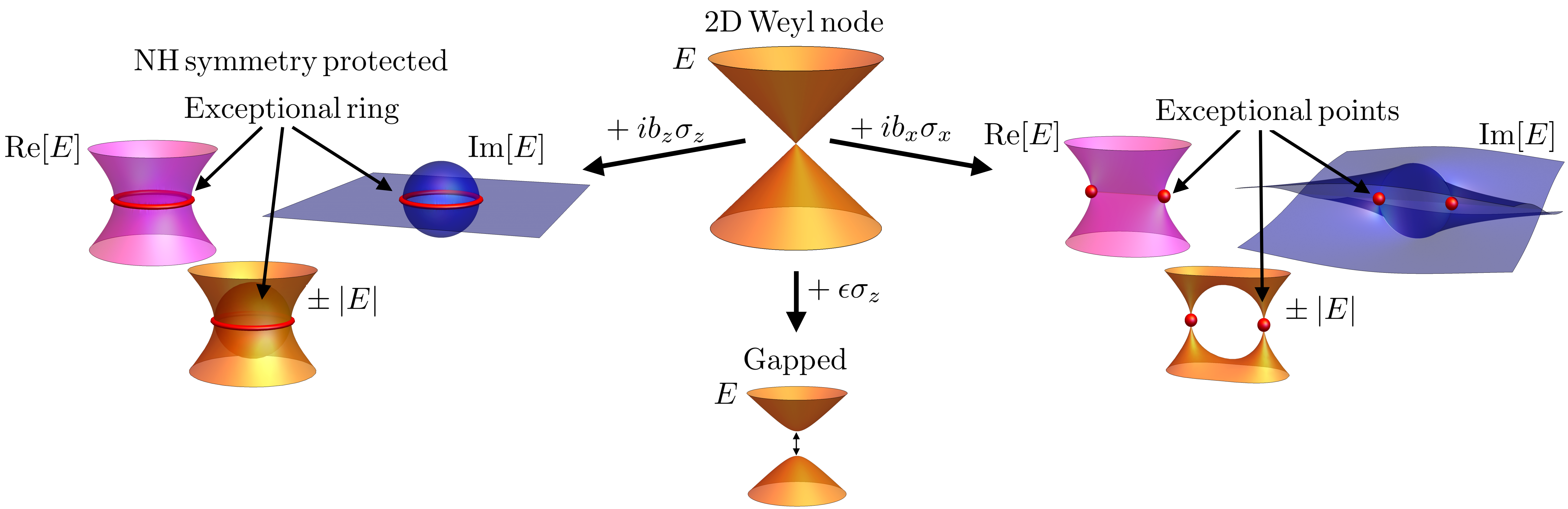
- Generically one equation less…
  - Exceptional (d-1)-dimensional surfaces
  - d-dimensional open “Fermi volumes”
- 2d example




$$H = (2 - \cos k_x - \cos k_y)\sigma_x + i\sigma_z/4$$

J.C. Budich, J. Carlström, F.K. Kunst and E.J. Bergholtz, Phys. Rev. B 99, 041406 (2019)
+ several subsequent postings…

# 2d NH nodal phases summarised

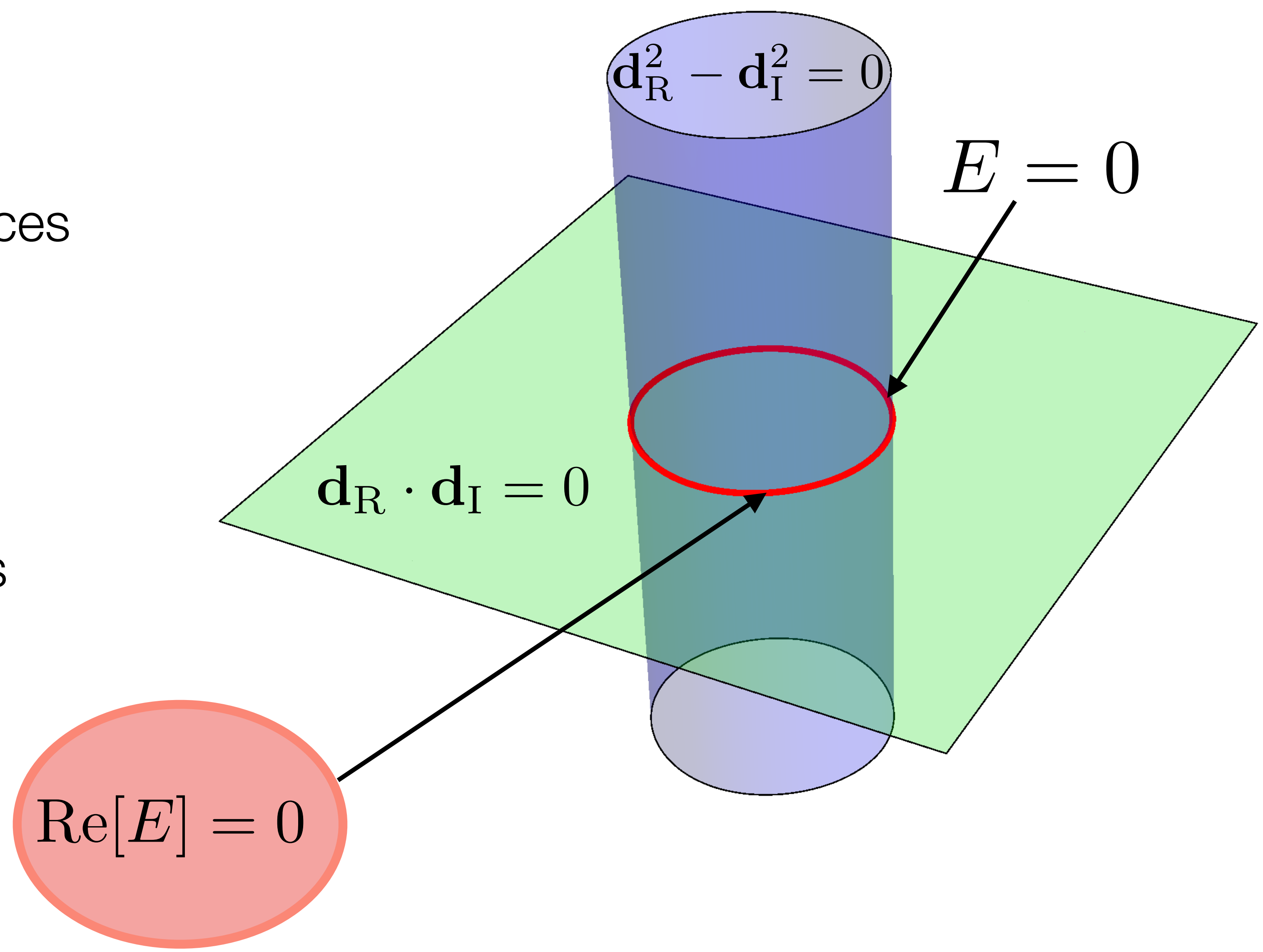



- Turns out that more bands further enriches the picture further (tbd next week)
- How about 3d?

# 3d: generic exceptional rings,…

- 3 parameters but only 2 constraints — generic line-like solutions!
- E=0 solutions form exceptional rings Y. Xu, S.-T. Wang, and L.-M. Duan, PRL 118, 045701 (2017)
- Think about this geometrically
  - Intersections between 2d surfaces
- Leads to unusual open Fermi surfaces
  - Terminated by exceptional lines

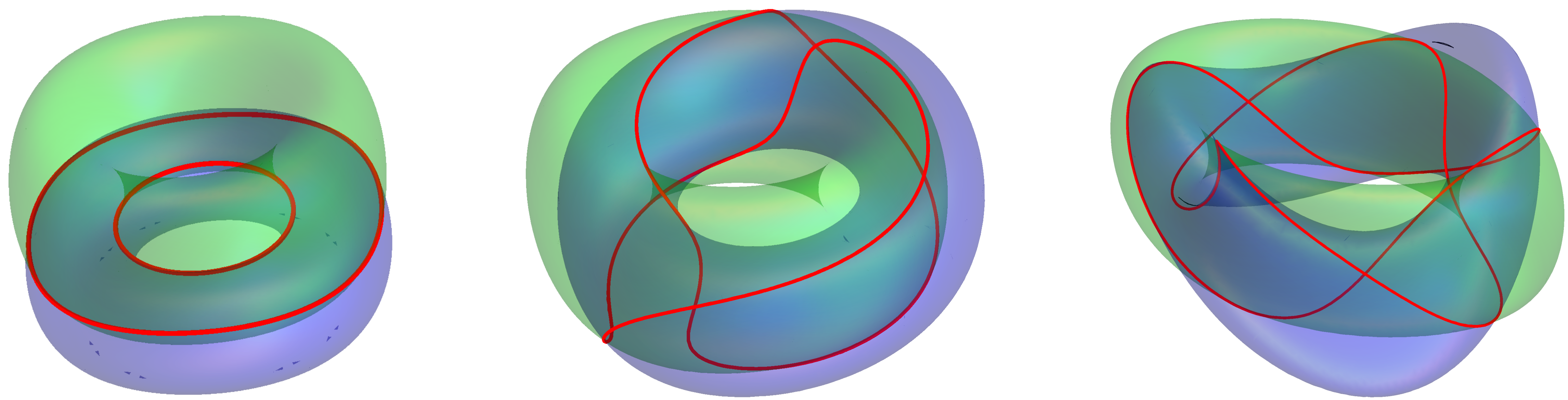
J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)



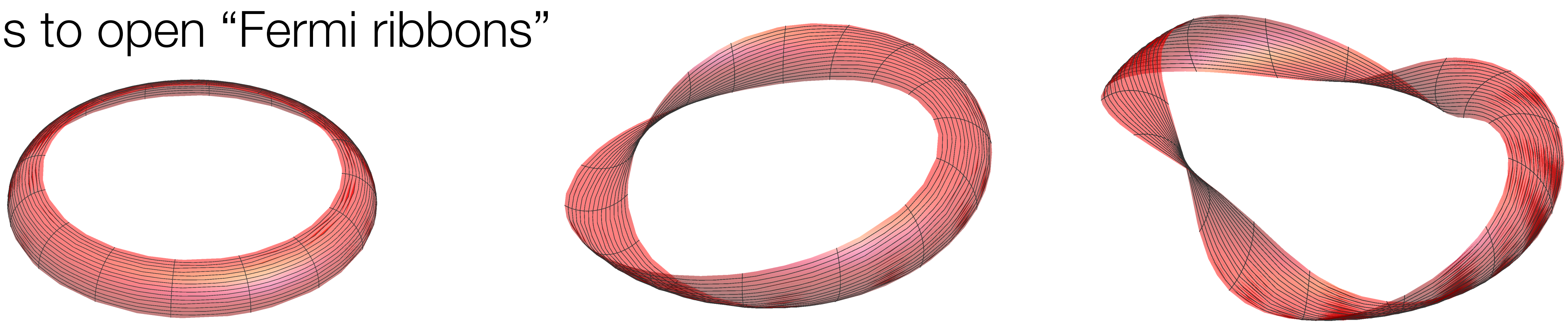
# Exceptional links and twisted "Fermi Ribbons"

- Exceptional links generated as generic intersections between more general 2d closed surfaces

J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)



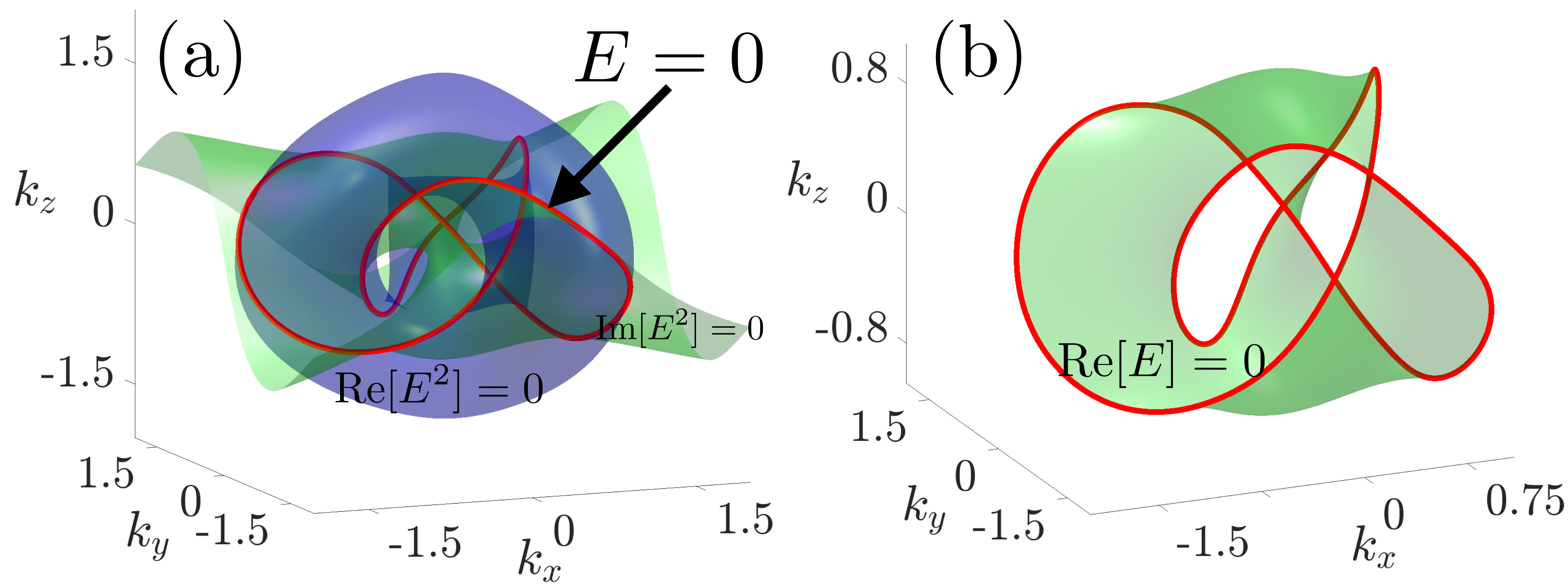
- Leads to open "Fermi ribbons"



  - Seifert surfaces, orientable

# Generalization: Knotted non-Hermitian metals

J. Carlström, M. Stålhammar, J.C. Budich and E.J. Bergholtz, Phys. Rev. B 99, 161115 (2019)




"Fermi-Seifert surfaces"

- Two notions of topology combined
  - Hermitian generic line-like nodes occur in D=4, but in D>3 all knots are trivial!

- Boundary states, hyperbolic knots, Alexander polynomials etc in followup works…

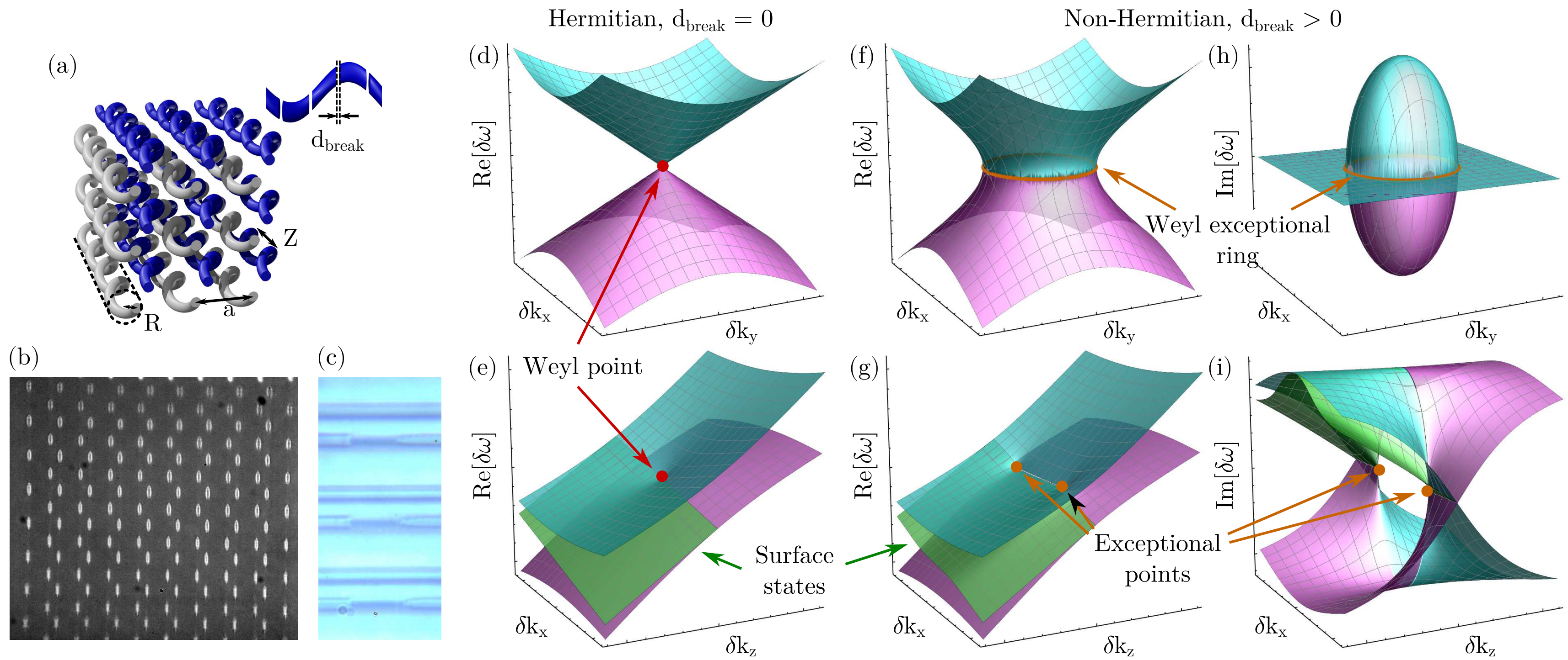
C.H. Lee et. al., Com. Phys. 4, 47 (2021)

M. Stålhammar, et. al., SciPost Phys. 7, 019 (2019)

# Exceptional rings & knots: Experiments

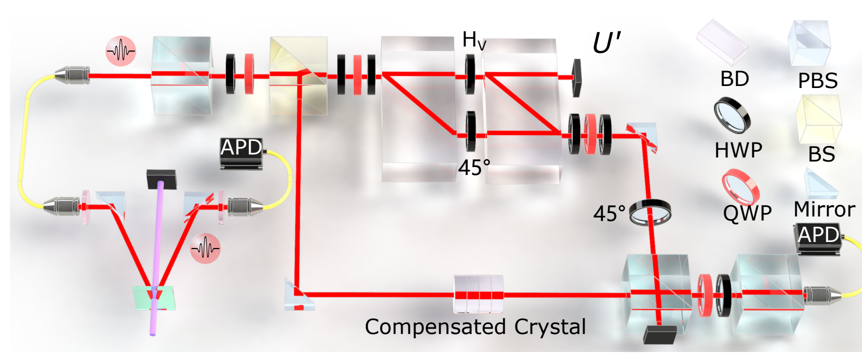
- Exceptional rings realised with coupled waveguides

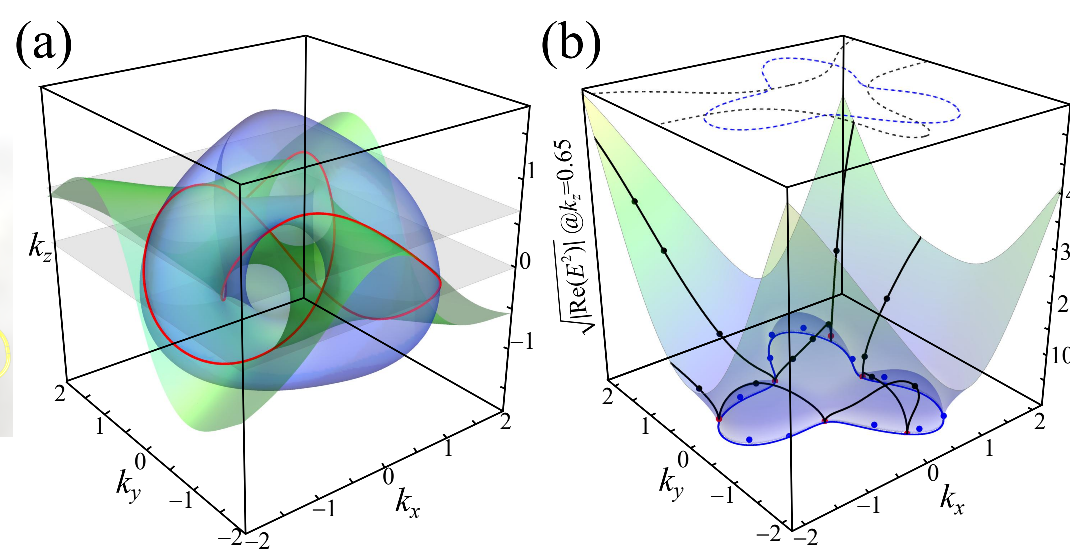
  Cerjan et. al.
  Nature Photonics 13, 623 (2019)




- Exceptional knots and Seifert surfaces in single-photon interferometry

  Wang et. al.
  Phys. Rev. Lett. 127, 026404 (2021)



# Minimal example 2: Large N

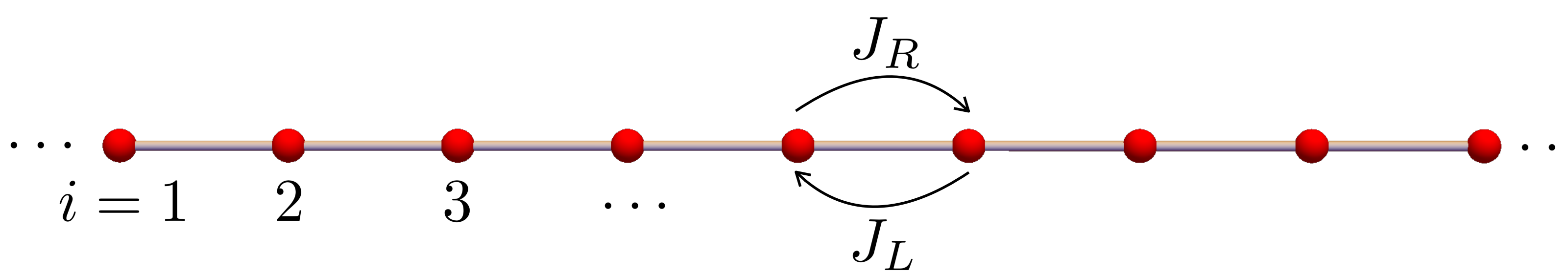
- Take home: Topology of complex energies & strong response to boundary conditions

# Hatano-Nelson model

N. Hatano and D.R. Nelson
Phys. Rev. Lett. 77, 570 (1996)

Single-band model with asymmetric hopping

$$H = \sum_i \left( J_L c_i^\dagger c_{i+1} + J_R c_{i+1}^\dagger c_i \right) \qquad J_L, J_R \in \mathbb{R} \qquad J_L \neq J_R$$



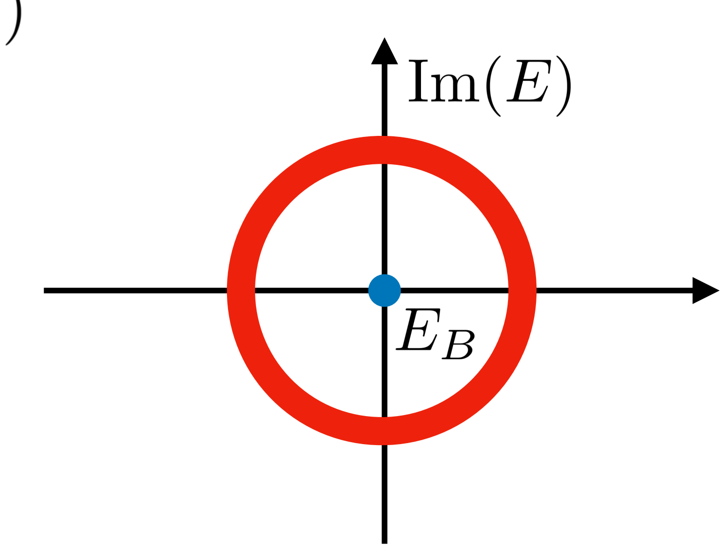

- Complex dispersion relation

$$E_k = (J_L + J_R)\cos(k) + i\,(J_L - J_R)\sin(k)$$

- Winding number distinguishes different phases with respect to a point gap

$$w = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \mathrm{d}k \, \partial_k \ln E_k$$

- Phase transition at $|J_L| = |J_R|$




Gong et. al.
Phys. Rev. X 8, 031079 (2018)

# Hatano-Nelson with open boundaries

N. Hatano and D.R. Nelson
Phys. Rev. Lett. 77, 570 (1996)

- The spectrum with open boundaries is completely different from the periodic system — states pile up at one of the boundaries!

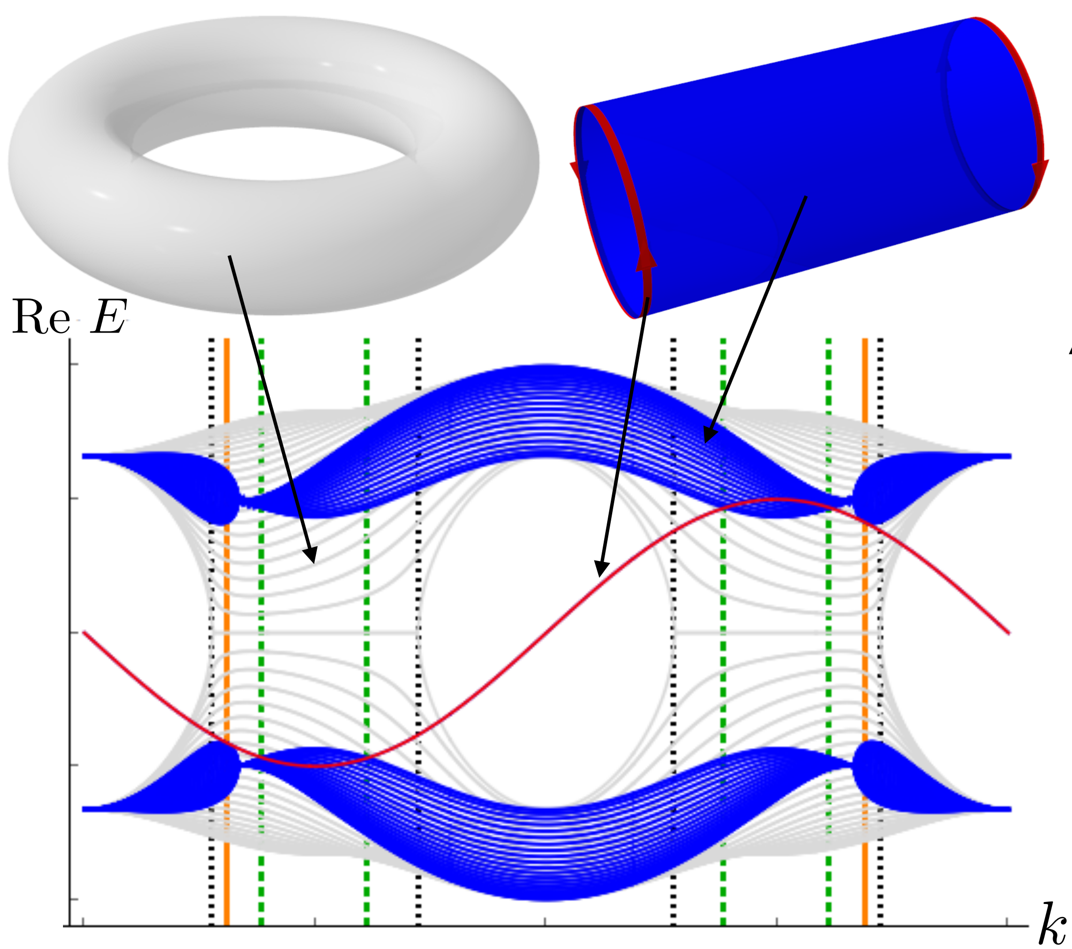
$$H_{\text{open}} = \begin{pmatrix} 0 & J_R & 0 & 0 & 0 \\ J_L & 0 & J_R & 0 & 0 \\ 0 & J_L & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & J_R \\ 0 & 0 & 0 & J_L & 0 \end{pmatrix}$$

  - Single Jordan block and order N exceptional point with at $J_L = 0$
  - Extreme sensitivity to boundary conditions: $E_k = 0$ vs $E_k = J_R e^{-ik}$

All states at the end site | Bloch states

- Actually, what, more precisely, do we mean by “states” above?

# Focus 2: Anomalous bulk-boundary correspondence




F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018)

Alternative approach :

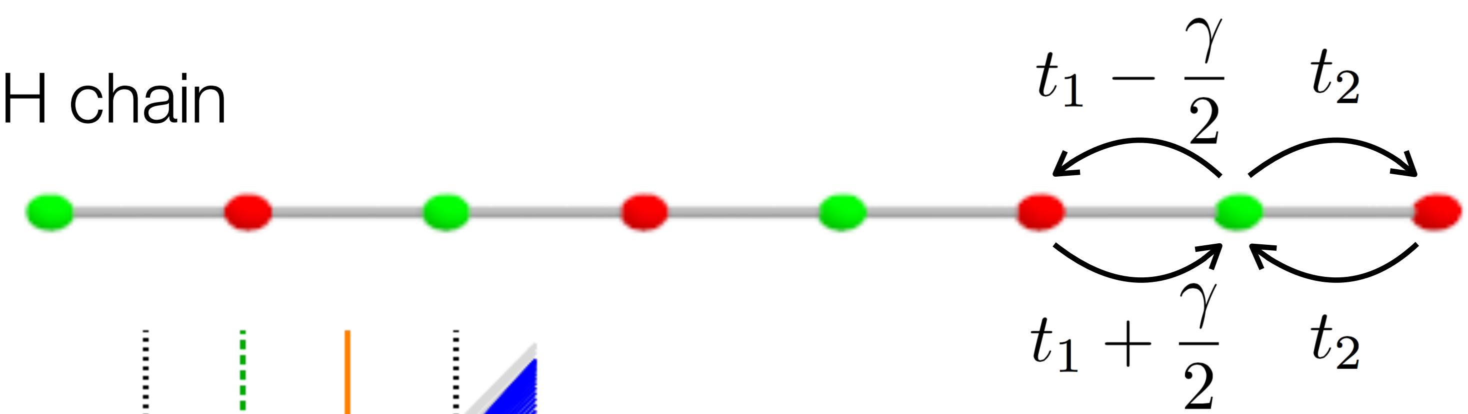
S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018)

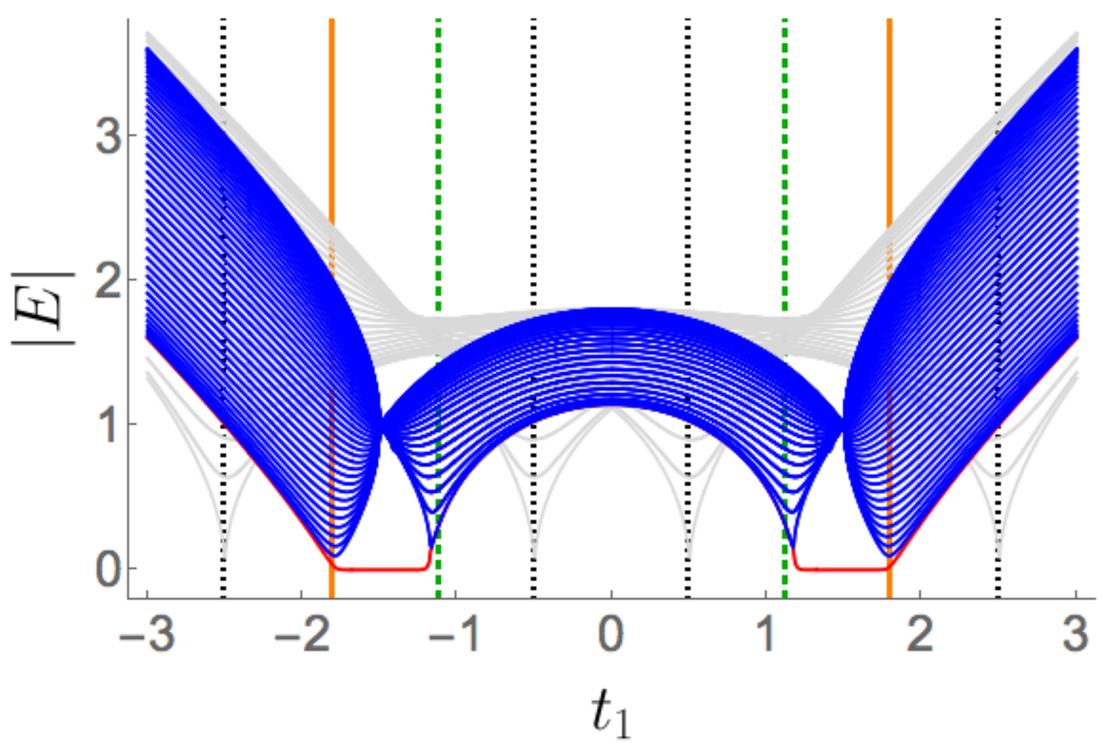
- Take home: Open and closed boundary conditions give very different physics — but cases can be understood and are experimentally relevant!

# Basic observation (cf. also minimal example 2)

- Open and periodic energy spectra can be dramatically different!

- Example: non-Hermitian SSH chain



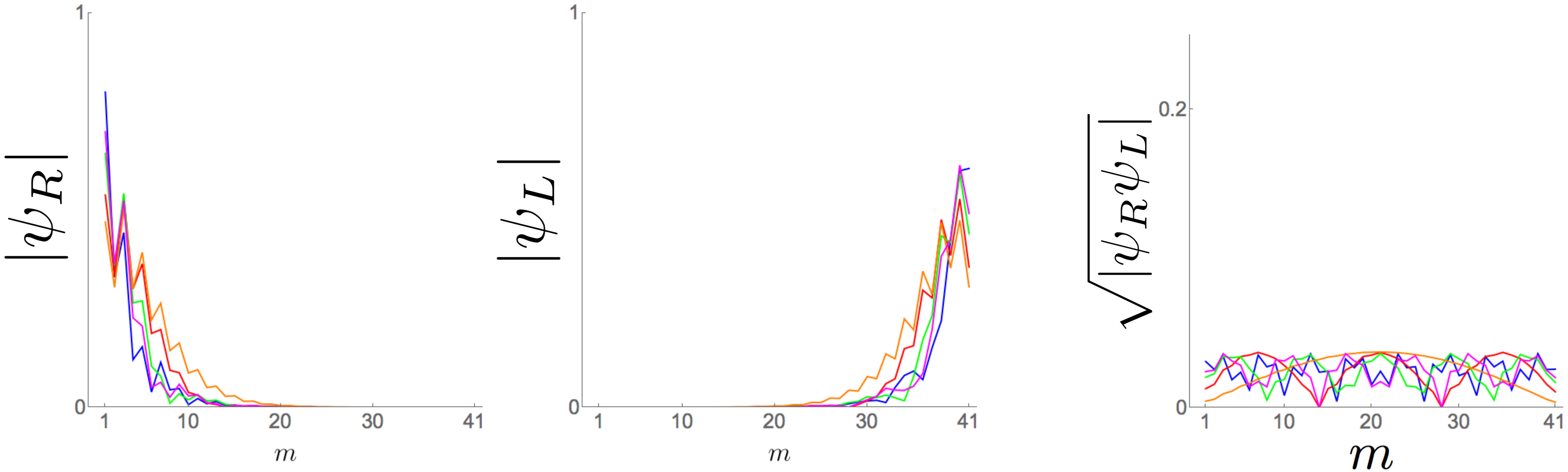




- Literature filled with topological invariants calculated with periodic boundary conditions — but these do not generally dictate the presence of boundary modes!

- Need to consider the open system from the outset…

# Non-Hermitian skin effect

- At the heart of the problem




- Left and right eigenstates pile up at opposite sides

- But their “product” does not

# Biorthogonal quantum mechanics

Brody, J Phys. A: Math. Theor. 47, 035305 (2013)

- By definition we have

$$H|u_n^R\rangle = E_n|u_n^R\rangle \quad \text{and} \quad H^\dagger|u_n^L\rangle = E_n^*|u_n^L\rangle$$

- Away from exceptional points one can get a <u>*complete orthonormal basis*</u> by choosing

$$\langle u_n^L|u_m^R\rangle = \delta_{nm}\,\langle u_n^L|u_n^R\rangle$$

  - Leading to

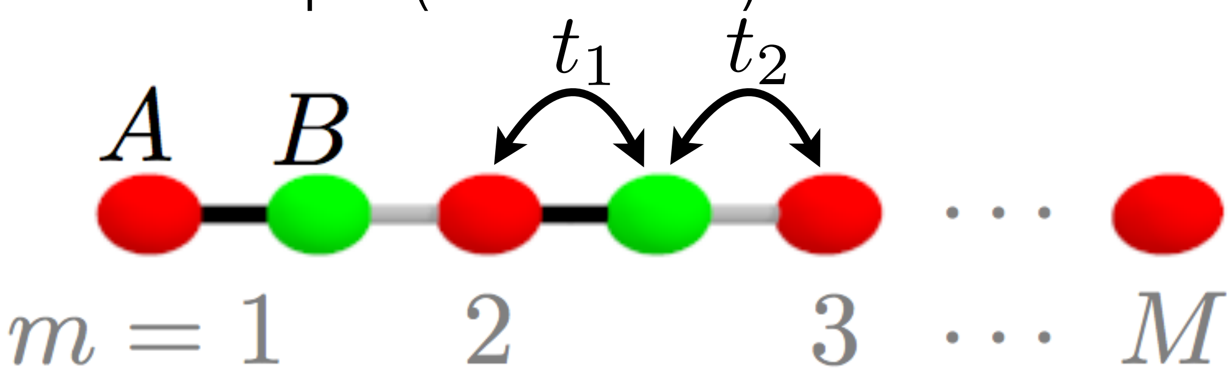
$$\sum_n \Pi_n = \mathbb{1} \quad \text{with} \quad \Pi_n = |u_n^R\rangle\,\langle u_n^L|$$

and $E_n = \langle u_n^L|H|u_n^R\rangle$ with $E_n \in \mathbb{C}$

- This provides the "product"…

# A step back, again: Exact boundary states

- Basic example (SSH chain)



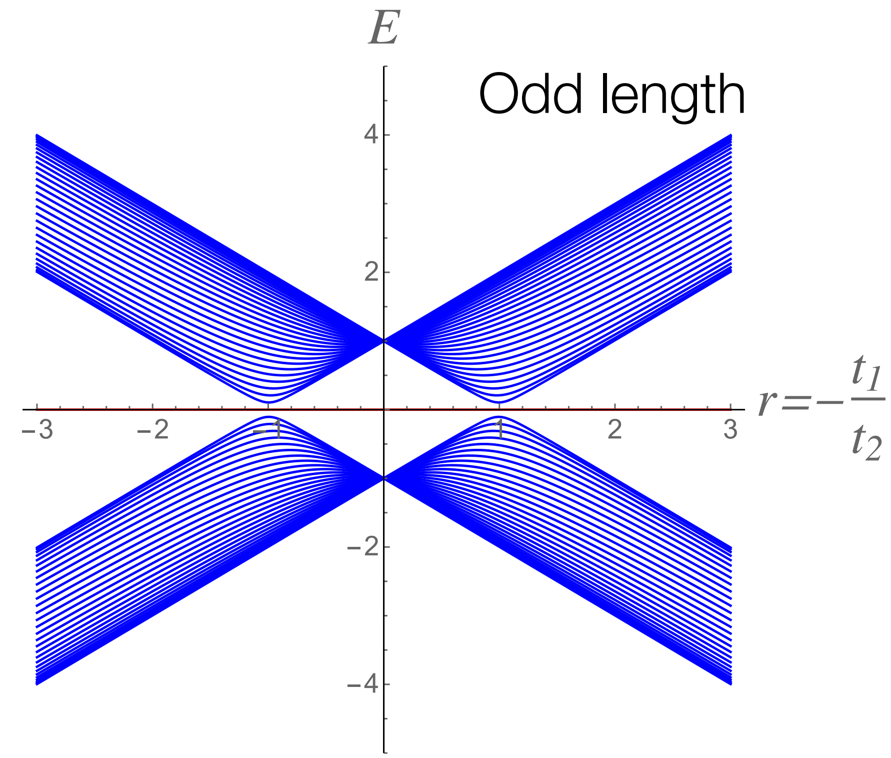

$$|\Psi_{\text{end}}\rangle = \mathcal{N} \sum_m \left(-\frac{t_1}{t_2}\right)^m c^\dagger_{A,m}|0\rangle$$

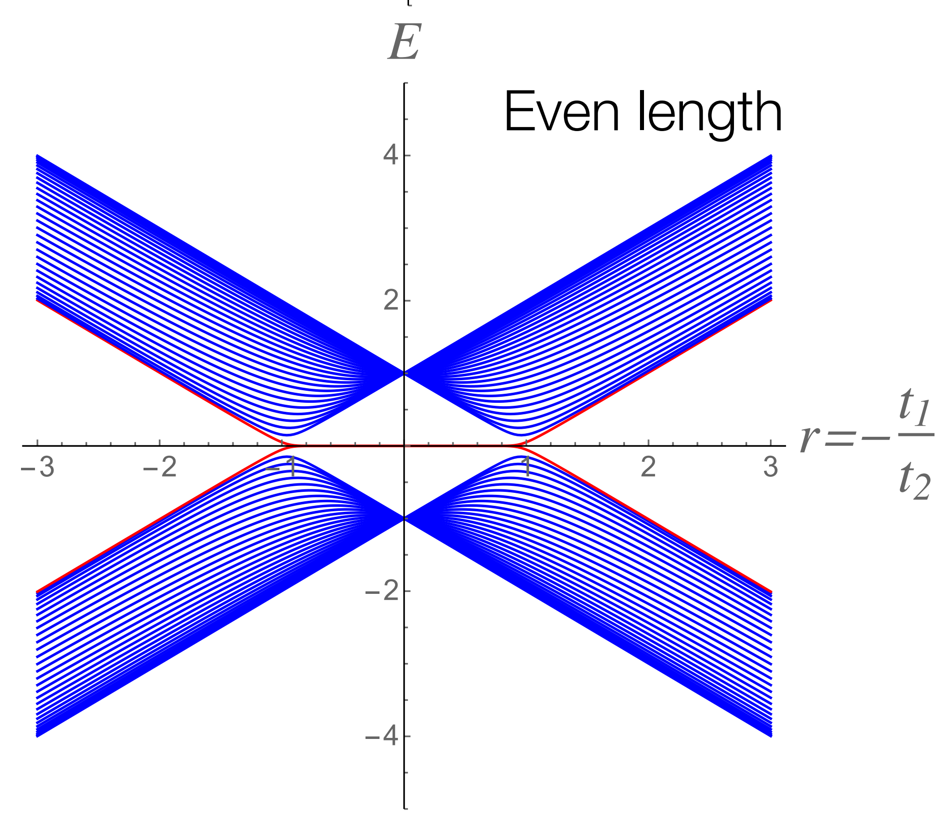
$$E = 0$$

$t_1$ $t_2$

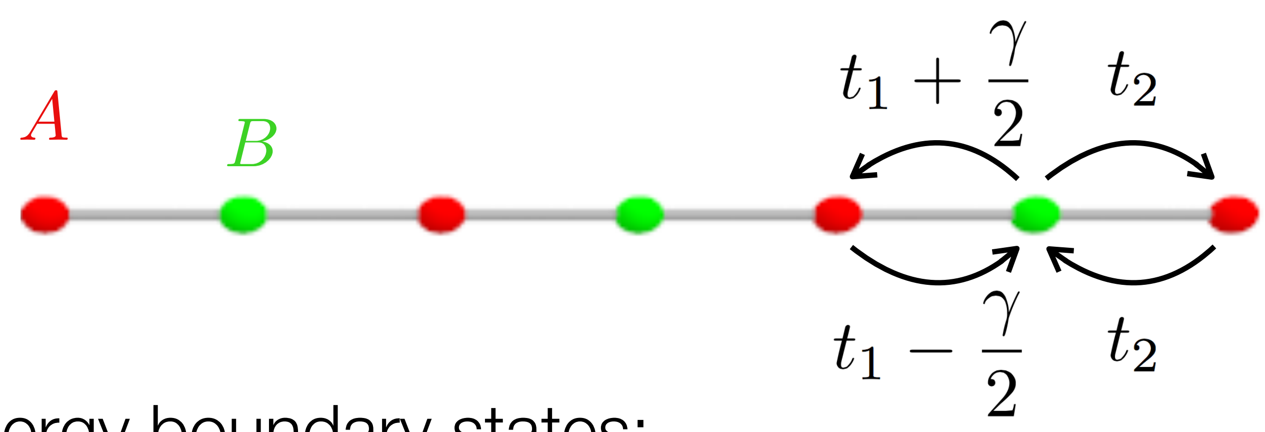
$$1 \qquad 0 = 1 \cdot t_1 + \left(-\frac{t_1}{t_2}\right) \cdot t_2 \qquad -\frac{t_1}{t_2}$$

- Phase transition when the boundary state delocalises!



# Back to the non-Hermitian SSH chain




- Exact zero energy boundary states:

$$|\psi_R\rangle = \mathcal{N}_R \sum_{m=1}^{M} r_R^m c^\dagger_{A,m} |0\rangle \qquad |\psi_L\rangle = \mathcal{N}_L \sum_{m=1}^{M} r_L^m c^\dagger_{A,m} |0\rangle$$

$$r_R = -\frac{t_1 - \frac{\gamma}{2}}{t_2} \neq r_L = -\frac{t_1 + \frac{\gamma}{2}}{t_2}$$

- Observation: when $|r_L^* r_R| = 1$ we have an exact zero energy biorthogonal bulk state!

$$\langle \Pi_m \rangle \equiv \langle \psi_L | \Pi_m | \psi_R \rangle \sim (r_L^* r_R)^m$$ (now with $\Pi_m = |e_{A,m}\rangle\langle e_{A,m}| + |e_{B,m}\rangle\langle e_{B,m}|$)

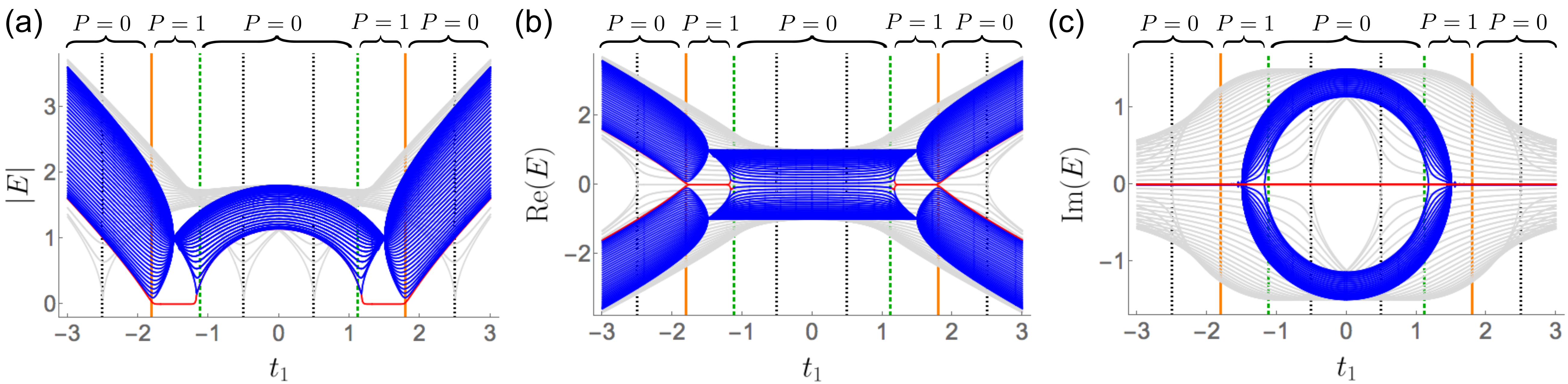
Not positive definite!

- Phase transitions and changes in zero-modes at $t_1 = \pm\sqrt{\frac{\gamma^2}{4} + t_2^2}, \pm\sqrt{\frac{\gamma^2}{4} - t_2^2}$ ?

# Biorthogonal polarisation and boundary modes

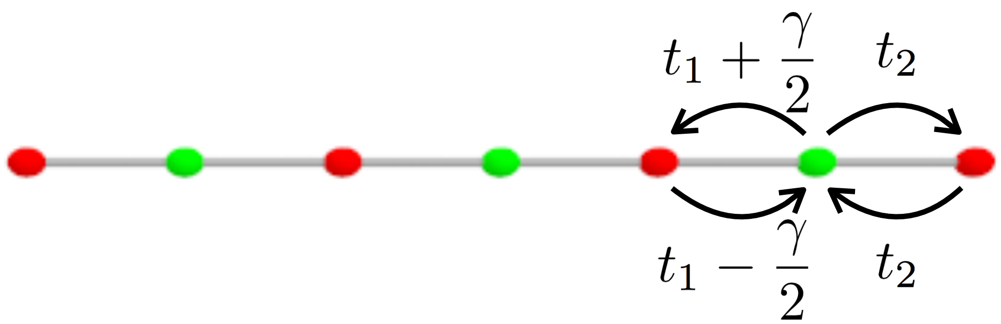
- We construct a "biorthogonal polarisation", $P$, which is quantised and jumps precisely when $|r_L^* r_R| = 1$

$$P \equiv 1 - \lim_{M\to\infty} \left\langle \psi_L \left| \frac{\sum_m m\,\Pi_m}{M} \right| \psi_R \right\rangle$$




- Predicts the correct phase transitions — strikingly different from bulk invariants and also from indicators involving only right or left eigenstates!
  - Works also for non-solvable models and multiple boundary modes

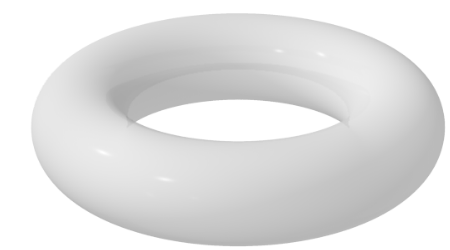
# It generalises directly: Non-Hermitian Chern insulators



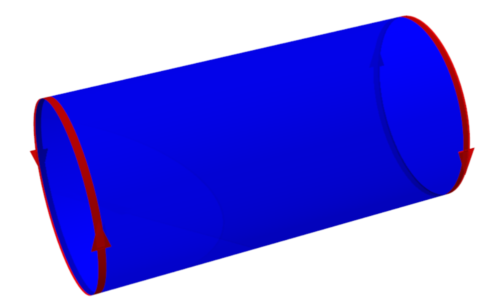

$$t_1 = t + \delta\cos(k) \qquad t_2 = t - \delta\cos(k)$$

$$d_z(k) = -\Delta\sin(k)$$

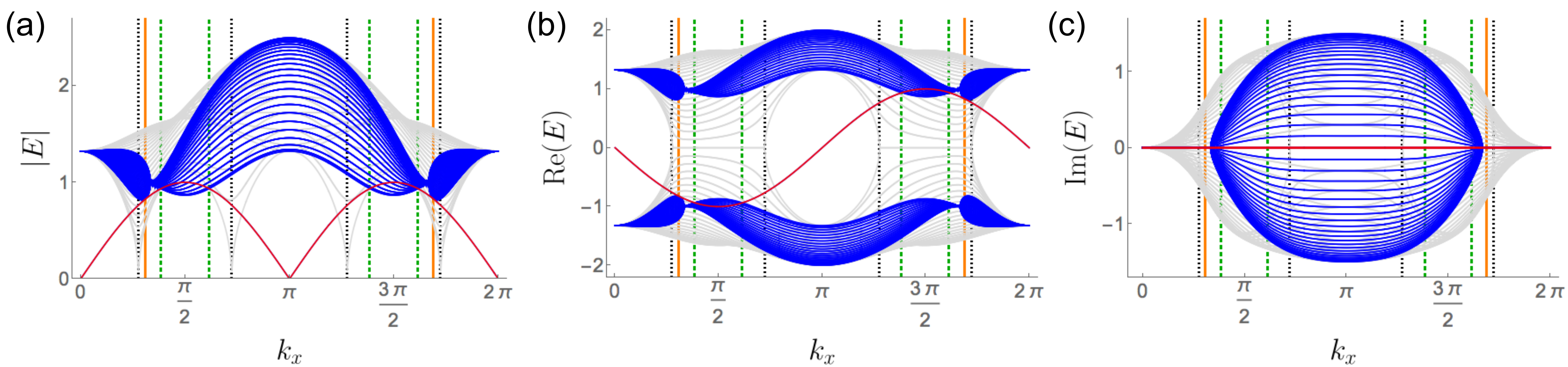
$$|\psi_R\rangle = \mathcal{N}_R \sum_{m=1}^{M} r_R^m c^\dagger_{A,m}|0\rangle \qquad |\psi_L\rangle = \mathcal{N}_L \sum_{m=1}^{M} r_L^m c^\dagger_{A,m}|0\rangle$$



$$r_R = -\frac{t_1 - \frac{\gamma}{2}}{t_2} \qquad r_L = -\frac{t_1 + \frac{\gamma}{2}}{t_2}$$

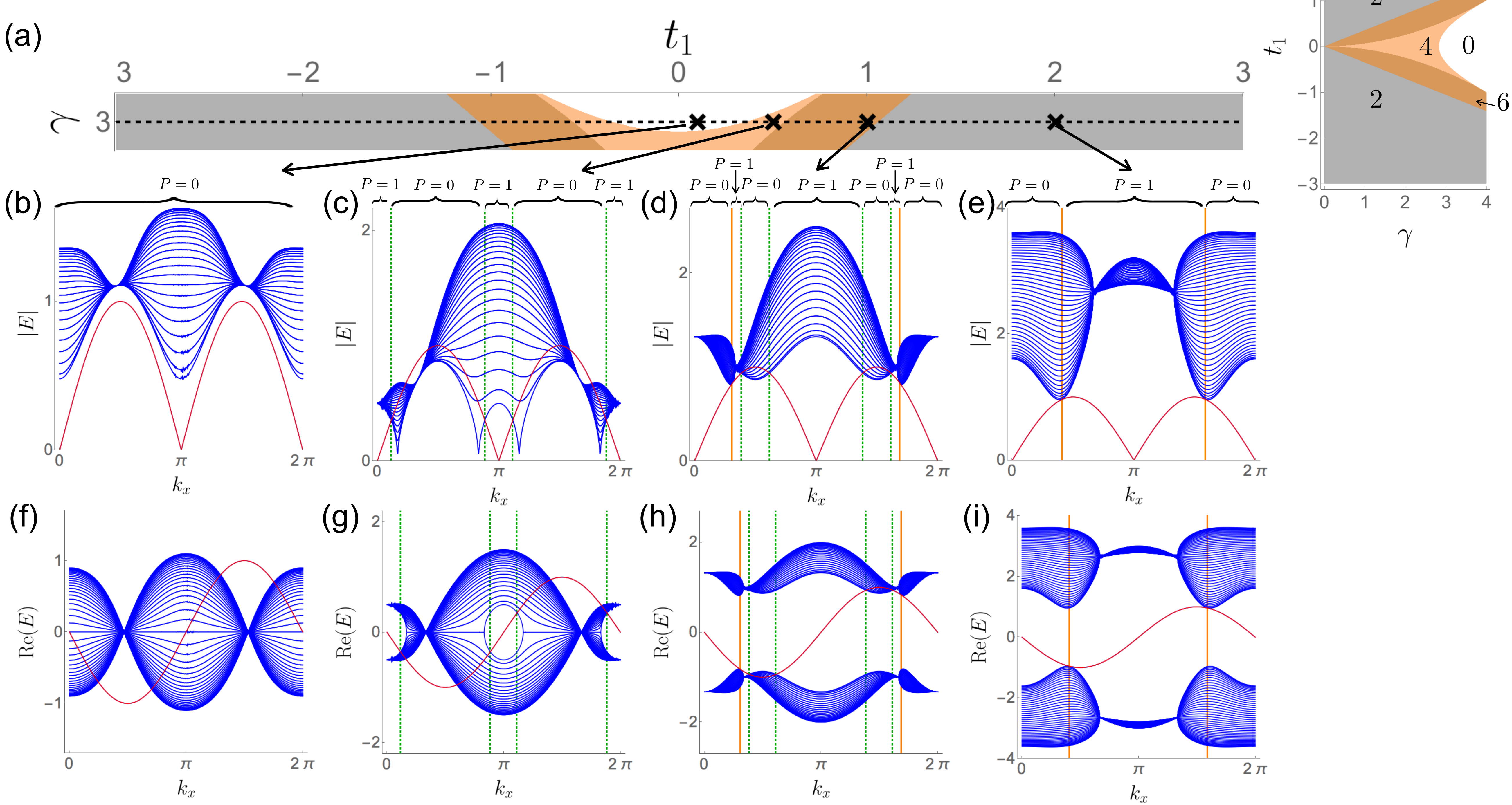


$$|r_L^* r_R| = 1 \Rightarrow \cos(k) = \frac{\gamma^2}{16t\delta}, \pm\sqrt{\frac{\gamma^2/8 - t^2}{\delta^2}}$$



# Chern insulator phase diagram

$$|r_L^* r_R| = 1 \qquad \cos(k) = \frac{\gamma^2}{16t\delta},\ \pm\sqrt{\frac{\gamma^2/8 - t^2}{\delta^2}}$$



# Why does it work?

- Spectrum from left *and* right eigenvectors

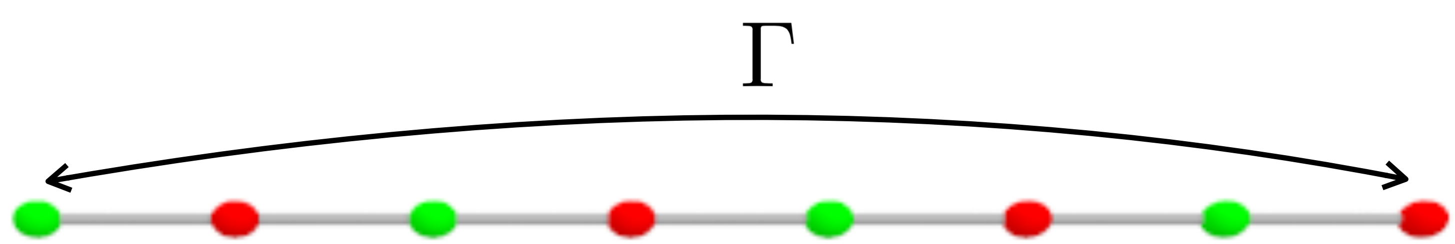
$$E_n = \langle u_n^L | H | u_n^R \rangle$$

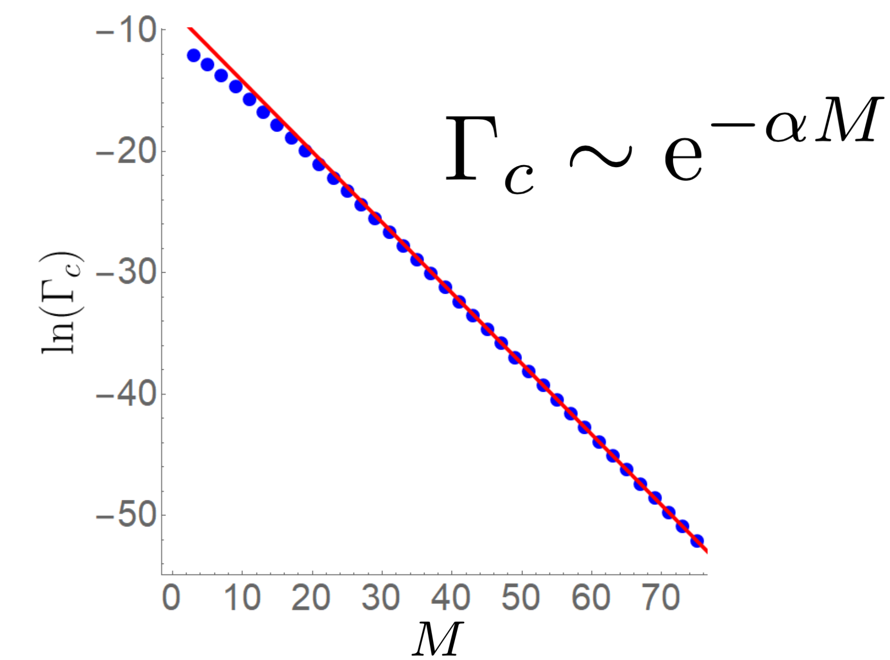
  - Extended/delocalised biorthogonal states play the same role as extended states does in Hermitian models where the distinction between right and left is gone

# Periodic vs. open boundary conditions

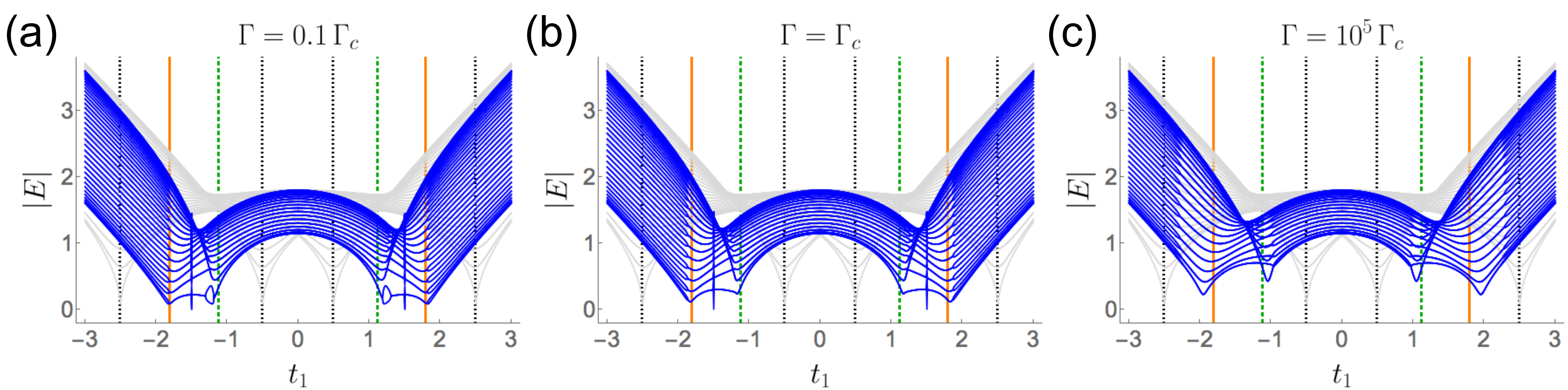
Inspired by: Xiong, Journal of Physics Communications 2, 035043 (2018)

- Effect of coupling the ends



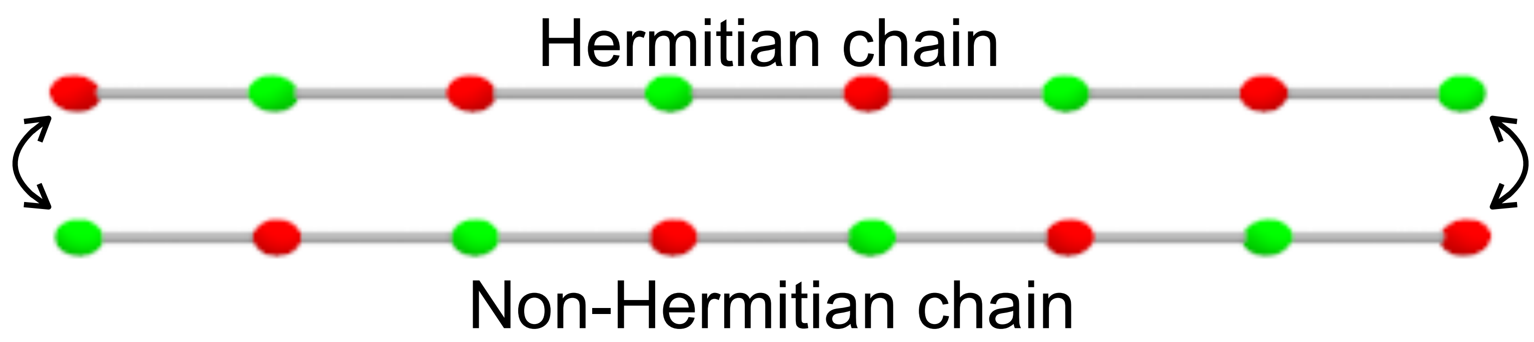



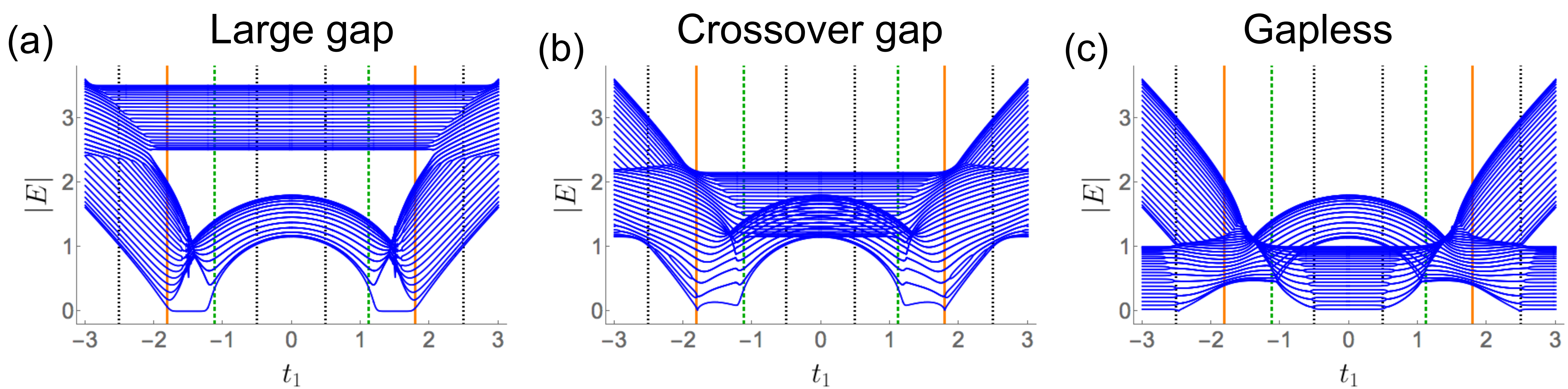

- Crossover at exponentially small $\Gamma$
- Intuitive from the perspective of the skin effect (but we'll get back to some subtleties…)



# Domain walls

- Physical mechanism: coupling ends via a Hermitian domain

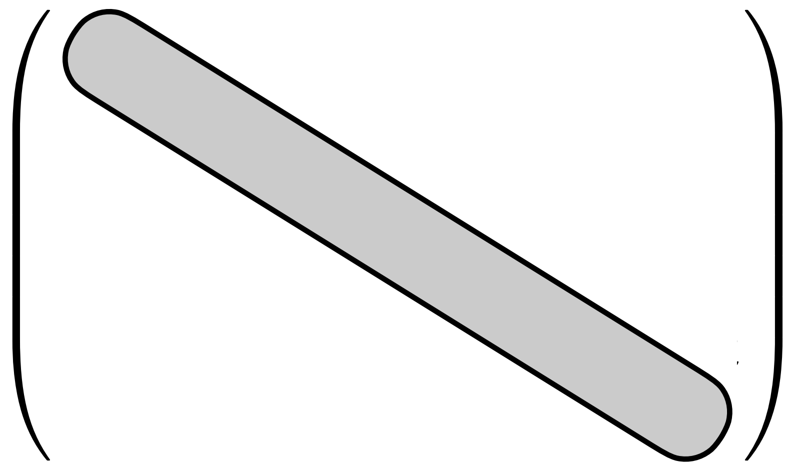


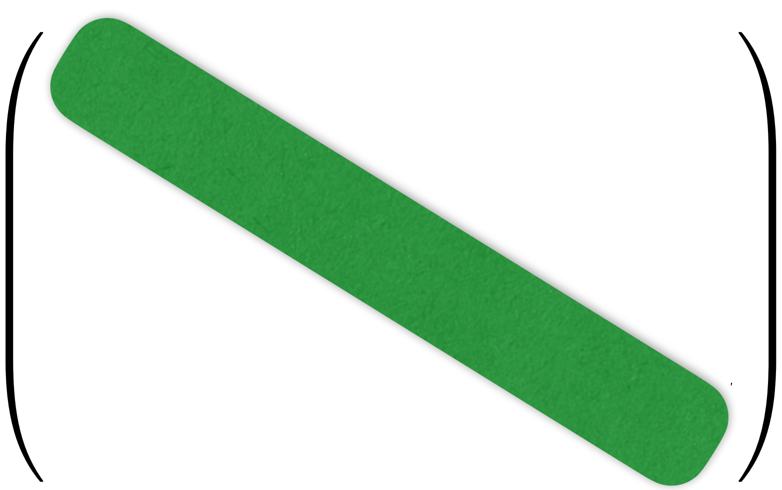

- Both periodic and open system physics can be realised depending on the strength of the effective coupling!
    - Also tuneable geometrically and/or by Wannier function engineering

# (In)stability of the spectra of large NH matrices

- Same thing, different perspective

$H_{\mathrm{open}} =$ 

  - The spectrum of $H_{\mathrm{open}}$ is stable to small perturbations like $H_1$ but not $H_2$

$H_1 =$ 

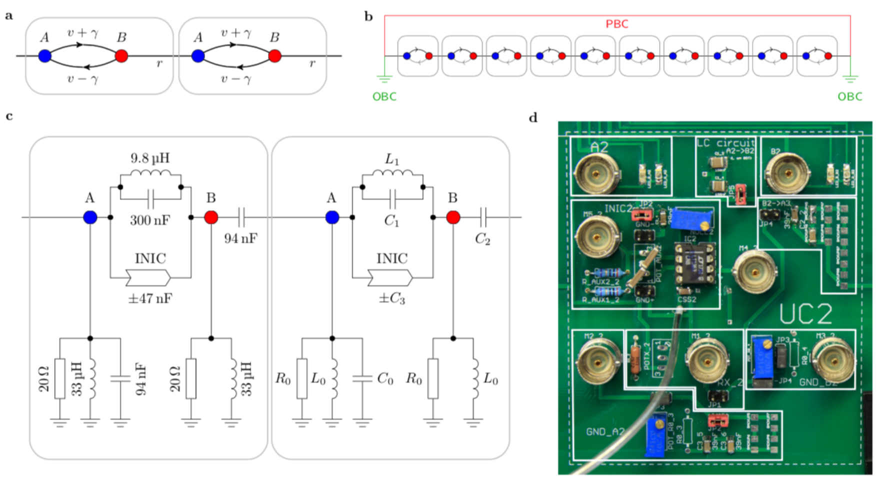
$H_2 =$ 

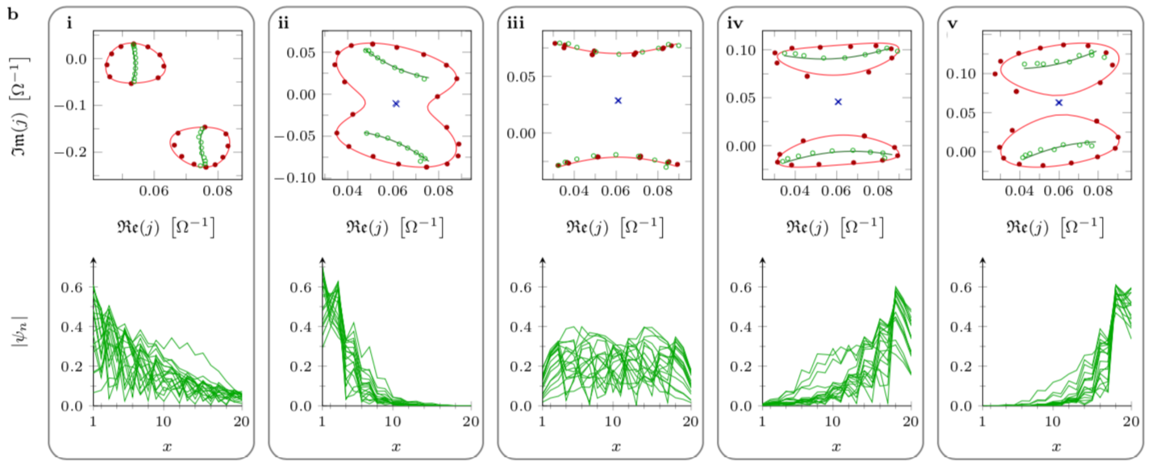
  - Stability in math and physics literature quite different concepts

# Experiments!

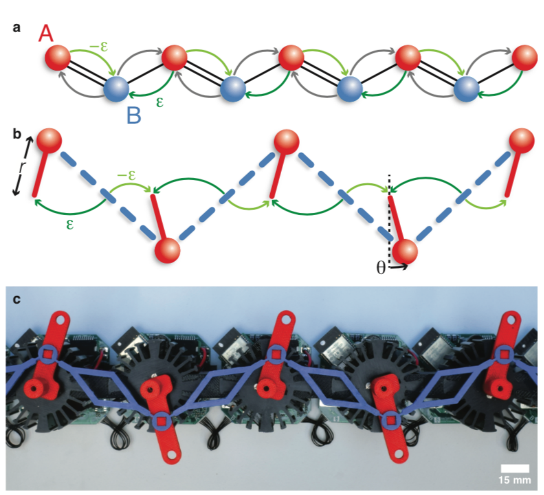
- A wave of experimental studies of the bulk-boundary correspondence followed…

Topolectric circuits: Helbig et al., arXiv:1907.11562.

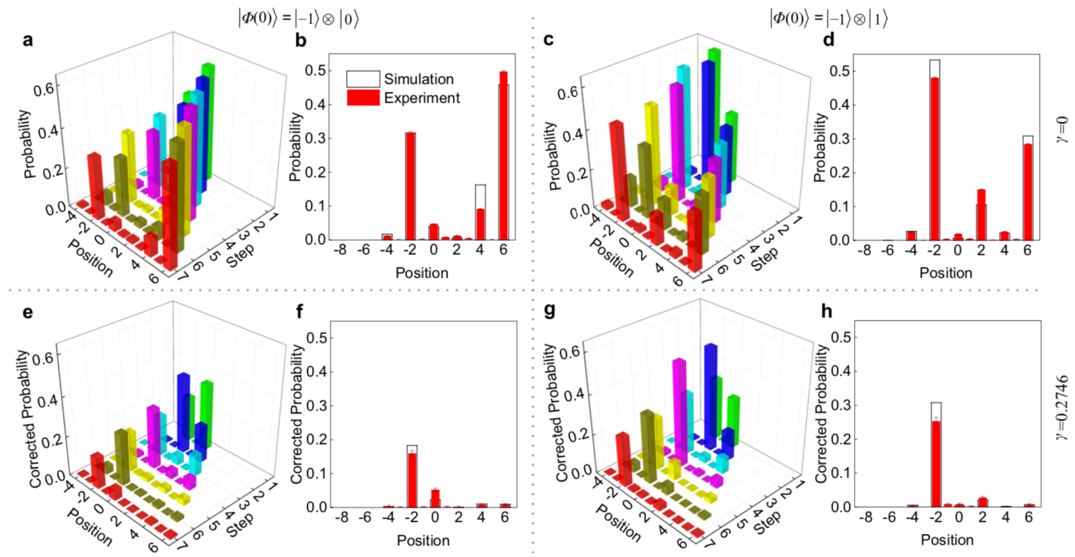




Mechanical/robotic system: Ghatak et al., arXiv:1907.11619.

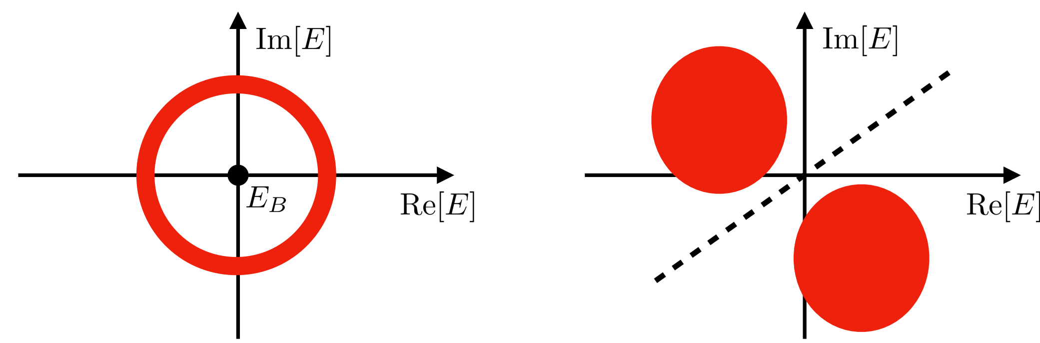


Quantum walks: Xiao et al., arXiv:1907.12566



# There is much more…

- New notions of gapped topological phases




Kawabata et. al.
PRX 9, 041015 (2019)

- Non-Hermitian sensing devices?

  - harnessing non-analytic dispersion relations?

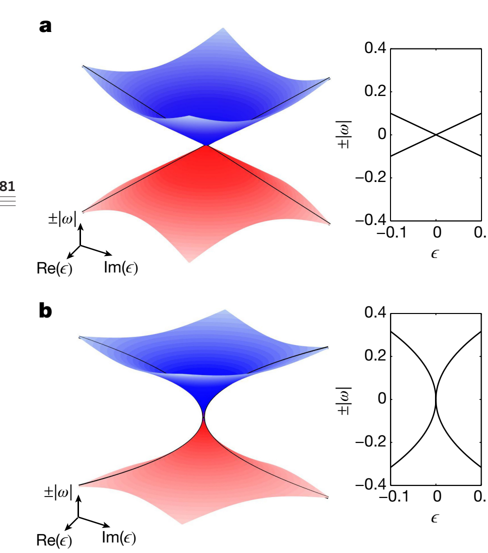
LETTER

doi:10.1038/nature23281

**Exceptional points enhance sensing in an optical microcavity**

Weijian Chen[1], Şahin Kaya Özdemir[1], Guangming Zhao[1], Jan Wiersig[2] & Lan Yang[1]

*Nature volume 548, pages*
192–196 (10 August 2017)




- Topological lasers?

  - gain needed, previously thought to be in conflict with topological phases

RESEARCH ARTICLE

TOPOLOGICAL PHOTONICS

**Topological insulator laser: Theory**

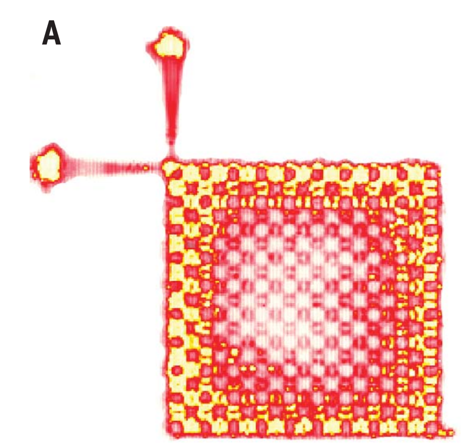
Gal Harari,[1*] Miguel A. Bandres,[1*] Yaakov Lumer,[2] Mikael C. Rechtsman,[3]
Y. D. Chong,[4] Mercedeh Khajavikhan,[5] Demetrios N. Christodoulides,[5] Mordechai Segev[1†]

RESEARCH ARTICLE

TOPOLOGICAL PHOTONICS

**Topological insulator laser: Experiments**

Miguel A. Bandres,[1*] Steffen Wittek,[2*] Gal Harari,[1*] Midya Parto,[2] Jinhan Ren,[2]
Mordechai Segev,[1†] Demetrios N. Christodoulides,[2†] Mercedeh Khajavikhan[2†]




Topological lasing mode

*Science* 16 Mar 2018:
Vol. 359, Issue 6381, eaar4005

*Science* 16 Mar 2018:
Vol. 359, Issue 6381, eaar4003

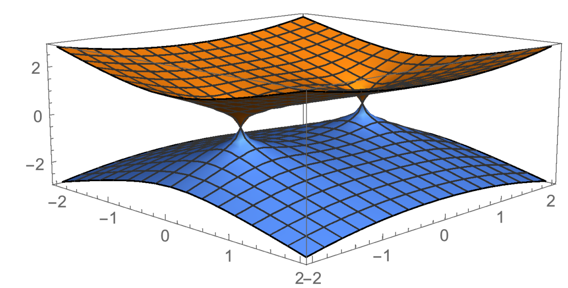
# Summary for today

- **Non-Hermiticity implies qualitatively new topological features**

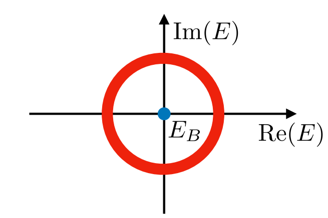
- Exceptional degeneracies, square roots

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

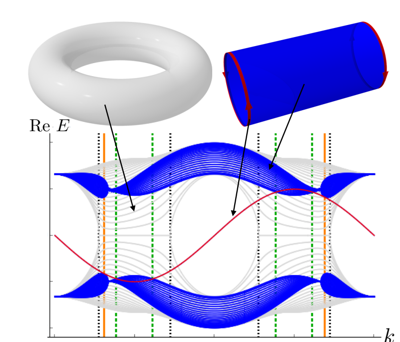
- Gapless nodal phases theoretically far more abundant and conceptually rich than in the Hermitian realm



- Topology of complex energies & strong response to boundary conditions




- Open and closed boundary conditions give very different physics — but cases can be understood and are experimentally relevant!



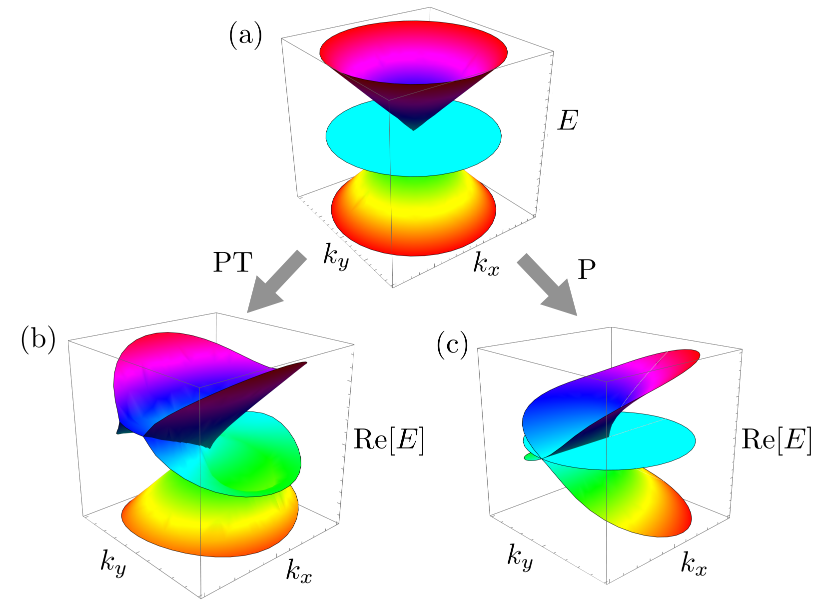

- **Booming field with lot's of fun to discover…**

*E.J. Bergholtz, J.C. Budich, and F.K. Kunst,* ***Rev. Mod. Phys. 93, 15005 (2021)***

# Next time (Midsummer Eve)

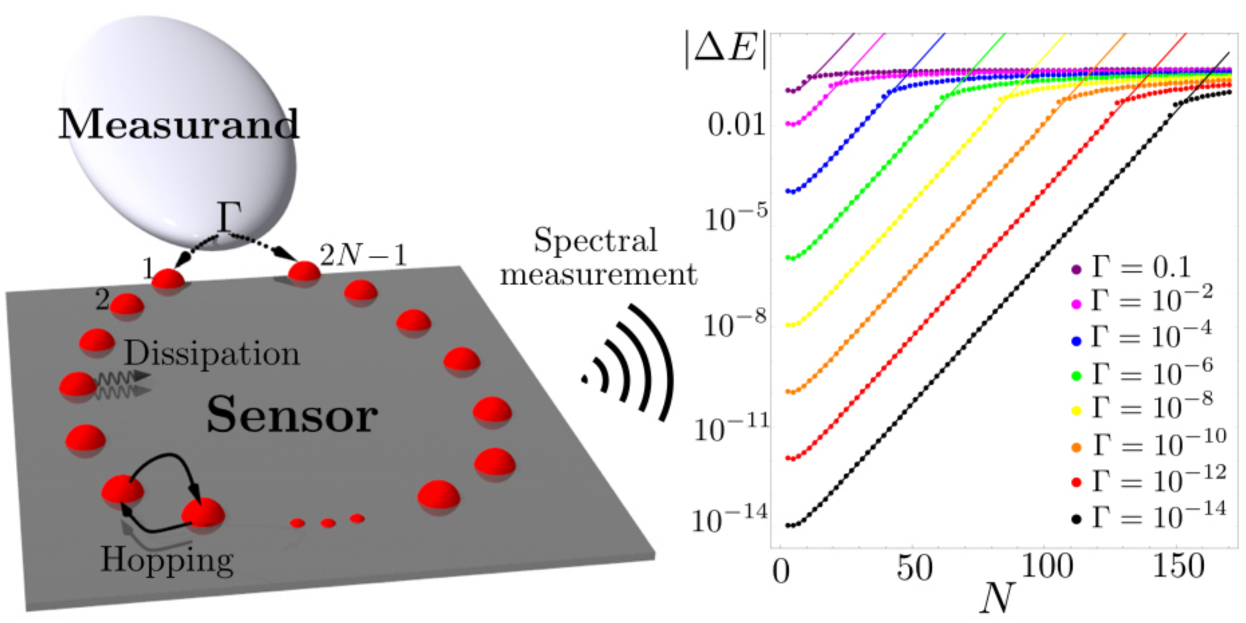
- More bands and symmetries

$$N = 3, 4, \ldots$$




- Non-Hermitian Topological Sensors

$$\text{large } N$$




- Classical -> Quantum
- Other recent developments and topics on demand…