Lectures on Fractional Statistics

based on *Fractional Statistics* by MG & Frank Wilczek

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1. Introduction

There used to be a universal belief that identical particles are either bosons or fermions.

Reason: Wave fcts. are1D reps. of the permutation group S_n
only 2 such reps., symmetric and antisymmmetric

$$\psi(\boldsymbol{r}_2, \boldsymbol{r}_1) = \pm\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$$

Also: Spin-statistics theorem: integer spins bosons, half-integer spins fermions

$$\left[S_i, S_j\right] = \mathrm{i}\hbar\epsilon_{ijk}S_k \quad \longrightarrow \quad \boldsymbol{S}^2 = \hbar^2 s(s+1) \quad \text{with} \quad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

The spin-statistics theorem, however, only applies to 3D: In 2D, we only have one generator L_z of angular momentum, and spin is not quantized.

The argument regarding the symmetry of , however, applies to any dimension.

2. Path integrals and the braid group

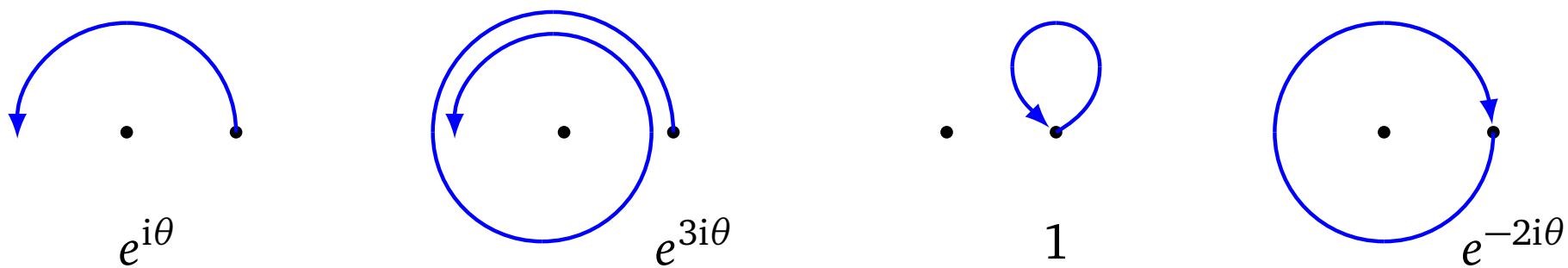
Leinaas & Myrheim 1977:

Fundamental quantity to consider is not the wave function, but relative amplitudes of paths belonging to topologically distinct sectors when particles are interchanged.

In 2D, we can define a winding number.

In 3D, the only topologically inequivalent choices are paths which interchange the particles or do not interchange them.

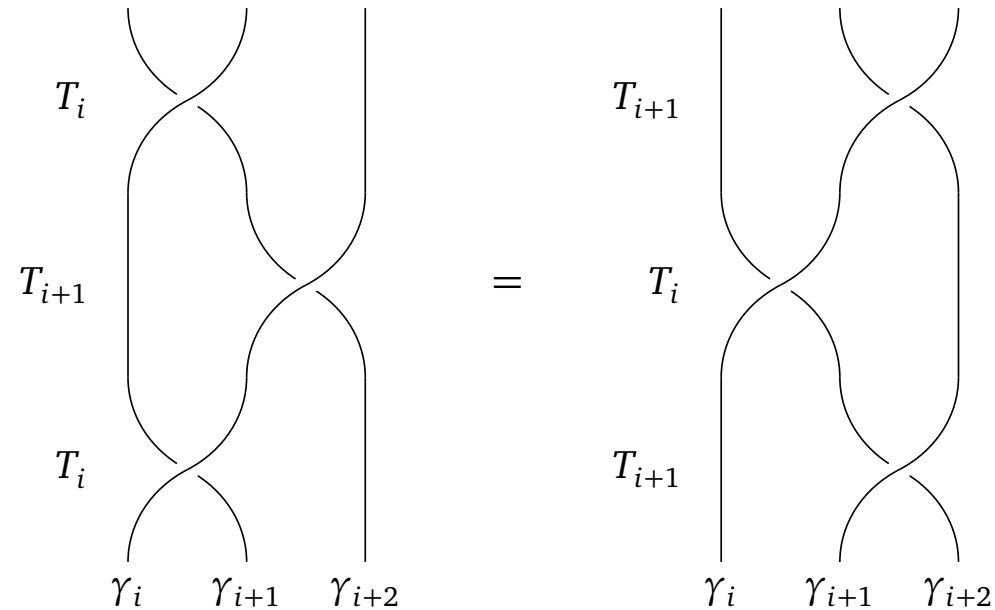
Topologically distinct windings of 2 particles (with and without interchanges):



Relevant group is the braid group B_n
with algebra

$$T_i T_j = T_j T_i \qquad \text{for } |i-j| > 1,$$

$$T_i T_j T_i = T_j T_i T_j \qquad \text{for } |i-j| = 1,$$



and 1D representation on $\mathbb{R}^2$

$$\tau(T_i) = e^{\mathrm{i}\theta}, \quad \theta \in \,]-\pi, \pi].$$

fermions: $\theta = \pi$, bosons $\theta = 0$, anyons: fractional phase factor $e^{\mathrm{i}\theta}$

relative angular momentum: 3D, quantized as $\hbar l$, l odd for fermions, even for bosons

2D: only one generator of canonical relative angular momentum: $L_z = -\mathrm{i}\hbar\partial_\varphi$

→ kinematic relative angular momentum quantized as: $l_z = \hbar\left(\text{even integer} - \dfrac{\theta}{\pi}\right)$

3. Charge flux-tube composites

epitomize anyons, due to Wilczek 1982:

consider particle w/ charge *q* and flux tube $\Phi = \frac{\theta}{\pi}\Phi_0$, where $\Phi_0 = \frac{2\pi\hbar c}{q}$, in the xy-plane:

$$\boldsymbol{a}(\boldsymbol{r}) = \frac{\Phi}{2\pi}\frac{\boldsymbol{e}_z \times \boldsymbol{r}}{r^2} = \frac{\Phi}{2\pi r}\boldsymbol{e}_\varphi \quad \longrightarrow \quad \boldsymbol{b}(\boldsymbol{r}) = \nabla \times \boldsymbol{a} = \Phi\delta(\boldsymbol{r})\boldsymbol{e}_z$$

Hamiltonian: $$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{a}(\boldsymbol{r})\right)^2 = -\frac{\hbar^2}{2m}\frac{1}{r}\partial_r r \partial_r + \frac{1}{2mr^2}L_z^2(\theta)$$

with kinetic relative angular momentum:

$$L_z(\theta) = e^{+\mathrm{i}\theta\varphi/\pi}\left(-\mathrm{i}\hbar\partial_\varphi\right)e^{-\mathrm{i}\theta\varphi/\pi} = \hbar\left(-\mathrm{i}\partial_\varphi - \frac{\theta}{\pi}\right)$$

move charge *q* counterclockwise around the flux tube → get Aharonov-Bohm phase

$$\frac{q}{\hbar c}\oint \boldsymbol{a}(\boldsymbol{r})d\boldsymbol{r} = \frac{q\Phi}{\hbar c} = 2\theta$$

corresponding to statistical parameter θ for a counterclockwise interchange.

transmute statistics of *N* particle system: attach charge *q* and flux $\Phi = \frac{\theta}{\pi}\Phi_0$ to each particle:

$$\boldsymbol{a}(\boldsymbol{r}_i) = \frac{\theta}{\pi}\frac{\hbar c}{q}\sum_{\substack{j \\ (j\neq i)}} \frac{\boldsymbol{e}_z \times (\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2}, \qquad H = \frac{1}{2m}\sum_i \left(\boldsymbol{p}_i - \frac{q}{c}\boldsymbol{a}(\boldsymbol{r}_i) + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_i)\right)^2$$

strictly speaking, we get the winding phases twice
(from each charge moving in the vector field from each flux tube)

More on this subtle issue below.

on closed surfaces (spheres, tori):

Dirac monopole condition applies to the sum of statistical and electromagnetic flux!

(This underlies how fundamental the connection between anyons and charge-flux tube compensates is.)

4. Chern-Simons construction

in 2+1 D:

couple U(1) conserved particle current $J^\mu = (\rho, \boldsymbol{J})$ to a fictitious gauge field $a^\mu(a^0, \boldsymbol{a})$

$$\Delta\mathcal{L} = -\frac{q}{c} J^\mu a_\mu + \frac{\mu}{2c}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho.$$

Euler-Lagrange $\partial_\nu \frac{\delta\mathcal{L}}{\delta(\partial_\nu a_\mu)} - \frac{\delta\mathcal{L}}{\delta a_\mu} = 0 \quad \longrightarrow \quad qJ^\mu = \mu\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \qquad (*)$

$\mu = 0$ component couples the magnetic field to the charge:

$$b_z(\boldsymbol{r}) = \partial_x a^y - \partial_y a^x = -\frac{q}{\mu}\rho(\boldsymbol{r})$$

and hence attaches flux $\Phi = -\frac{q}{\mu}$ to each particle.

phase generated by $\Delta\mathcal{L}$ in the path integral as we move one particle around another:

$$\exp\left(\frac{\mathrm{i}}{\hbar}\Delta S\right) = \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\,\Delta\mathcal{L}\right)$$

use symmetrical gauge: $a^0 = 0,\ \boldsymbol{a} = \frac{\Phi}{2\pi r}\boldsymbol{e}_\varphi$

with the particle current $\boldsymbol{J}(\boldsymbol{r}) = \rho\boldsymbol{v} = \rho r\partial_t\varphi\,\boldsymbol{e}_\varphi$ the first term in $\Delta\mathcal{L}$ just gives the

Aharonov -Bohm phase for the motion of one particle in the field of the other:

$$\frac{q}{\hbar c}\frac{\Phi}{2\pi}\int \mathrm{d}\boldsymbol{r}\,\rho\int \mathrm{d}t\,\partial_t\varphi = \frac{q}{\hbar c}\frac{\Phi}{2\pi}\varphi$$

Interchange with winding angle $\varphi = \pi,$ counting both phases from both particles:

$$\exp\left(\frac{\mathrm{i}q\Phi}{\hbar c}\right) = \exp(2\mathrm{i}\theta)$$

Substituting $(*)$ into $\Delta\mathcal{L}$, we see that the second term gives just -1/2 times this contribution!

(= field correction; due to Goldhaber, MacKenzie, Wilczek 1989)

Physical interpretation of the field correction:

recall that the flux in the CS construction is proportional to the charge, $\Phi = -\frac{q}{\mu}$.

consider adiabatic attachment of flux, and hence also charge.

(ficticious) electric field generated by the flux: $$\oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{s} = E_\varphi \cdot 2\pi r = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

change in the kinematical angular momentum:

$$\Delta l_z = \int F_\varphi r\,\mathrm{d}t = -\frac{1}{2\pi c}\int q(\Phi)\frac{\partial \Phi}{\partial t}\,\mathrm{d}t = \frac{\mu}{2\pi c}\int \Phi\,\mathrm{d}\Phi = -\frac{1}{2\pi c}\frac{q\Phi}{2} = -\frac{\hbar}{2}\frac{\theta}{\pi}$$

is 1/2 of what we had when we considered flux-charge composites naively in Section 3!

Reason: $q \propto \Phi$, torque increases linearly even for $\partial_t \Phi = \mathrm{const.}$

We will see below that this is precisely the situation we have in the FQHE!

5. Abelian anyons in fractionally quantized Hall states

5.1. Landau levels

particles of charge $-e$ in the xy-plane, perpendicular magn. field: $\boldsymbol{B} = -B\boldsymbol{e}_z$

complex coordinates $z = x + \mathrm{i}y\,, \quad \partial = \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right)$

$\omega_c = \frac{eB}{Mc}$

kinetic Hamiltonian $H = \dfrac{1}{2M}\left(\boldsymbol{p} + \dfrac{e}{c}\boldsymbol{A}(\boldsymbol{r})\right)^2 = \hbar\omega_c\left(a^\dagger a + \dfrac{1}{2}\right)$

$l = \sqrt{\frac{\hbar c}{eB}}$

w/ ladder operators $a = \dfrac{l}{\sqrt{2}}\left(2\bar{\partial} + \dfrac{z}{2l^2}\right), \quad a^\dagger = \dfrac{l}{\sqrt{2}}\left(-2\partial + \dfrac{1}{2l^2}\bar{z}\right)$

$\left[a, a^\dagger\right] = 1$

basis states spanning the lowest LL w/ energy $\frac{1}{2}\hbar\omega_c$: $\psi_m(z) = z^m\, e^{-\frac{1}{4l^2}|z|^2}$

describe rings w/ radius $r_m = \sqrt{2m}\,l$ $\longrightarrow$ $\dfrac{\text{number of states}}{\text{area}} = \dfrac{m}{\pi r_m^2} = \dfrac{1}{2\pi l^2}$

magnetic flux required for each state: $2\pi l^2 B = \frac{2\pi\hbar c}{e} = \Phi_0$ (i.e., one Dirac quantum)

from now on: $l = 1$

circular droplet of *N* electrons in the LLL:

$$\psi(z_1, \ldots, z_N) = \mathcal{A}\left\{z_1^0 z_2^1 \ldots z_N^{N-1}\right\} \cdot \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2} = \prod_{i<j}^{N}(z_i - z_j) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}$$

Most general *N* particle state in the LLL:

$$\psi(z_1, \ldots, z_N) = f(z_1, \ldots, z_N) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}$$

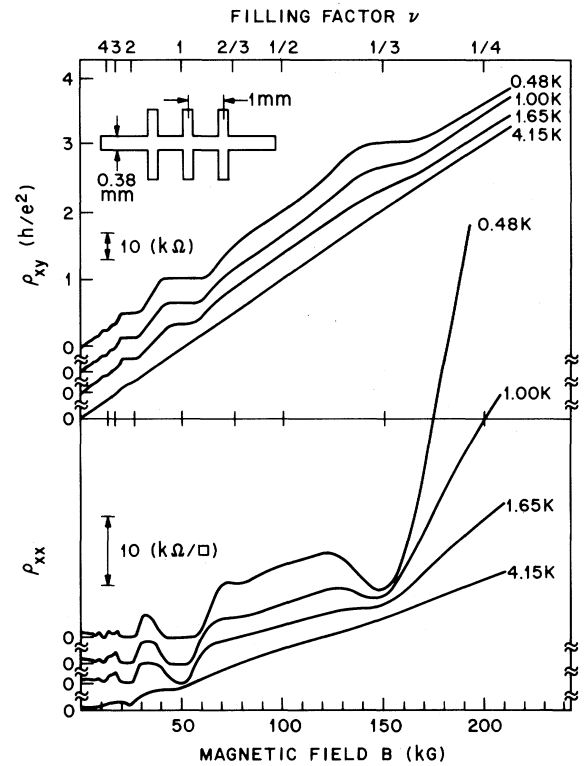
PBCs: $\psi(z_1, z_2, \ldots, z_N)$ when viewed as fct. of z_1 while $z_2, \ldots, z_N$ are fixed, has as many zeros as there are states in the LLL (or flux Dirac quanta through the closed surface).

Most general *N* fermion wave function in the LLL:

$$\psi(z_1, \ldots, z_N) = P(z_1, \ldots, z_N) \prod_{i<j}^{N}(z_i - z_j) \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}$$

where $P(z_1, \ldots, z_N)$ is a completely symmetric polynomial.

Tsui, Störmer, Gossard 1982



5.2. The Laughlin wave function

explains plateau in the Hall resistivity $\rho_{\mathrm{xy}} = 3h/e^2$

of a 2D electron gas, i.e., at Landau level filling

$\nu = 1/3$ where $\dfrac{1}{\nu} = \dfrac{\partial N_\Phi}{\partial N}$

w/ N_Φ # of Dirac flux quanta, N # of particles

at $\nu = 1/3$, we have 3 times as many zeros as particles.

Laughlin 1983: to suppress configurations in which the particles come close to each other, put all the zeros where the other particles are:

$$\psi_m(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2} \qquad \text{for} \quad \nu = 1/m$$

→ has the correlations of a liquid

uniquely defining property: vanishes as r^m as two particles approach each other.

ψ_m is ground state of model H which excludes relative angular momenta $l_z = 1, \ldots, m-2$

for realistic (e.g. Coulomb interactions):

$P(z_1, \ldots, z_N)$ loosely attaches to zeros to the electron coordinates

("approximate" superfermions)

5.3. Fractionally charged quasiparticle excitations

Gedankenexperiment: adiabatically insert one flux quantum $\Phi_0 = \frac{hc}{e}$ in $\boldsymbol{e}_z$ direction, then remove this flux tube via a singular gauge transformation

→ final state will be an eigenstate of the initial Hamiltonian

flux insertion induces electric field: $\oint \boldsymbol{E}\, \mathrm{d}\boldsymbol{s} = E_\varphi \cdot 2\pi r = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$

LLL will not lead to an increase in kin. energy, but to a transverse current:

$J_\mathrm{r} = \sigma_\mathrm{xy} E_\varphi$ where $\sigma_\mathrm{xy} = \frac{1}{m}\frac{e^2}{h}$ sign convention: $\boldsymbol{B} = -B\boldsymbol{e}_\mathrm{z}$, $\rho_\mathrm{xy} < 0$, $\sigma_\mathrm{xy} > 0$.

Charge transported away from the flux tube: $\Delta Q = 2\pi r \int J_{\mathrm{r}}\,\mathrm{d}t = -\frac{1}{m}\frac{e^2}{hc}\int \mathrm{d}\Phi = -\frac{e}{m}$

since $\nu = \frac{1}{m}$, charge $-\frac{e}{m}$ occupies 1 state in the LLL.

basis of ang. momentum eigenstates around ξ → basis states evolve according to

$$(z-\xi)^m\, e^{-\frac{1}{4}|z|^2} \underset{\text{flux insertion}}{\longrightarrow} |z-\xi|\cdot(z-\xi)^m\, e^{-\frac{1}{4}|z|^2} \underset{\text{singular gauge transformation}}{\longrightarrow} (z-\xi)^{m+1}\, e^{-\frac{1}{4}|z|^2}$$

Laughlin wave fct. evolves into:

$$\psi_\xi^{\mathrm{QH}}(z_1,\ldots,z_N) = \prod_{i=1}^{N}(z_i-\xi)\prod_{i<j}^{N}(z_i-z_j)^m\prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}$$

→ quasihole of charge $+\frac{e}{m}$ at ξ

Quasielectron: insert the flux in the other direction, reduce angl. mom, has charge $-\frac{e}{m}$

5.4. Fractional statistics of the quasihole excitation

view the quasiholes as “particles”, w/ Hilbert space spanned by the parent wave fcts.

charge $e^* = +\frac{e}{m}$, effective flux quanta $\Phi_0^* = \frac{2\pi\hbar c}{e^*} = m\Phi_0$, length $l^* = \sqrt{\frac{\hbar c}{e^* B}} = l\sqrt{m}$

→ expect single quasihole wave fct. to describe charge e^* particle in the LLL:

$$\phi(\bar{\xi}) = f(\bar{\xi})\, e^{-\frac{1}{4m}|\xi|^2} \qquad \text{(complex conjugation because charge now > 0)} \qquad (*)$$

electron wave fct. w/ 2 quasiholes:

$$\psi(z_1,\ldots,z_N) = \int \mathrm{D}[\xi_1,\xi_2]\, \underline{\phi_{p,m}(\bar{\xi}_1,\bar{\xi}_2)}\, \underline{\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1,\ldots,z_N)}$$

with $\underline{\phi_{p,m}(\bar{\xi}_1,\bar{\xi}_2)} = (\bar{\xi}_1 - \bar{\xi}_2)^{p+\frac{1}{m}} \prod_{k=1,2} e^{-\frac{1}{4m}|\xi_k|^2}$ *p* even integer, and

$$\underline{\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1,\ldots,z_N)} = (\xi_1 - \xi_2)^{\frac{1}{m}} \prod_{k=1,2} \left(e^{-\frac{1}{4m}|\xi_k|^2} \prod_{i=1}^{N} (z_i - \xi_k) \right) \prod_{i<j}^{N} (z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}.$$

where $\int \mathrm{D}[\xi_1, \xi_2] \equiv \int \ldots \int \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2$

Both $\underline{\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2)}$ and $\underline{\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}(z_1, \ldots, z_N)}$ are multiple valued functions of

$\bar{\xi}_1 - \bar{\xi}_2$ $\qquad$ $\xi_1 - \xi_2$

Reason: Hilbert space spanned by $\psi^{\mathrm{QHs}}_{\xi_1,\xi_2}$ has to be normalized and be analytic in ξ_1, ξ_2 ;

$\phi_{p,m}$ should be of the form $(*)$, while $\psi(z_1, \ldots, z_N)$ has to be well defined.

$$\underline{\phi_{p,m}(\bar{\xi}_1, \bar{\xi}_2)} = (\bar{\xi}_1 - \bar{\xi}_2)^{p+\frac{1}{m}} \prod_{k=1,2} e^{-\frac{1}{4m}|\xi_k|^2}$$

→ indicates $l_z = -\hbar\left(p + \frac{1}{m}\right)$

→ quasiholes are anyons with $\theta = \frac{\pi}{m}$

Charge-flux tube composite with $e^* = +\frac{e}{m}$, $\Phi_0 = \frac{2\pi\hbar c}{e}$ yields phase $\theta = \frac{\pi}{m}$ only once!

Reason: when we create QHs, we create the flux and the charge simultaneously.

(is exactly the situation we had in the CS-theory above)