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http://quantuminstitute.yale.edu/

Introduction to Circuit QED

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U.S. DEPARTMENT OF ENERGY
Office of Science
AFOSR
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Sweden is a great country:
Even the grocery stores are quantum

Lecture notes on circuit QED (150 pages)
2011 Les Houches Summer School

https://girvin.sites.yale.edu/lectures

Lecture series on quantum error correction and fault tolerance

arXiv:2111.08894: Introduction to Quantum Error Correction and Fault Tolerance

Videos of above lectures: https://girvin.sites.yale.edu/lectures

Lecture 1: Introduction to Circuit QED

- ‘Blackbox’ Quantization (BBQ)
- Dispersive Coupling and Readout
- Strong Dispersive Limit
- Photon ‘Number Splitting ‘
- ~~Photon Number Parity Measurement~~

Lecture 1: Introduction to Circuit QED

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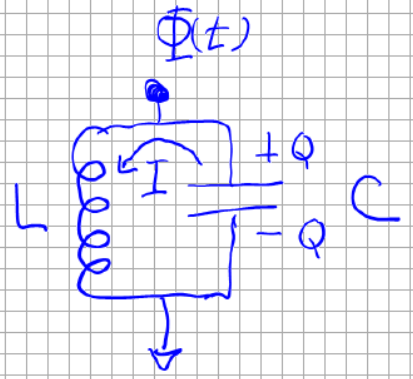
Introduction to Circuit QED

Artificial atoms and microwave photons

How to be a quantum electrical engineer

LC oscillator



Define generalized flux

$$\Phi(t) \equiv \int^t d\tau\, V(\tau)$$

$$\dot{\Phi} = V$$

Faraday induction (up to a minus sign)

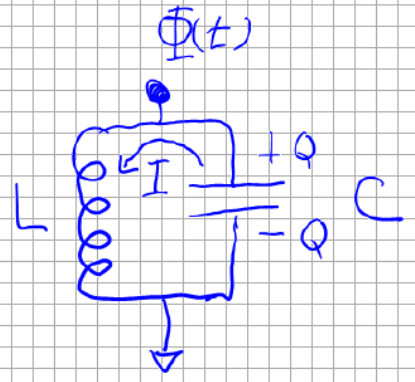
electrostatic energy $\frac{1}{2} C \dot{\Phi}^2$

magnetic energy $\frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2$ $\quad (\Phi = IL)$

Lagrangian $\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

velocity coordinate



momentum $Q \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}} = C\dot{\Phi} = CV$

charge Q is momentum canonically conjugate to flux.

Hamiltonian $H = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$

harmonic oscillator with "mass" $m = C$

"spring constant" $k = 1/L$

resonance frequency $\omega_R = \sqrt{\frac{k}{m}} = \frac{1}{\sqrt{LC}}$

Hamilton eqn's of motion

$\dot{\Phi} = \frac{\partial H}{\partial Q} = \frac{Q}{C} = V$ ✓ Faraday induction

$\dot{Q} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$ ✓ charge conservation

$$\ddot{\Phi} = \frac{\dot{Q}}{C} = -\frac{1}{LC}\Phi$$

$$I = I_0 \sin(\omega_R t + \theta)$$

$$V = I_0 Z_R \cos(\omega_R t + \theta)$$

characteristic impedance

$$I = -\dot{Q} = -C\dot{V} = +\underbrace{\omega_R C Z_R}_{=1} I_0 \sin(\omega_R t + \theta)$$

$$\boxed{Z_R = \frac{1}{\omega_R C} = \sqrt{\frac{L}{C}}}$$

$Z_R \sim 50 - 500\,\Omega$ because impedance of free space

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$$

quantum of impedance $Z_K = \frac{h}{e^2} \approx 25{,}812\,\Omega$

$$\alpha \equiv \frac{e^2}{\hbar c}\frac{1}{[4\pi\epsilon_0]} \approx \frac{1}{137}$$

$$\boxed{Z_0 = 2\alpha Z_K}$$

Quantizing the oscillator

$$[\hat{Q}, \hat{\Phi}] = -i\hbar \qquad \hat{\Phi} = \Phi_{ZPF}(a + a^{+}) \qquad \hat{Q} = -i\, Q_{ZPF}(a - a^{+})$$

$$[a, a^{+}] = 1 \qquad Q_{ZPF}\, \Phi_{ZPF} = \frac{\hbar}{2}$$

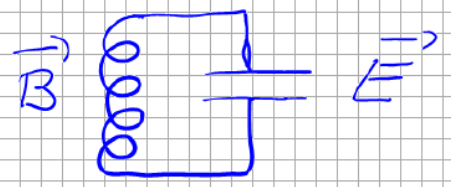
virial thm $\langle 0 | \frac{\hat{Q}^2}{2C} | 0 \rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Rightarrow Q_{ZPF} = \sqrt{\frac{\hbar}{2 Z_R}}$

$$\langle 0 | \frac{\hat{\Phi}^2}{2L} | 0 \rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Longrightarrow \Phi_{ZPF} = \sqrt{\frac{\hbar}{2} Z_R}$$

$Q_{ZPF}\, \Phi_{ZPF} = \frac{\hbar}{2}$ ✓ $\quad \psi(\Phi) = \langle \Phi | 0 \rangle$ is a minimum uncertainty packet

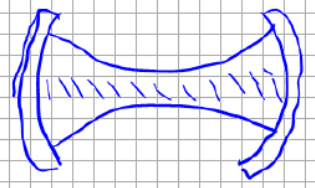
$$\frac{Q_{ZPF}}{e} = \sqrt{\frac{h}{4\pi e^2} \frac{1}{Z_R}} = \sqrt{\frac{Z_K}{4\pi Z_R}} \sim \sqrt{\frac{137}{4\pi}} \sim 3$$

$$H = \hbar\omega_R\left(a^\dagger a + \tfrac{1}{2}\right)$$



$a^\dagger$ creates a <u>photon</u>

can also represent 3D free space modes

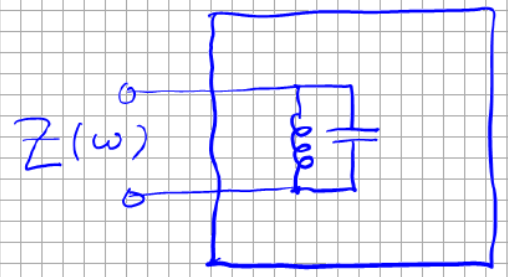


1D optical fibers, co-planar wave guides, etc.

Black Box Quantization (BBQ)

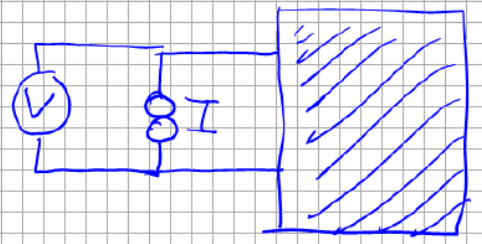
PRL 108, 240502 (2012)

To quantize an oscillator, we only need to know ω_R and Z_R.

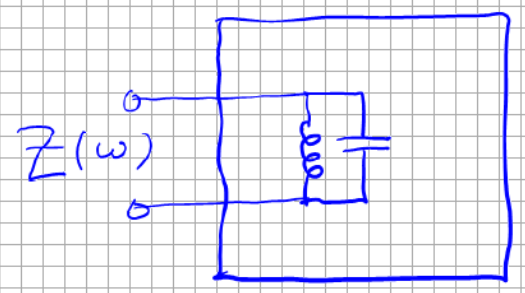


Measure $Z(\omega)$ via $V(\omega) = I(\omega)Z(\omega)$

or compute using a finite-element Maxwell solver e.g. HFSS, Comsol, etc.



Important that ideal current source and voltmeter have <u>infinite</u> impedance.



For a single LC resonator

$$\frac{1}{Z(\omega)} = j\omega C + \frac{1}{j\omega L}$$

[we are engineers! $j = -i$]

$$V(t) = V_0\, e^{j\omega t}$$

$$Z(\omega) = -j\, Z_R \frac{\omega\,\omega_R}{(\omega-\omega_R)(\omega+\omega_R)}$$

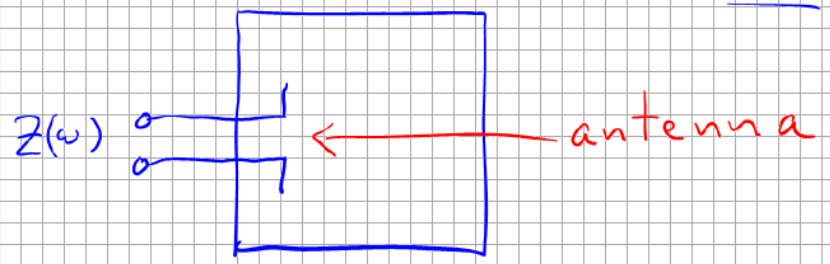
$Z(\omega)$ has poles at $\pm\omega_R$

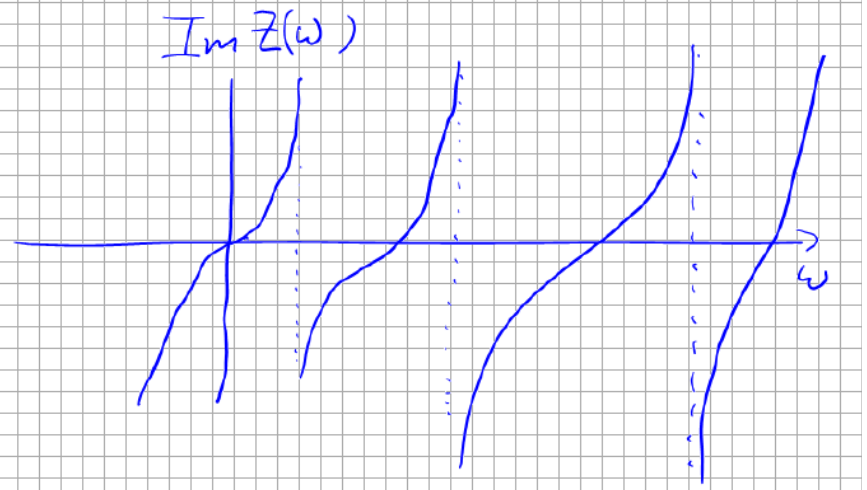
Residue: $\lim_{\omega \to +\omega_R} (\omega-\omega_R)\, Z(\omega) = -\frac{j\, Z_R\, \omega_R}{2}$

Poles of $Z(\omega)$ determine black box resonances under open circuit conditions. $Z(\omega)$ is a linear response function $V(\omega) = Z(\omega)\, I(\omega)$

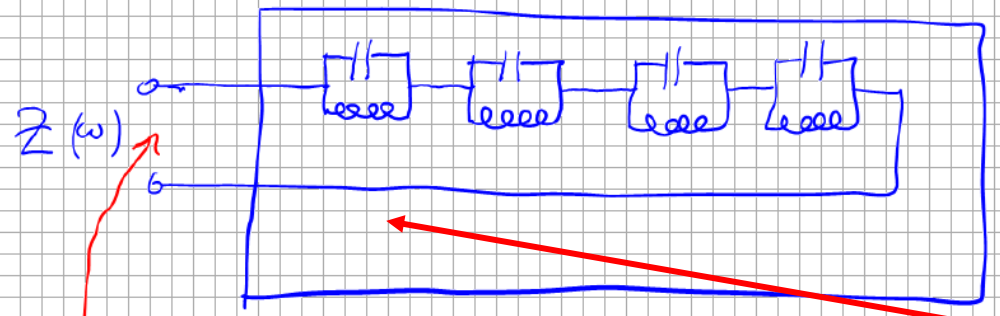
Residues determine characteristic impedances.

General black box has many resonances





Foster equivalent circuit



$$\lim_{\omega \to \omega_\ell} (\omega - \omega_\ell)\left[\sum_k Z^{(k)}(\omega)\right] = -\frac{j Z_k^{(\ell)} \omega_\ell}{2}$$

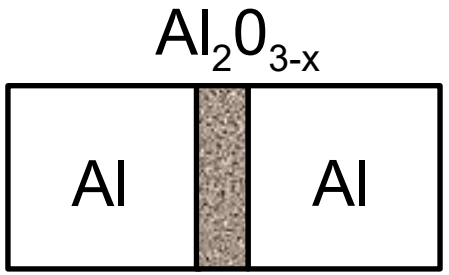
$$Z(\omega = 0) = \infty$$

$$\hat{\Phi} = \sum_k \Phi^{(k)}_{ZPF}\left(a_k^\dagger + a_k\right)$$

Sum over normal modes of cavity

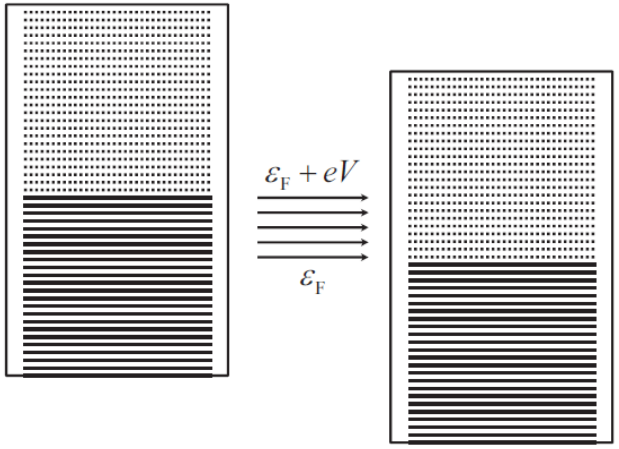
A simple approach to the Josephson Effect

~~Josephson~~ Normal Tunnel Junctions

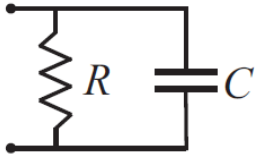


Normal tunnel junction

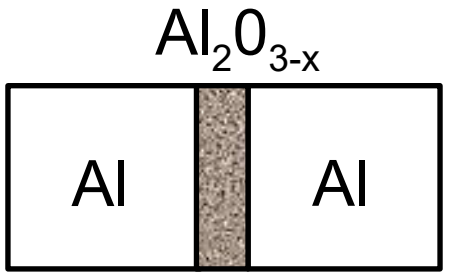
a)



b)

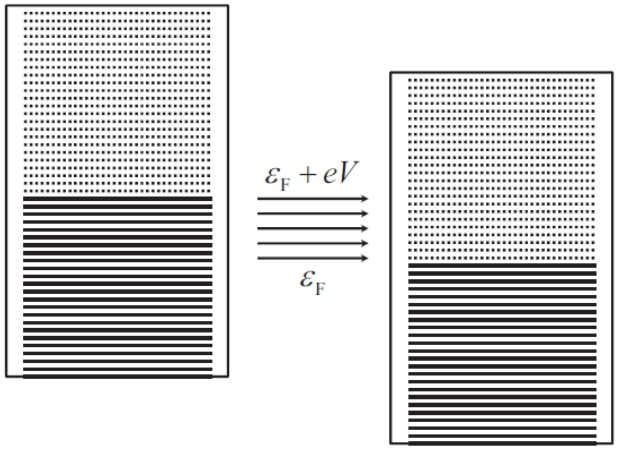


Josephson Tunnel Junctions

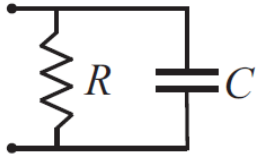


Normal tunnel junction

a)

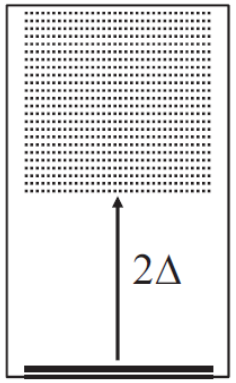


b)

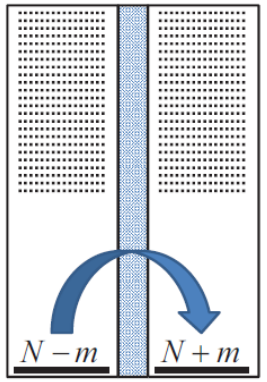


Superconducting tunnel junction

a)

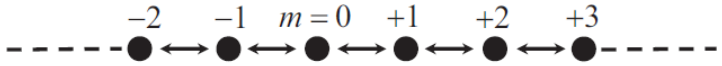


b)



Unique ground state for N pairs on an island

c)



Total number m of Cooper pairs that have tunneled uniquely determines the non-degenerate low-energy quantum state of a pair of islands.

Josephson Tunnel Junctions

$$|\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m \, |m\rangle$$

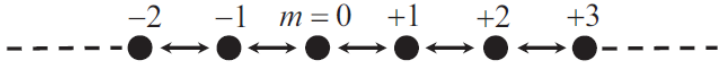
Exactly the same Hilbert space (not necessarily the Hamiltonian) as a 1D tight-binding model (integer *position m*)

position basis $|m\rangle$

plane waves in 1st BZ (only) $\quad |\varphi\rangle = \sum_m e^{i\varphi m} |m\rangle$

linear momentum $-\pi < \varphi < +\pi$

c)



Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

integer $m \Leftrightarrow \varphi$ compact

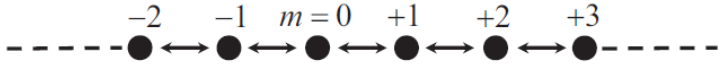
Josephson Tunnel Junctions

$$|\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m |m\rangle$$

Exactly the same Hilbert space (not necessarily the Hamiltonian) as a 1D tight-binding model (integer *position*)

Or:
same Hilbert space as a quantum rotor (integer *angular momentum*)

c)

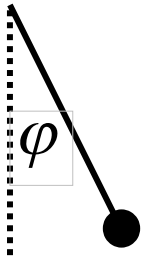


Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

position basis $|m\rangle$

plane waves in 1st BZ (only) $|\varphi\rangle = \sum_m e^{i\varphi m} |m\rangle$

linear momentum $-\pi < \varphi < +\pi$



angular momentum basis $|m\rangle$

position basis $|\varphi\rangle = \sum_m e^{im\varphi} |m\rangle$

angular position $-\pi < \varphi < +\pi$

$$\psi_m(\varphi) = \langle \varphi | m \rangle = e^{im\varphi}$$

integer $m \Leftrightarrow \varphi$ compact

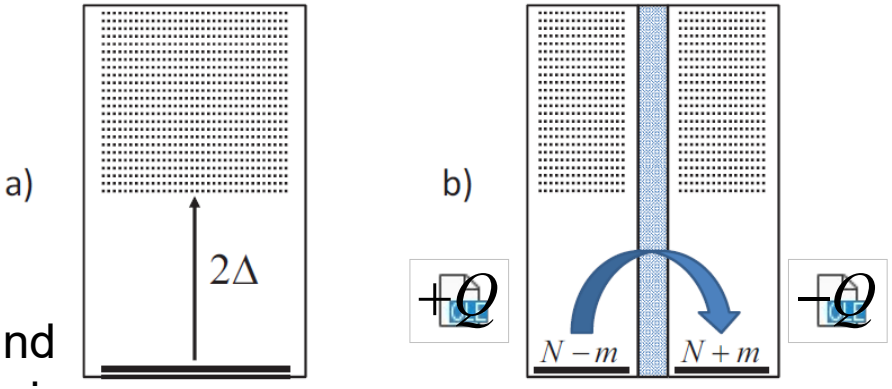
Josephson Tunnel Junction as a capacitor

(N.B. ignoring offset charge, see J. Martinis lecture 1 and my Les Houches notes)

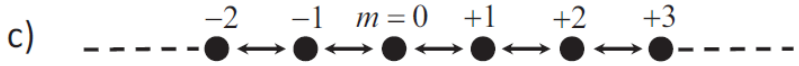
$$Q = (2e)m$$

$$U = \frac{Q^2}{2C} = 4\frac{e^2}{2C}m^2 = 4E_c\, m^2$$

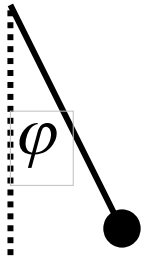
Superconducting tunnel junction



Unique ground state for N pairs on an island



Quantum Rotor



$$T = \frac{L^2}{2I} = -\frac{1}{2I}\frac{d^2}{d\varphi^2}$$

$$T|m\rangle = \frac{m^2}{2I}|m\rangle$$

Charging energy looks like rotor K.E.

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Cooper Pair Tunneling (Josephson Effect)

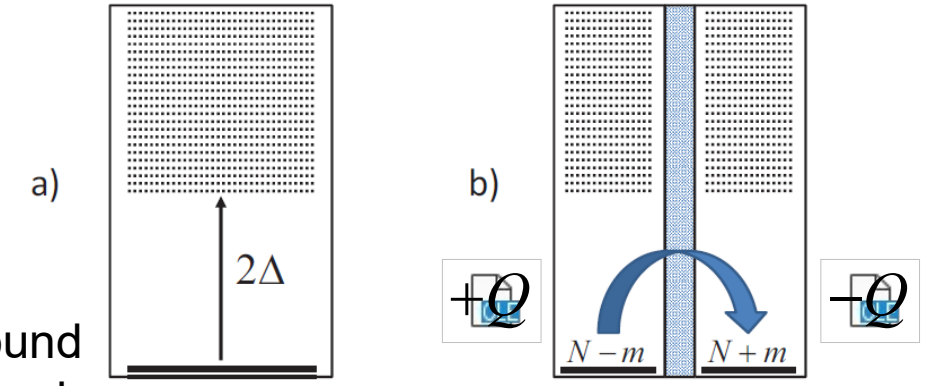
$$H_J = -\frac{E_J}{2}\sum_m \{|m+1\rangle\langle m| + |m+1\rangle\langle m|\}$$

[tight-binding hopping matrix element that changes position by ± 1]

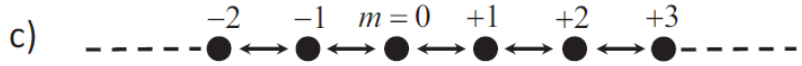
$$H_J = -E_J \cos\varphi$$

[gravitational potential producing a torque that changes the angular momentum by ± 1]

Superconducting tunnel junction

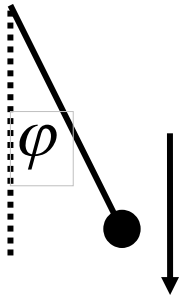


Unique ground state for N pairs on an island



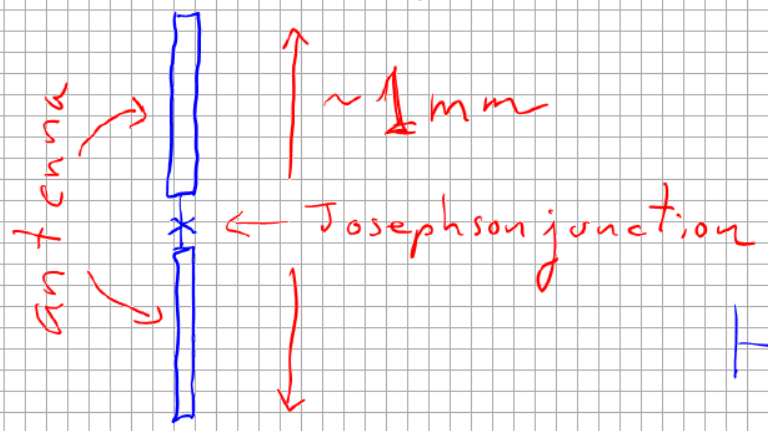
Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Quantum Rotor



'gravity'

"Transmon" qubit



dipole moment

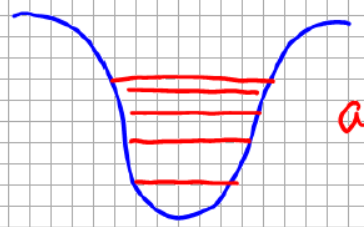
$\sim Q_{ZPF} \times 1\,mm$

$$\boxed{\hat{Q} = (2e)m}$$

$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\left(\underbrace{\frac{2e}{\hbar}\hat{\Phi}}\right)$$

$C_\Sigma \equiv C_J + C_{geometric}$

$\varphi \equiv$ SC order parameter phase



anharmonic oscillator

$\hbar\dot{\varphi} = 2eV = 2e\dot{\Phi}$

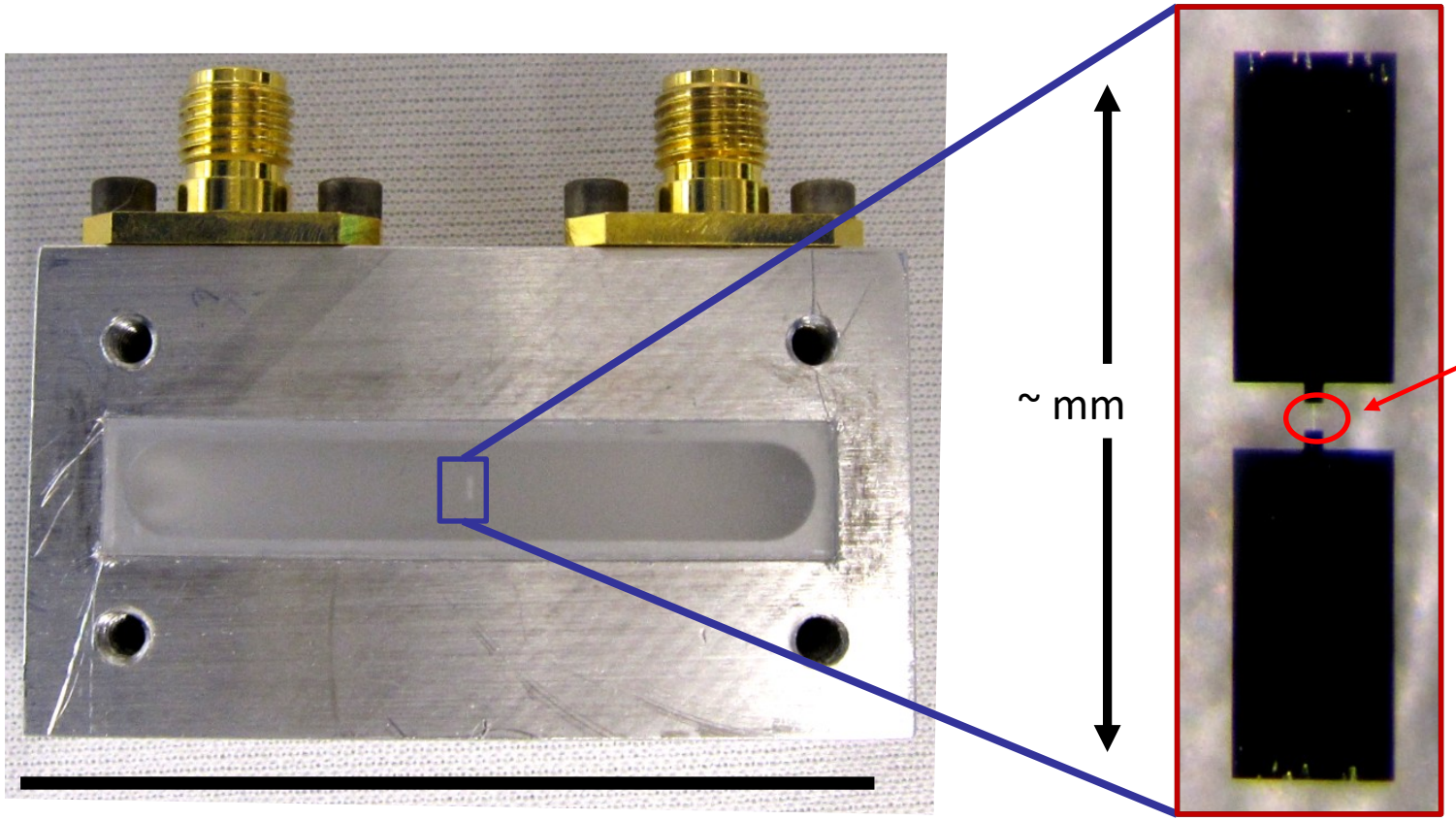
Josephson relation

Subtlety: $\hat{Q} = (2e)\hat{n}$ is discrete <u>not</u> continuous.

For $E_J \gg E_C \equiv \frac{e^2}{2C_\Sigma}$ $\langle \varphi^2 \rangle \ll 2\pi$ so can expand the cosine and safely ignore the subtleties.

Typically $\frac{E_J}{E_C} \sim 10^2$

Transmon Qubit in 3D Cavity



50 mm

$$\frac{Q_{\mathrm{ZPF}}}{2e} \sim \frac{1}{\sqrt{16\pi\alpha}} \sim 1-3$$

bit flip

$$V_{\mathrm{dipole}} = g\sigma^{x}(a + a^{\dagger})$$

$$g = \frac{d \bullet E_{rms}}{h}$$

$$|d| = 2e \times 1\ \mathrm{mm} \approx 10^{7}\,\mathrm{Debye!!}$$

Huge dipole moment: strong coupling

g ~ 100 MHz

$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\left(\frac{2e}{\hbar}\Phi\right) = H_0 + V$$

$$H_0 = \frac{\hat{Q}^2}{2C_\Sigma} + \underbrace{\frac{E_J}{2}\left(\frac{2e}{\hbar}\right)^2}_{\frac{1}{2L_J}} \Phi^2$$

Josephson plasma oscillation

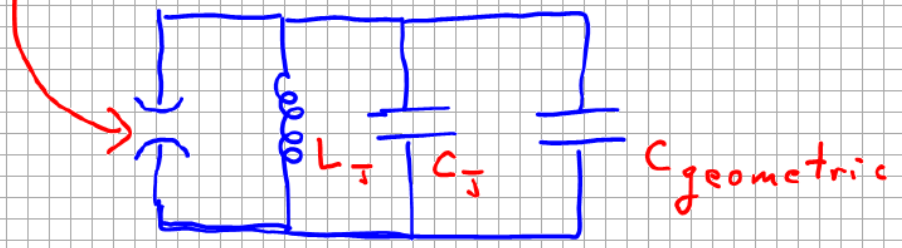
Josephson inductance

$$\hbar\omega_J = \frac{1}{\sqrt{L_J C_\Sigma}} = \sqrt{8E_J E_C} \qquad E_C \equiv \frac{e^2}{2C_\Sigma}$$

$$\frac{\omega_J}{2\pi} \sim 5\text{-}10\ \text{GHz}$$

$$V(\Phi) = E_J\left\{-\frac{1}{4!}\left(\frac{2e}{\hbar}\Phi\right)^4 + \frac{1}{6!}\left(\frac{2e}{\hbar}\Phi\right)^6 - \cdots\right\}$$

Expand cosine, remove quadratic part



$$V \simeq -\frac{E_C}{2}\left(a^\dagger a^\dagger a a + 2 a^\dagger a\right)$$

(rotating wave approximation)

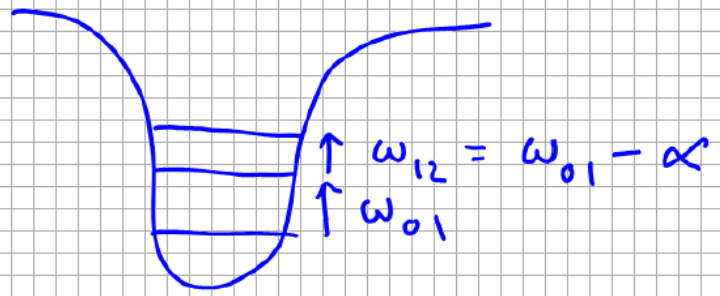
$$H = \hbar\omega_{01}\, a^\dagger a - \frac{\hbar\alpha}{2}\, a^\dagger a^\dagger a a$$

$$\hbar\omega_{01} = \hbar\omega_J - E_c$$

$$\hbar\alpha = E_c$$

$\alpha \sim 200\ \mathrm{MHz}$ $\qquad$ $\omega_{01} \sim 5\ \mathrm{GHz}$

'weakly' anharmonic oscillator

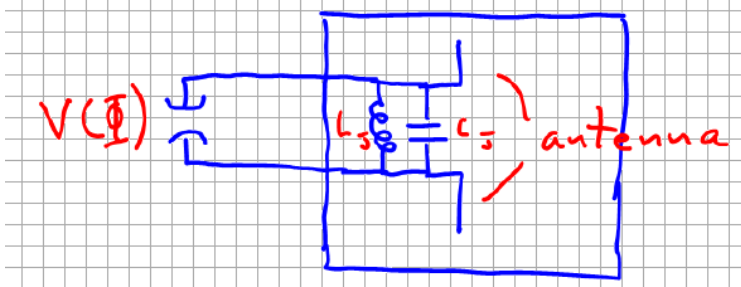


For Rabi driving at ω_{01} with amplitude $\Omega_R \ll \alpha$ anharmonicity looks <u>large</u> and transmon $\simeq$ 2-level atom

Black-Box Quantization (SC qubits coupled to resonators)

Phys. Rev. Lett. **108**, 240502 (2012)

Transmon coupled to resonator (BBQ)



$$\hat{\Phi} = \sum_k \Phi^{(k)}_{ZPF} \left(a_k + a_k^\dagger\right)$$

Sum on cavity and transmon modes.

Diagonalize quadratic H_0; Express V in that basis

$$V = -\sum_l \delta\omega_l A_l^\dagger A_l + \sum_{jk} \chi_{jk} \hat{n}_j \hat{n}_k \qquad \text{(RWA)}$$

Most anharmonic mode identified as qubit (say $j=0$)

χ_{00} big $\qquad \chi_{0k}$ next biggest $\qquad \chi_{lk}$ smallest $\quad l,k \neq 0$

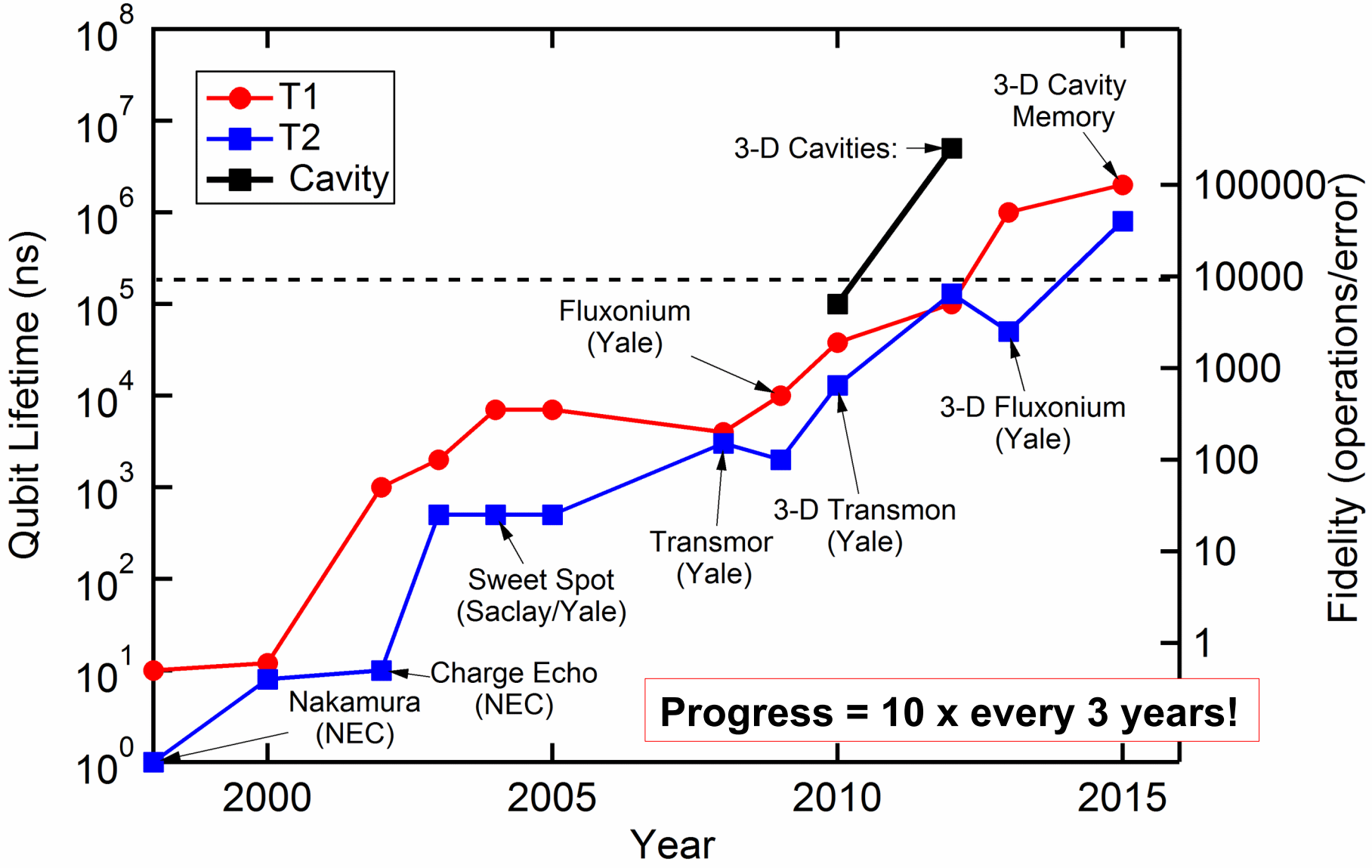
$\hat{n}_0 = 0,1 \longleftrightarrow \sigma^z = \pm 1$ two-level approximation

$$\chi_{0k} \hat{n}_0 \hat{n}_k \longrightarrow \frac{\chi_{0k}}{2} \sigma^z \hat{n}_k$$

cross Kerr $\longrightarrow$ Jaynes-Cummings in dispersive limit

mode frequency depends on qubit state

Remarkable Progress in Coherence



Lecture 1: Introduction to Circuit QED

- ‘Blackbox’ Quantization (BBQ)
- Dispersive Coupling and Readout
- Strong Dispersive Limit
- Photon ‘Number Splitting ‘
- ~~Photon Number Parity Measurement~~

Qubit-Cavity cross-Kerr for two lowest levels of dressed transmon.

$$\chi A^{\dagger} A B^{\dagger} B$$

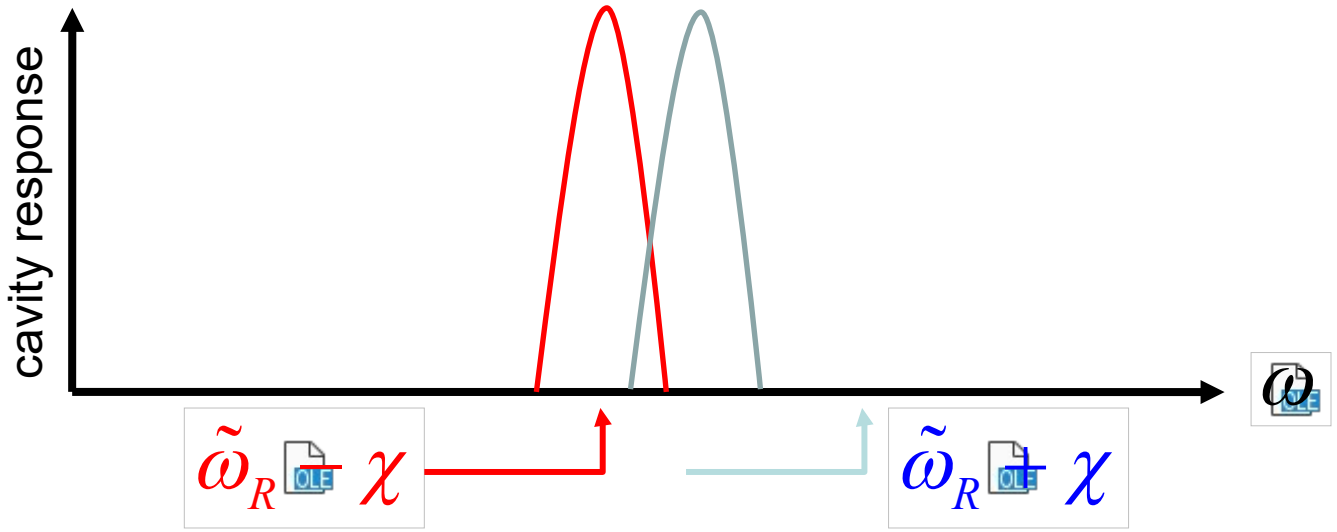
$$A^{\dagger} A = \hat{n}$$

$$B^{\dagger} B = \frac{1+\sigma^{Z}}{2} = 0,1$$

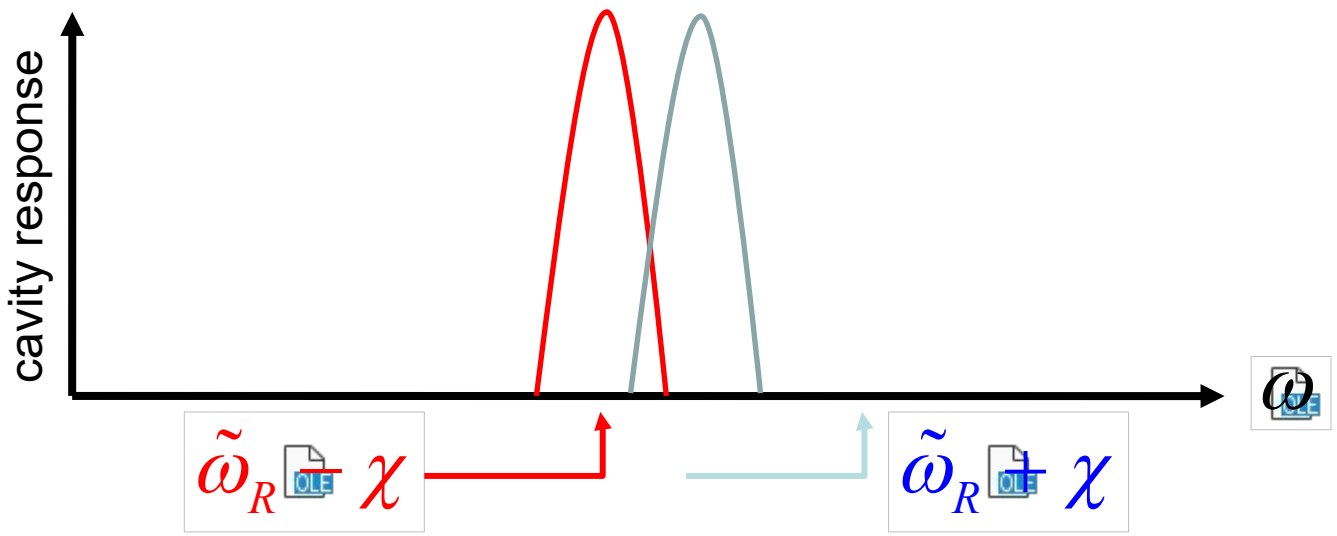
$$H = \left(\tilde{\omega}_R + \chi \sigma^{Z}\right)\hat{n} + \frac{\omega_Q}{2}\sigma^{Z}$$

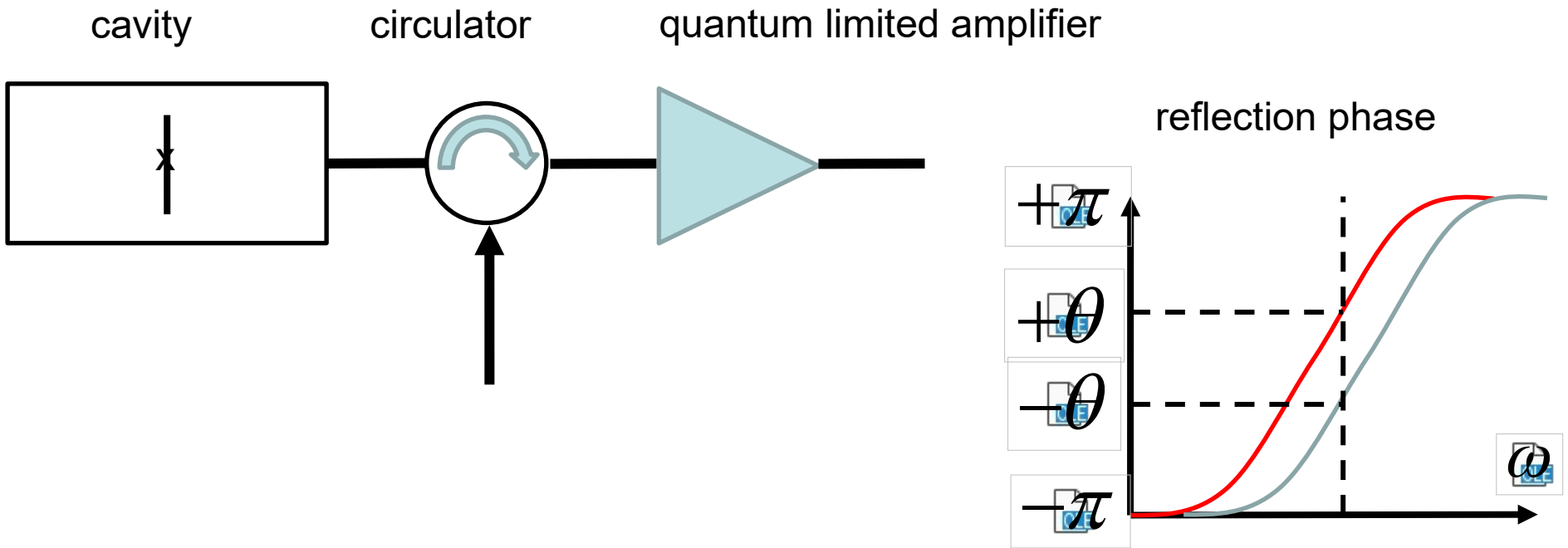
'Dispersive' coupling

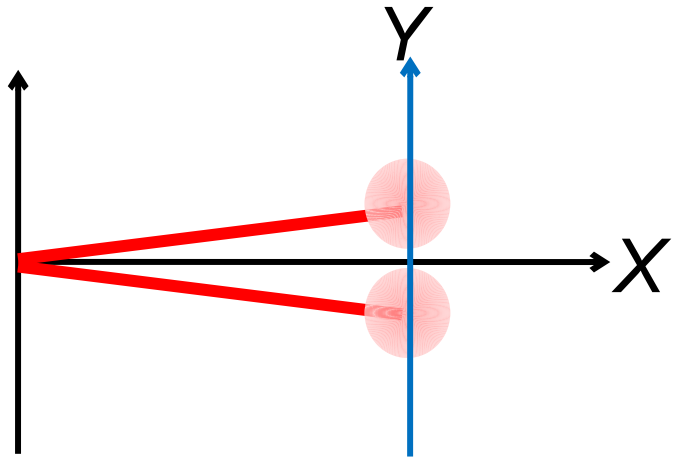
Can read out qubit state by measuring cavity resonance frequency



Can read out qubit state by measuring cavity resonance frequency



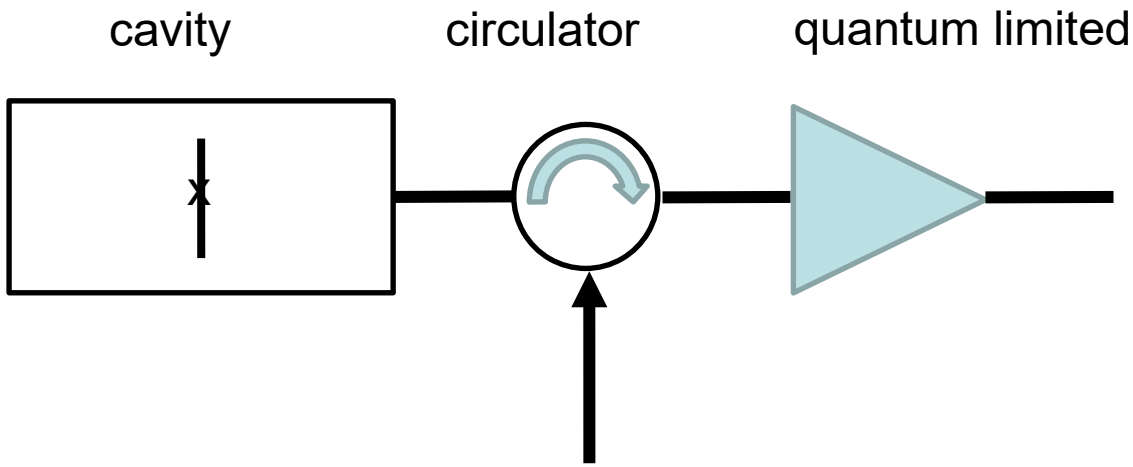


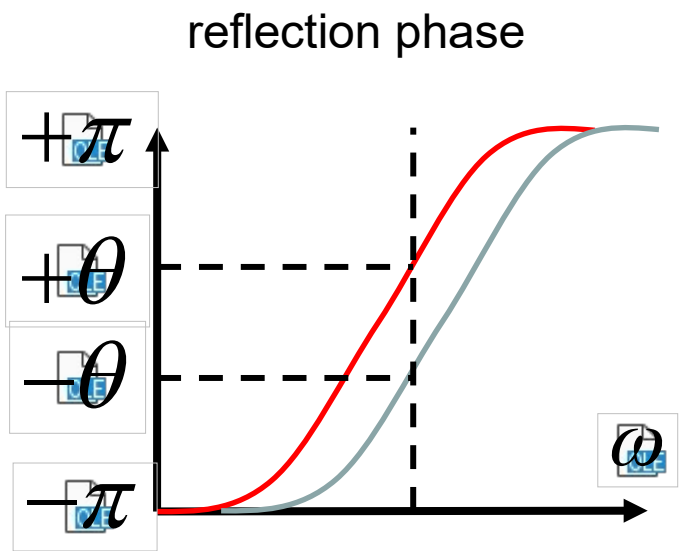


$$|\psi_{\text{in}}\rangle = \{a|\uparrow\rangle + b|\downarrow\rangle\}|\alpha\rangle$$

$$|\psi_{\text{out}}\rangle = a|e^{+i\theta}\alpha\rangle|\uparrow\rangle + b|e^{-i\theta}\alpha\rangle|\downarrow\rangle$$

State of qubit is <u>entangled</u> with the ‘meter’ (microwave phase)
Then ‘meter’ is read with amplifier.

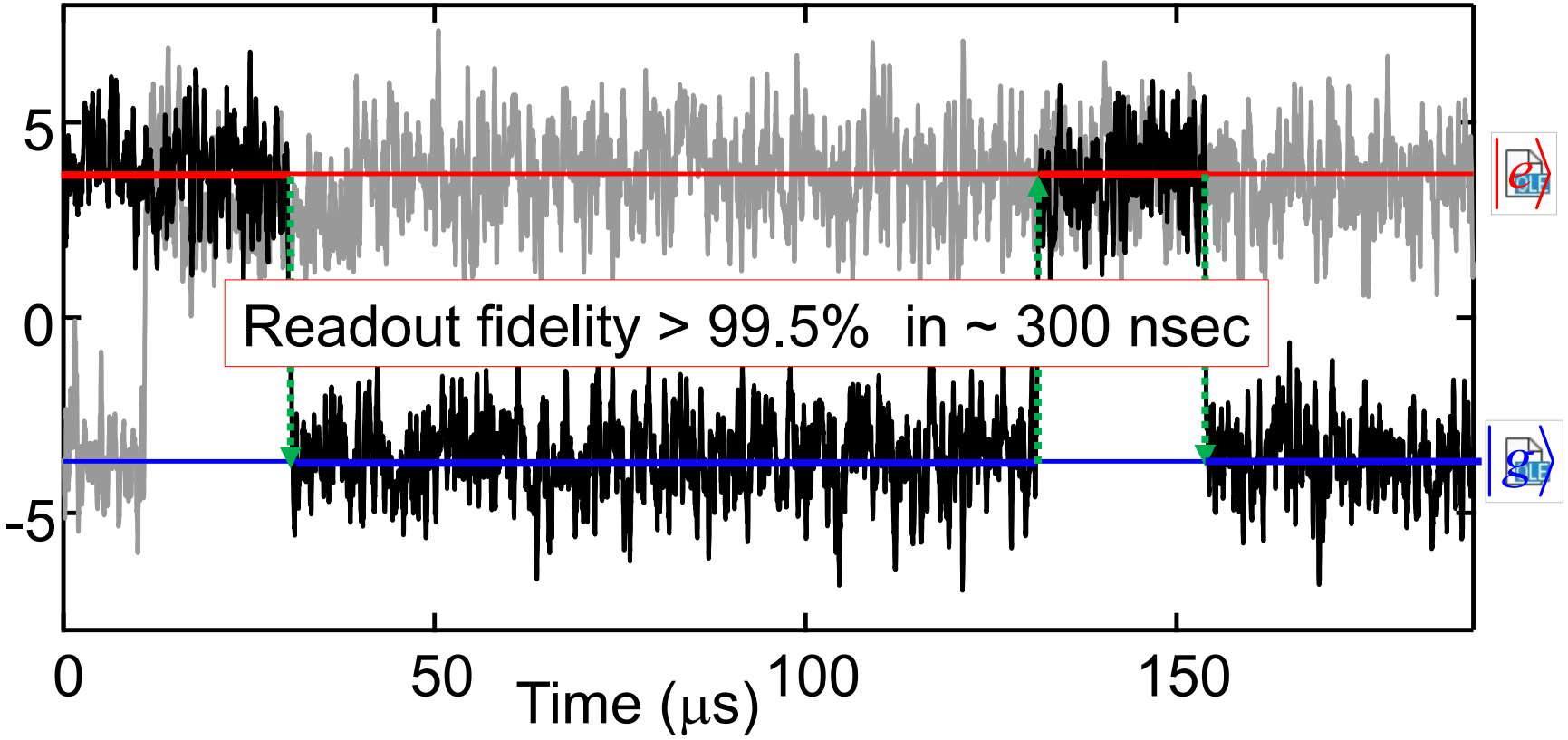




Quantum Jumps of a 3D Transmon Qubit

Results from Devoret group, Yale: Hatridge et al., Science 2013*

dispersive circuit QED readout + JJ paramp



Many groups now working with JJ paramps & feedback, including:
Berkeley, Delft, JILA, ENS/Paris, IBM, Wisc., Saclay, UCSB, …

*First jumps: R. Vijay et al., 2011 (Berkeley)

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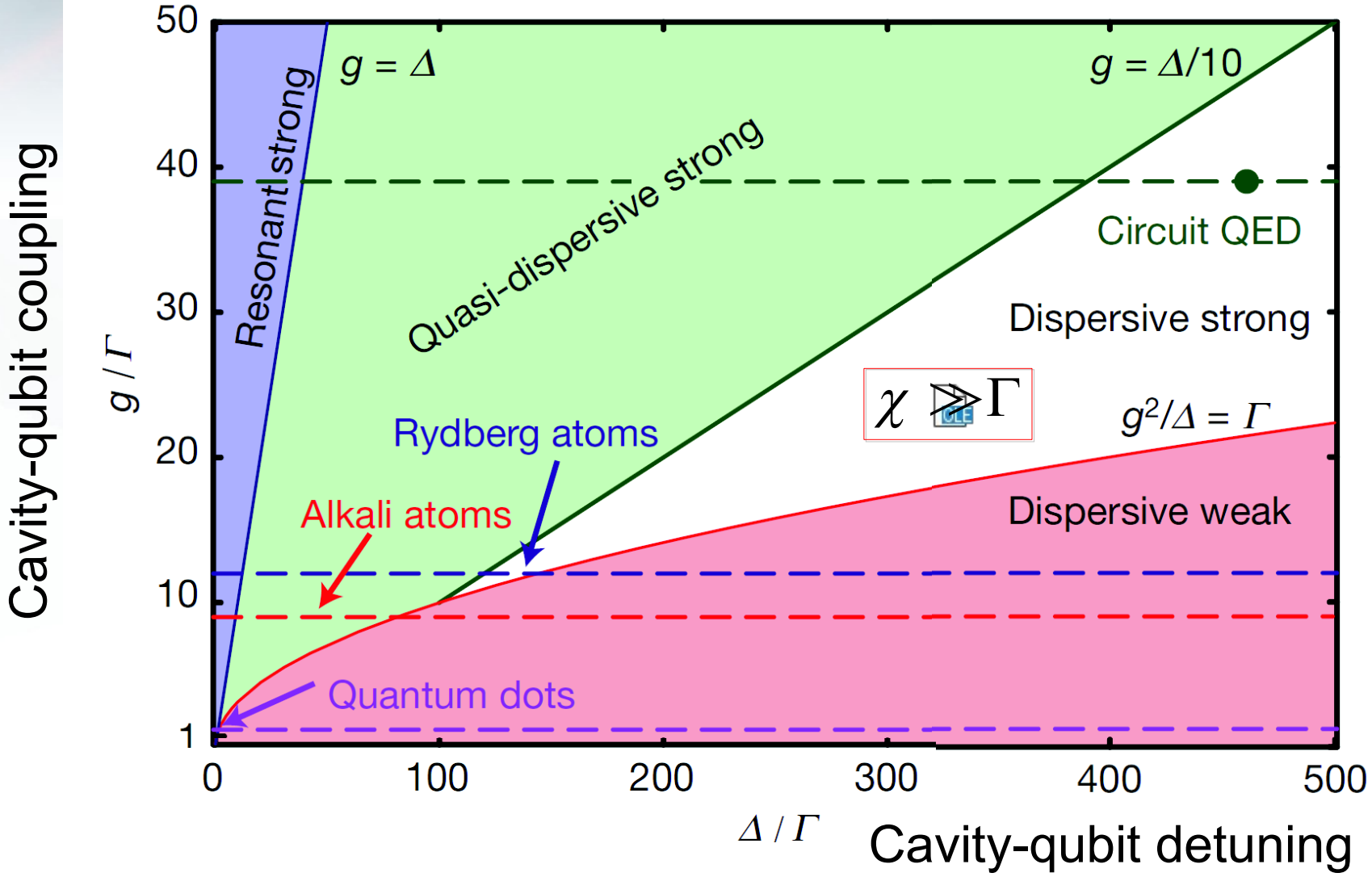
cQED 'phase diagram'

$\Gamma \equiv (\kappa, \gamma)$, linewidth of cavity or qubit



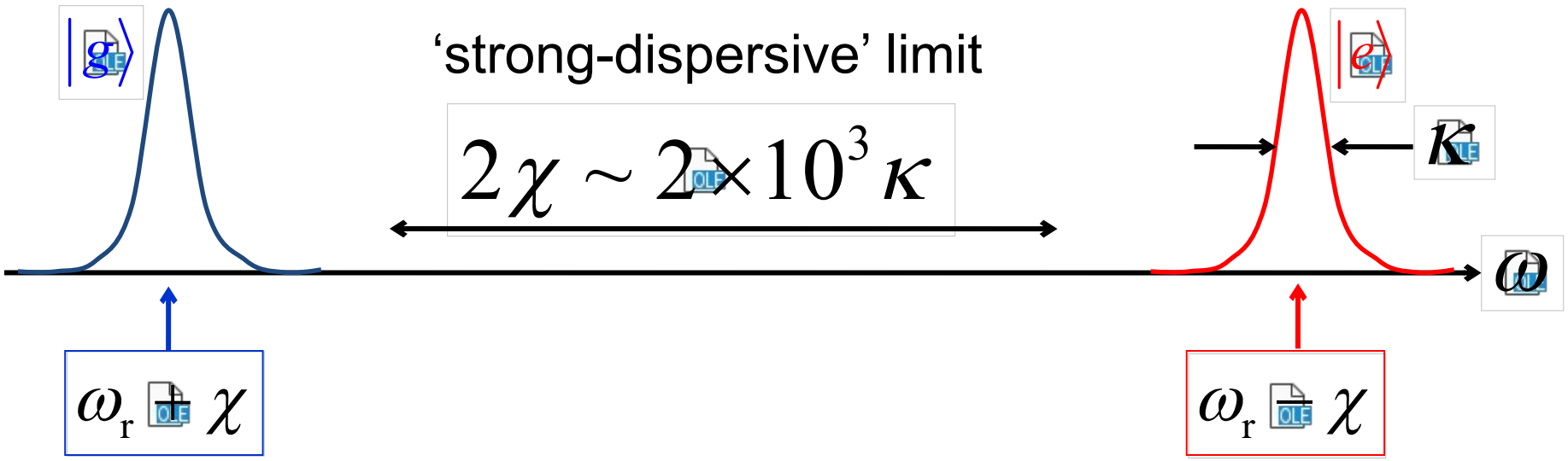
Strong Dispersive Hamiltonian

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + \chi\sigma^z a^\dagger a + H_{\text{damping}}$$

$$\chi >> \kappa, \Gamma$$

resonator qubit dispersive coupling

$$\text{cavity frequency} = \omega_r + \chi\sigma^z$$



Lecture 1: Introduction to Circuit QED

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Using strong-dispersive coupling to measure the photon number distribution in a cavity

Strong Dispersive Hamiltonian

$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + \chi \sigma^{z} a^{\dagger} a + H_{\mathrm{damping}}$$

$$\chi >> \kappa, \Gamma$$

resonator qubit dispersive coupling

Reinterpretation of same Hamiltonian: Quantized Light Shift of Qubit Transition Frequency

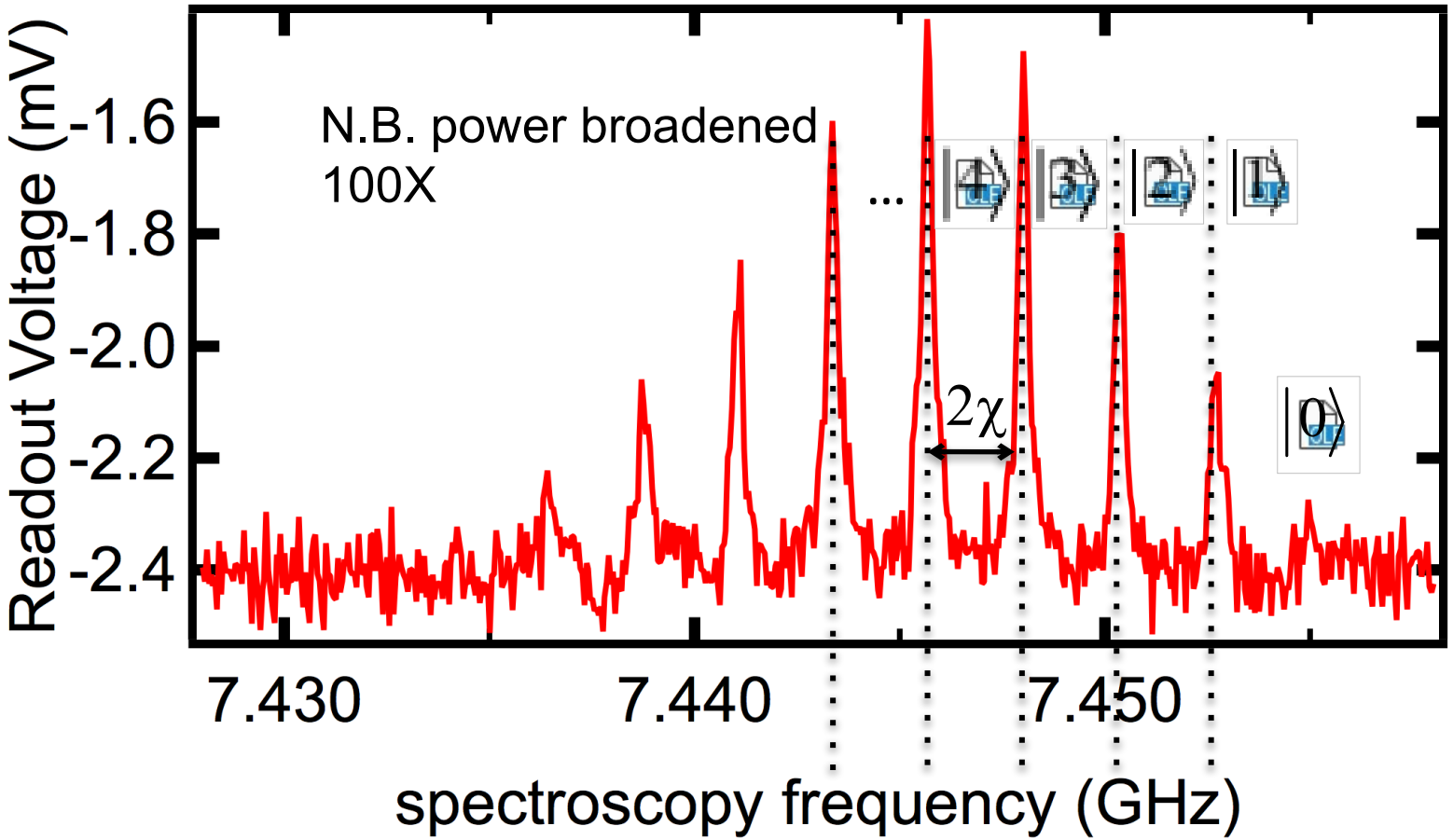
$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{1}{2} \sigma^{z} \left[\omega_{\mathrm{q}} + 2 \chi a^{\dagger} a \right] + H_{\mathrm{damping}}$$

Coherent state is closest thing to a classical sinusoidal RF signal

$$\boxed{\psi(\Phi) = \psi_0(\Phi - \alpha)}$$

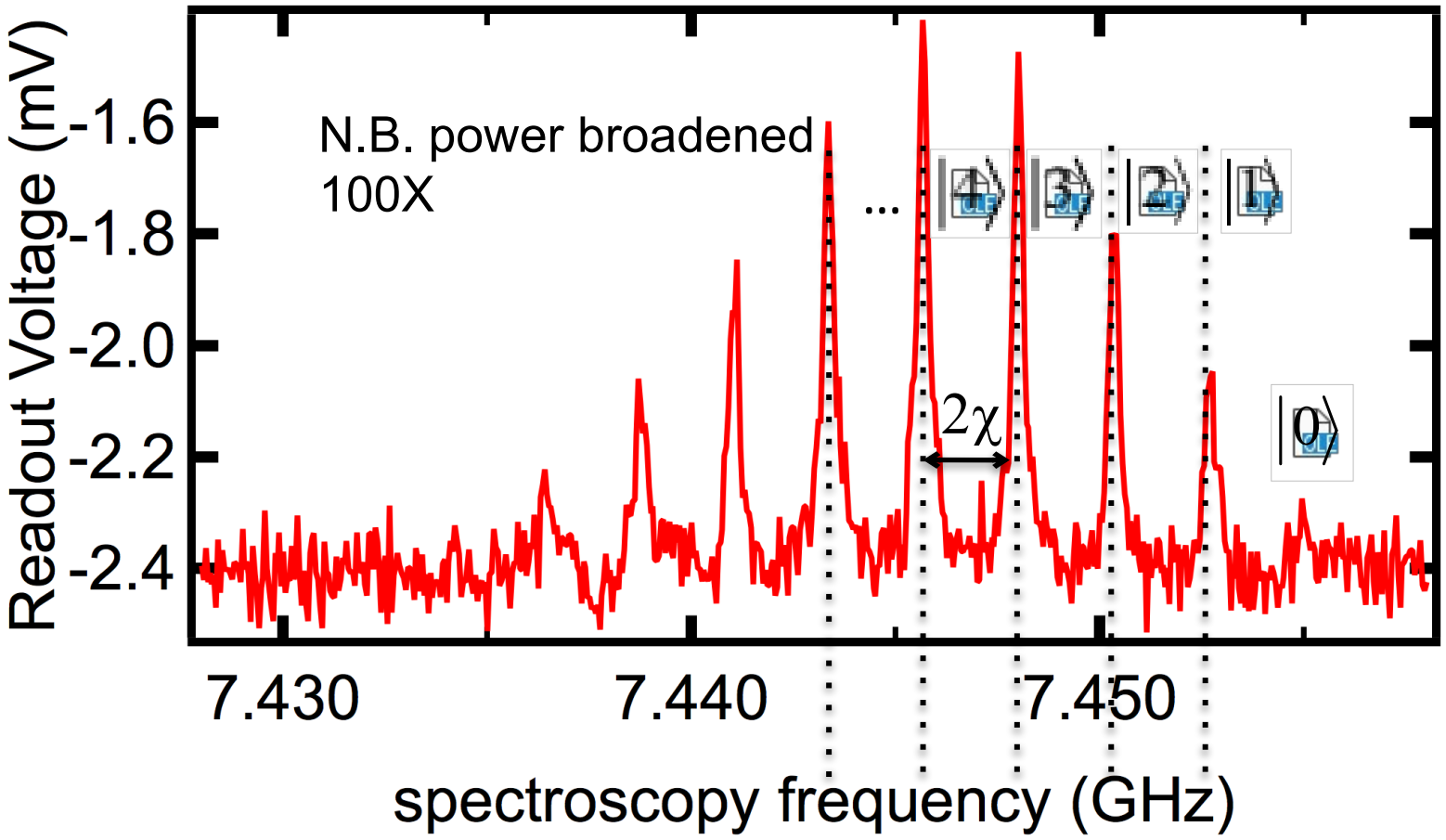
- quantized light shift of qubit frequency (coherent microwave state)

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



- quantized light shift of qubit frequency (coherent microwave state)

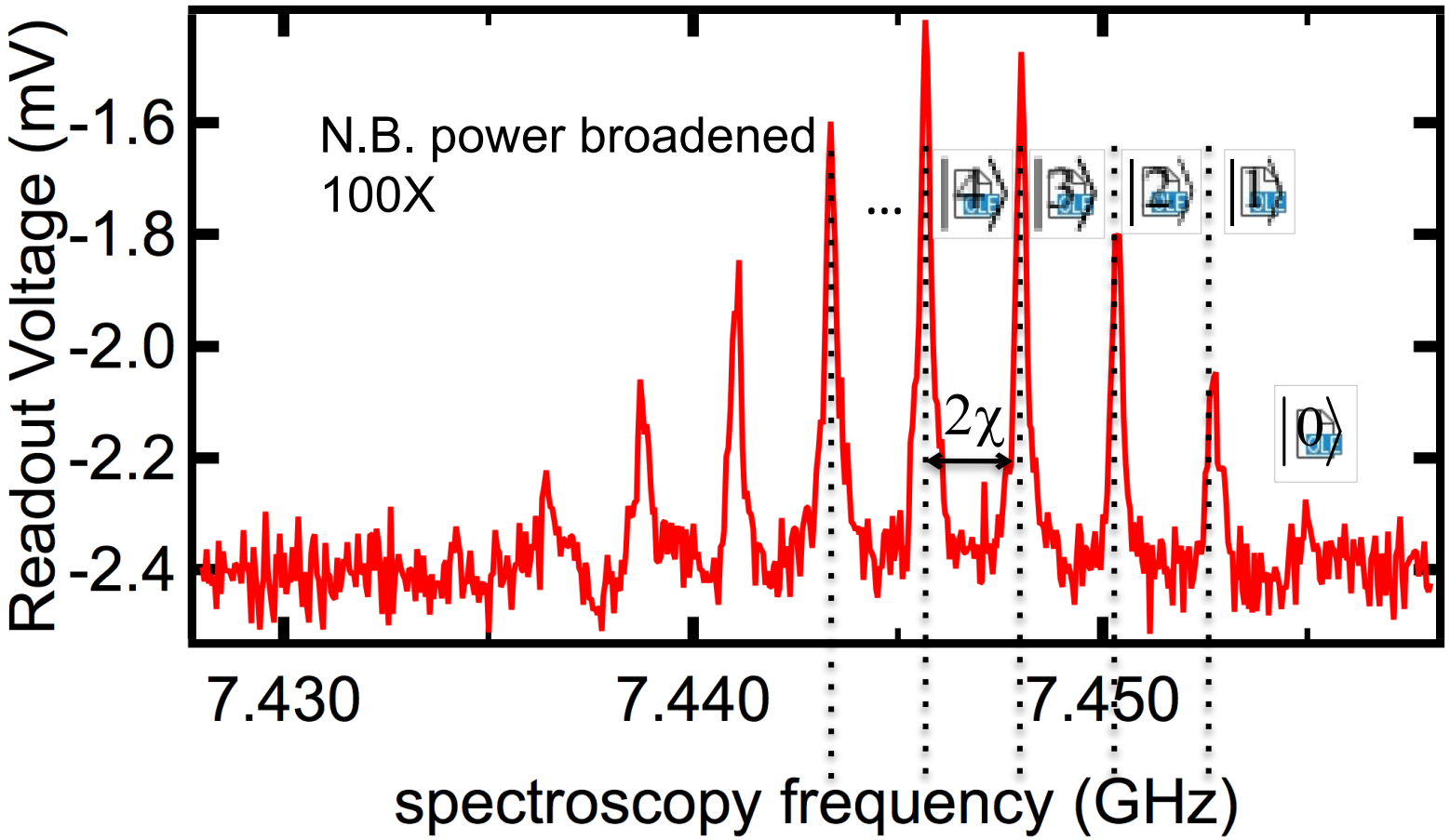
$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



Microwaves are particles!

- quantized light shift of qubit frequency (coherent microwave state)

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



New low-noise way to do axion dark matter detection by QND photon counting

Zheng et al. arXiv:1607.02529 → A. Chou: PRL **126**, 141302 (2021)

Summary: Introduction to Circuit QED

- ‘Blackbox’ Quantization (BBQ)
- Dispersive Coupling and Readout
- Strong Dispersive Limit
- Photon ‘Number Splitting ‘

Photon number parity

$$\hat{P} = (-1)^{a^\dagger a} = \sum_{n=0}^{\infty} |n\rangle (-1)^n \langle n|$$

Remarkably <u>easy</u> to measure using our quantum engineering toolbox

and

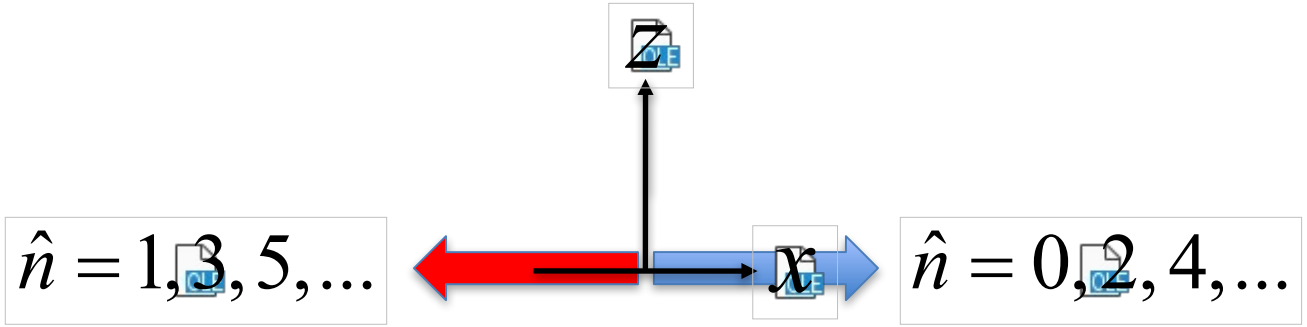
Measurement is 99.8% QND

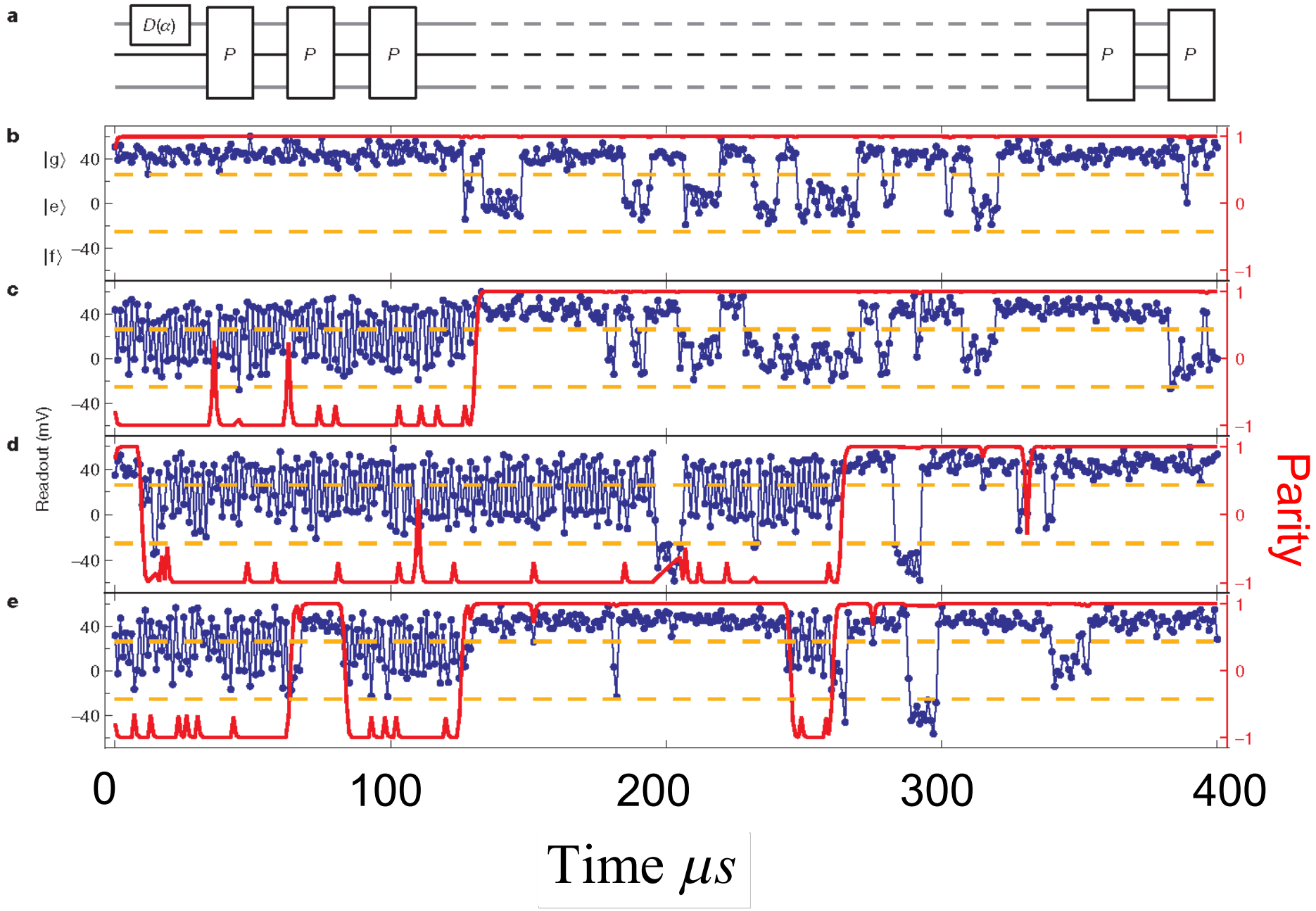
Measuring Photon Number Parity

- use quantized light shift of qubit frequency

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$

$$e^{-i2\chi\hat{n}t\frac{\sigma^z}{2}} = e^{-i\pi\hat{n}\frac{\sigma^z}{2}}$$





Nature **511**, 444 (2014)

400 consecutive parity measurements (99.8% QND)