Topological Superconductivity

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Quantum Matter Theory

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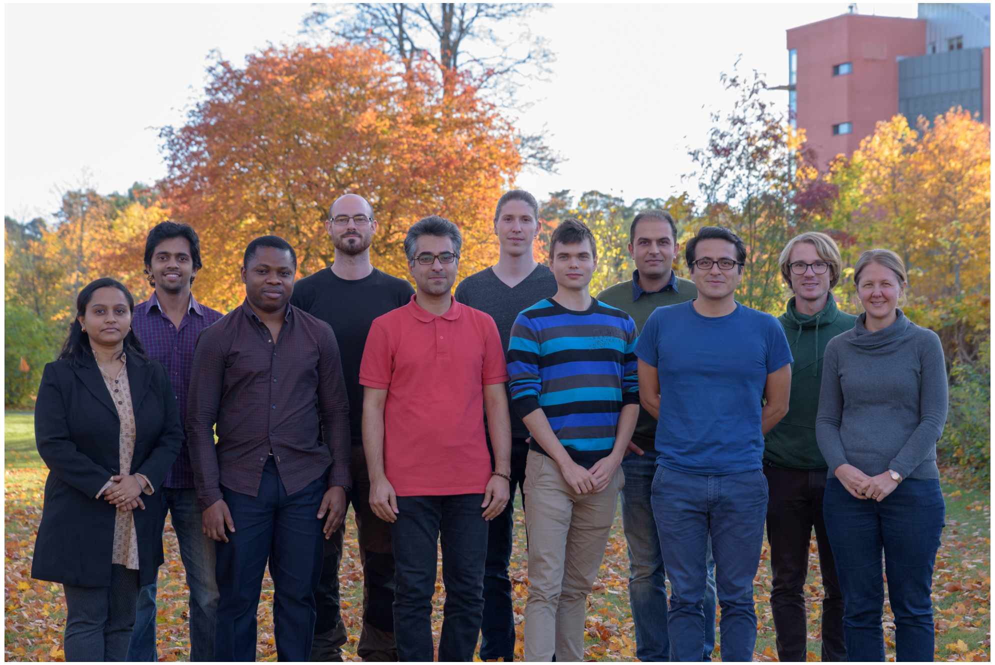
Acknowledgements



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Content

- Introduction to superconductivity
  - BCS, BdG, group theory
- Topological superconductivity
  - Chiral superconductors
  - “Spinless” superconductors → Majorana fermions

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Introduction to Superconductivity

What is it?

How do we describe it?

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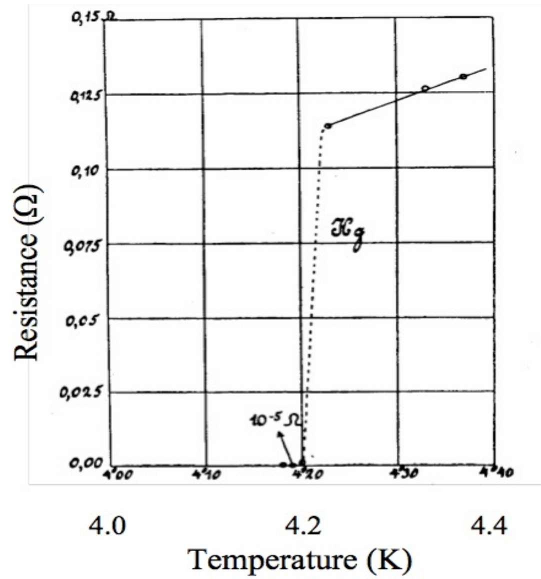
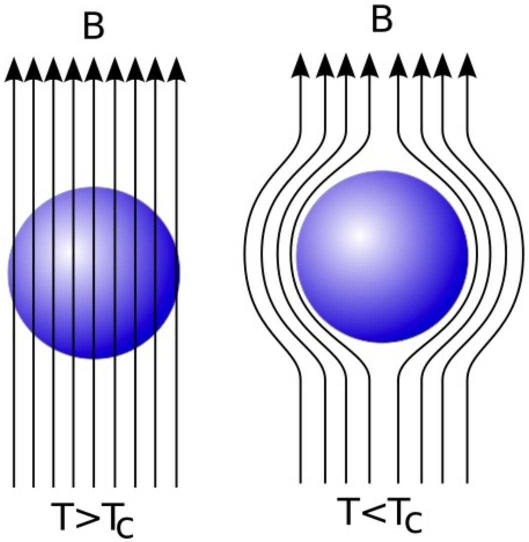
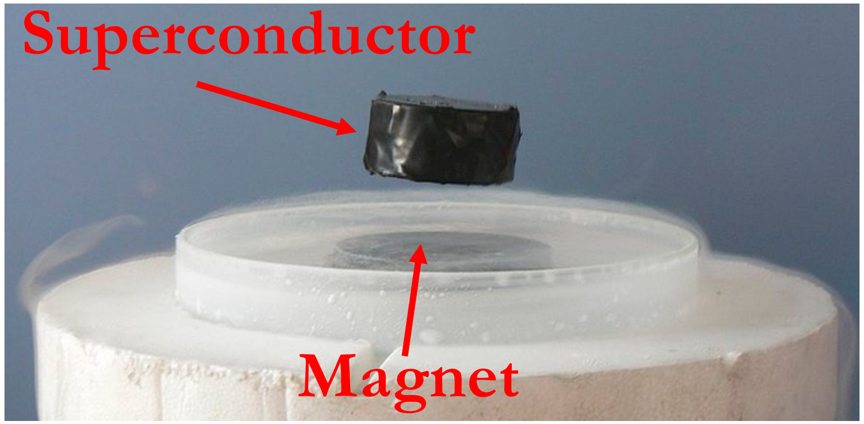
What is Superconductivity?

– Electric transport without resistance

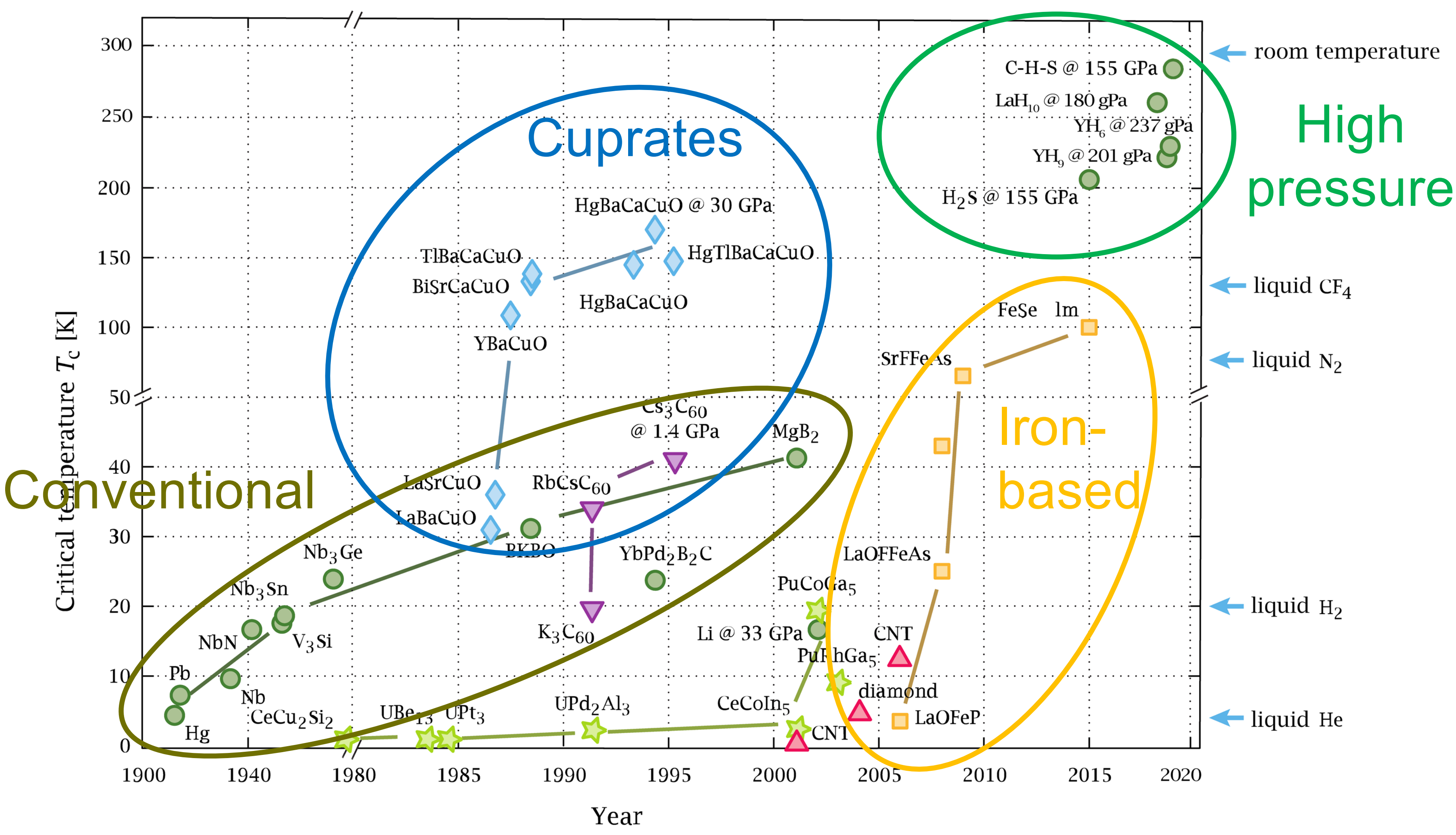


– Meissner effect





Superconductors



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How?

- But how do electrons move without resistance?
  - All electrons in coherent quantum state with fixed phase (condensate)

$$\Psi = \Delta_0 e^{i\boxed{\varphi}}$$

- **Bardeen-Cooper-Schrieffer (BCS) theory**
  - Condensation of electron (Cooper) pairs (with fermionic wave function)
  - Many-body state, but possible to describe within mean-field theory

BCS Hamiltonian

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Pairing Hamiltonian:

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}\sigma} c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

Kinetic (band) energy

Electron pairing

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -g_0/V, & (|\epsilon_{\mathbf{k}}| < \omega_D) \\ 0 & (\text{otherwise}) \end{cases}$$

Mean-field theory with $F_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$ (pair amplitude at $\mathbf{k}$)

$$\sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} [(c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} - F^{\dagger}_{\mathbf{k}}) + F^{\dagger}_{\mathbf{k}}][(c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - F_{\mathbf{k}'}) + F_{\mathbf{k}'}] \approx$$

$$\sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} (F_{\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + F^{\dagger}_{\mathbf{k}} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - F^{\dagger}_{\mathbf{k}} F_{\mathbf{k}'}]$$

Ignore fluctuations
$(c^{\dagger} c^{\dagger} - F^{\dagger})(cc - F)$

Set **order parameter** $\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} F_{\mathbf{k}'} = \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle$

$$\rightarrow \quad H_{\mathrm{MF}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c^{\dagger}{}_{-\mathbf{k}\downarrow} c^{\dagger}_{\mathbf{k}\uparrow} + \mathrm{H.c.} + \sum_{\mathbf{k}} \frac{V}{g_0} \Delta^{\dagger}_{\mathbf{k}} \Delta_{\mathbf{k}}$$

See e.g. Tinkham: Introduction to superconductivity

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Matrix Formulation (BdG)

Define the Nambu spinor $\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$ $\quad \psi^{\dagger}{}_{\mathbf{k}} = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right)$

$$\rightarrow \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}\left(c^{\dagger}{}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} - c_{-\mathbf{k}\downarrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} + 1\right) = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right) \begin{bmatrix} \epsilon_{\mathbf{k}} & 0 \\ 0 & -\epsilon_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ \bar{c}^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}$$

constant

TRS: $\varepsilon_{\mathrm{k}} = \varepsilon_{-\mathrm{k}}$

$$\rightarrow \epsilon_{\mathbf{k}} \sum_{\sigma} c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \left[\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} \Delta\right] = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right) \boxed{\begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta \\ \bar{\Delta} & -\epsilon_{\mathbf{k}} \end{bmatrix}} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$$

Bogliubov-de Gennes (BdG) formulation

2x2 matrix problem

→ Solve by finding eigenvalues and vectors

Eigenstates = Quasiparticles

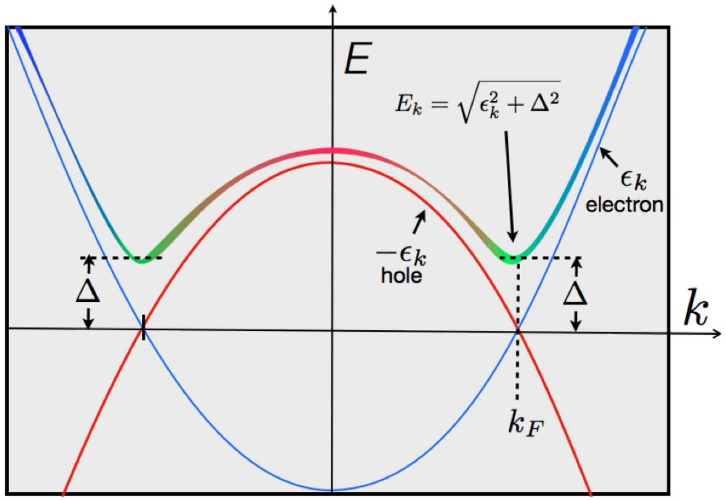
QP energies (eigenvalues): $E_{\mathbf{k}} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$

QP operators (eigenvectors):
$$\begin{cases} a^{\dagger}{}_{\mathbf{k}\uparrow} = \psi^{\dagger}{}_{\mathbf{k}} \cdot \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \end{pmatrix} = c^{\dagger}{}_{\mathbf{k}\uparrow} u_{\mathbf{k}} + c_{-\mathbf{k}\downarrow} v_{\mathbf{k}} \\ a_{-\mathbf{k}\downarrow} = \psi^{\dagger}{}_{\mathbf{k}} \cdot \begin{pmatrix} -v_{\mathbf{k}}^* \\ u_{\mathbf{k}}^* \end{pmatrix} = c_{-\mathbf{k}\downarrow} u_{\mathbf{k}}^* - c^{\dagger}{}_{\mathbf{k}\uparrow} v_{\mathbf{k}}^* \end{cases}$$

Bogoliubov tranformation

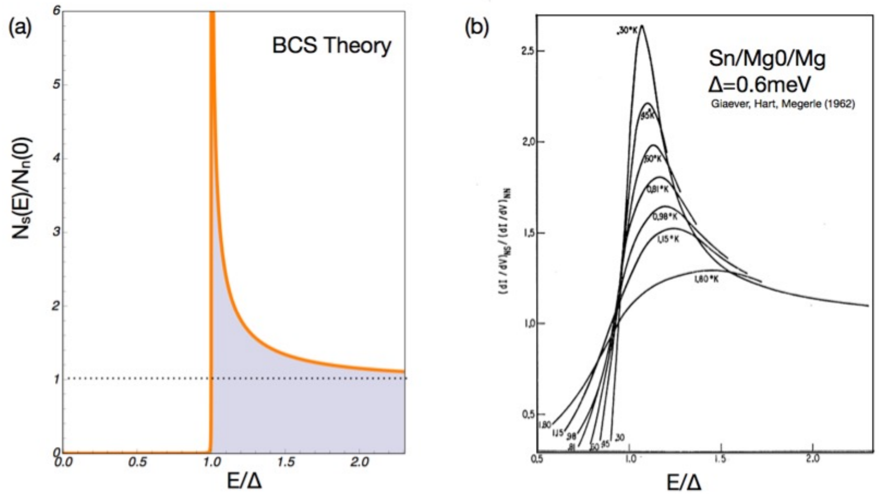
$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left[1 + \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]} \qquad v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left[1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right]}$$

Band structure



P. Coleman: Introduction to Many Body Physics

Density of states (DOS)



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Superconducting Order

Self-consistent order parameter:

$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}'} v^*_{\mathbf{k}'} \langle a^\dagger_{\mathbf{k}'\uparrow} a_{\mathbf{k}'\uparrow} + a^\dagger_{-\mathbf{k}'\downarrow} a_{\mathbf{k}'\downarrow} - 1 \rangle$$

$$\langle a^\dagger_{\mathbf{k}'\sigma} a_{\mathbf{k}'\sigma} \rangle = (1 + e^{E_{\mathbf{k}'}/k_B T})^{-1}$$ Fermi-Dirac distribution

Generalized order: fermionic, odd under particle exchange:

$$\Delta_{\alpha\beta}(\mathbf{k}) = -\Psi_{\beta\alpha}(-\mathbf{k})$$

$$\Delta_{\alpha\beta}(\mathbf{k}) = \Delta e^{i\varphi} \underbrace{\eta(\mathbf{k})}_{\text{orbital}} \underbrace{\chi_{\alpha\beta}}_{\text{spin}}$$

$$\chi_{\alpha\beta} \rightarrow \begin{cases} \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) & (S=0) \\ |\uparrow\uparrow\rangle & (S=1, S_z=1) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & (S=1, S_z=0) \\ |\downarrow\downarrow\rangle & (S=1, S_z=-1) \end{cases}$$

→ η even function in $\mathbf{k}$ (for $S=0$)

→ η odd function in $\mathbf{k}$ (for the three $S=1$ states)

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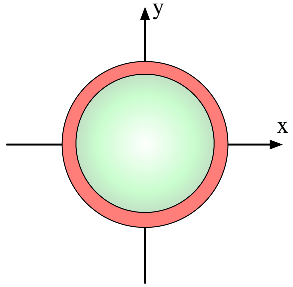
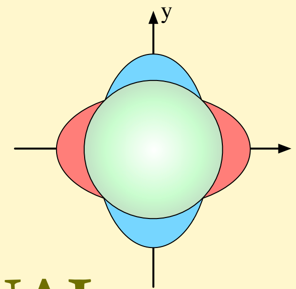
Spatial Symmetries

- Conventional superconductors:
  - Spin-singlet, *s*-wave (η k-independent)

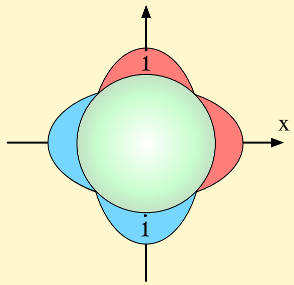


- Cuprate (high-Tc) superconductors:
  - Spin-singlet, *d*-wave ($\eta = k_x^2 - k_y^2$)



UNCONVENTIONAL

- *p*-wave superconductors:
  - Spin-triplet, *p*-wave (^{3}He, Sr_2RuO_4?)
  - Topological “spinless” superconductors with Majorana fermions



$k_x + ik_y$

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Superconducting Pairing

$V_{\mathbf{k},\mathbf{k}'}$ (and the band structure) determine the pairing symmetry, but often very hard to determine

- Lattice fluctations (phonon): spin-singlet *s*-wave **CONVENTIONAL**
- Antiferromagnetic spin fluctuations: spin-singlet *d*-wave (extended *s*-wave)
- Ferromagnetic spin fluctuations: spin-triplet *p*-wave
- Strong on-site repulsion (Heisenberg interaction): spin-singlet *d*-wave
- …

Can we determine the possible pairing symmetries in a material without knowing $V_{\mathbf{k},\mathbf{k}'}$?

Yes, by a general group theory analysis

See e.g. Sigrist and Ueda, RMP **63**, 239 (1991)

General Hamiltonian

General Hamiltonian: $$\mathcal{H}=\sum_{\mathbf{k},s}\varepsilon(\mathbf{k})a^{\dagger}_{\mathbf{k}s}a_{\mathbf{k}s}+\frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',s_1,s_2,s_3,s_4}V_{s_1s_2s_3s_4}(\mathbf{k},\mathbf{k}')a^{\dagger}_{-\mathbf{k}s_1}a^{\dagger}_{\mathbf{k}s_2}a_{\mathbf{k}'s_3}a_{-\mathbf{k}'s_4}$$

Mean-field order : $$\Delta_{ss'}(\mathbf{k})=-\sum_{\mathbf{k}',s_3,s_4}V_{s'ss_3s_4}(\mathbf{k},\mathbf{k}')\langle a_{\mathbf{k}'s_3}a_{-\mathbf{k}'s_4}\rangle$$

→ $$\tilde{\mathcal{H}}=\sum_{\mathbf{k},s}\varepsilon(\mathbf{k})a^{\dagger}_{\mathbf{k}s}a_{\mathbf{k}s}+\frac{1}{2}\sum_{\mathbf{k},s_1,s_2}[\Delta_{s_1s_2}(\mathbf{k})a^{\dagger}_{\mathbf{k}s_1}a^{\dagger}_{-\mathbf{k}s_2}-\Delta^{*}_{s_1s_2}(-\mathbf{k})a_{-\mathbf{k}s_1}a_{\mathbf{k}s_2}]$$

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Matrix Formulation

4-component notation (Nambu): $\mathbf{a}_{\mathbf{k}} = (a_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, a^{\dagger}_{-\mathbf{k}\uparrow}, a^{\dagger}_{-\mathbf{k}\downarrow})^T$

$$\rightarrow \tilde{\mathcal{H}} = \mathbf{a}^{\dagger}_{\mathbf{k}} \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \hat{\Delta}(\mathbf{k}) \\ \hat{\Delta}^{\dagger}(\mathbf{k}) & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix} \mathbf{a}_{\mathbf{k}}$$

Spin-singlet pairing: $\hat{\Delta}(\mathbf{k}) = i\hat{\sigma}_y \psi(\mathbf{k}) = \begin{bmatrix} 0 & \psi(\mathbf{k}) \\ -\psi(\mathbf{k}) & 0 \end{bmatrix}$

ψ even function of **k**

$$\left(\psi(\mathbf{k})[c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow} - c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}] \right)$$

Spin-triplet pairing: $\hat{\Delta}(\mathbf{k}) = i(\mathbf{d}(\mathbf{k})\cdot\hat{\boldsymbol{\sigma}})\hat{\sigma}_y$

$$= \begin{bmatrix} -d_x(\mathbf{k}) + i d_y(\mathbf{k}) & d_z(\mathbf{k}) \\ d_z(\mathbf{k}) & d_x(\mathbf{k}) + i d_y(\mathbf{k}) \end{bmatrix}$$

d vector odd function of **k**

$$\begin{pmatrix} \mathrm{m_z} = 0\text{:} \quad d_z(\mathbf{k})[c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow} + c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\downarrow}] \\ \mathrm{m_z} = 1\text{:} \quad [-d_x(\mathbf{k}) + i d_y(\mathbf{k})]c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k}\uparrow} \end{pmatrix}$$

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General Solution

QP energy (eigenvalue): $E_{\mathbf{k}} = \sqrt{\varepsilon(\mathbf{k})^2 + |\psi(\mathbf{k})|^2}$

$$E_{\mathbf{k}} = \sqrt{\varepsilon(\mathbf{k})^2 + |\mathbf{d}(\mathbf{k})|^2 \pm \mathbf{q}(\mathbf{k})|}$$

$$\left(\begin{array}{c} \hat{\Delta}\hat{\Delta}^{\dagger} = |\mathbf{d}|^2\hat{\sigma}_0 + \mathbf{q}\cdot\hat{\boldsymbol{\sigma}} \\ \mathbf{q} = i(\mathbf{d}\times\mathbf{d}^*) \\ \text{Finite } \mathbf{q} = \text{non-unitary} \end{array}\right)$$

Self-consistency equation, linear close to T_c:

$$v\Delta_{s_1 s_2}(\mathbf{k}) = -\sum_{s_3 s_4} \langle V_{s_2 s_1 s_3 s_4}(\mathbf{k},\mathbf{k}')\Delta_{s_3 s_4}(\mathbf{k}')\rangle_{\mathbf{k}'}$$

$$\frac{1}{v} = N(0)\int_0^{\varepsilon_c} d\varepsilon \frac{\tanh\left[\frac{\beta_c \varepsilon(k)}{2}\right]}{\varepsilon(\mathbf{k}} = \ln(1.14\beta_c\varepsilon_c)$$

- Largest eigenvalue gives T_c
- Eigenfunction (Δ) belongs to irreducible representation (irrep) of symmetry group

→ Possible SC symmetries belong to **irreps of symmetry group of H**

→ SC state always breaks U(1), can also break

- Crystal lattice, spin-rotation, time-reversal, … symmetries

Basis Gap Functions: D_{4h}

- D_{4h} = tetragonal symmetry (cuprates with $k_z = 0$)

Irreducible representation Γ	Basis function	
	(a) Spin-singlet	
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$	s-wave, extended s-wave
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-k_y^2)$	
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_x^2-k_y^2$	d(x^2-y^2)-wave
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_xk_y$	d(xy)-wave
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$	
	$\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$	
	(b) Spin-triplet	
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\ \hat{\mathbf{z}}k_z$	
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$	
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_x$	
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{x}}k_y+\hat{\mathbf{y}}k_x$	
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\ \hat{\mathbf{z}}k_x$	p(x)- and p(y)-wave degenerate
	$\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\ \hat{\mathbf{z}}k_y$	

Sigrist and Ueda, RMP **63**, 239 (1991)

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Basis Gap Functions: D_{6h}

- D_{6h} = hexagonal symmetry (graphene, Bi_2Se_3 TIs with $k_z = 0$,)

Irreducible representation Γ	Basis functions
	(a) Spin-singlet
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$ — s-wave, extended s-wave
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-3k_y^2)(k_y^2-3k_x^2)$
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_zk_x(k_x^2-3k_y^2)$
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_zk_y(k_y^2-3k_x^2)$
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$ $\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$
Γ_6^+	$\psi(\Gamma_6^+,1;\mathbf{k})=k_x^2-k_y^2$ $\psi(\Gamma_6^+,2;\mathbf{k})=2k_xk_y$ — d(x²-y²)-wave and d(xy)-wave degenerate
	(b) Spin-triplet
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\hat{\mathbf{z}}k_z$
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{z}}k_x(k_x^2-3k_y^2),$ $k_z[(k_x^2-k_y^2)\hat{\mathbf{x}}-2k_xk_y\hat{\mathbf{y}}]$
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{z}}k_y(k_y^2-3k_x^2),$ $k_z[(k_y^2-k_x^2)\hat{\mathbf{y}}-2k_xk_y\hat{\mathbf{x}}]$
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\hat{\mathbf{z}}k_x$ $\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\hat{\mathbf{z}}k_y$
Γ_6^-	$\mathbf{d}(\Gamma_6^-,1;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_y$ $\mathbf{d}(\Gamma_6^-,2;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$

Sigrist and Ueda, RMP **63**, 239 (1991)

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Multiple Order Parameters

Superconducting state highly unconventional if multiple components at T_c

- Two-dimensional irreps often gives $\Delta_1 + i\Delta_2$ at $T < T_c$
  - Only combination with fully gap → Highest energy gain
  - Singlet $d(x^2\text{-}y^2)+id(xy)$-wave for hexagonal lattices (graphene?)
  - Triplet ($m_z = 0$) $p(x)+ip(y)$-wave for square lattices

Topological (chiral) superconductors

Break time-reversal symmetry (TRS)

Fully gapped bulk energy spectrum

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Introduction to Superconductivity

What is it?

- A charged superfluid of Cooper pairs (2 electrons) with fermionic character
- Cooper pairs formed by effective attractive interaction

How do we describe it?

- BCS theory (mean-field theory of condensation)
- BdG matrix formalism
- Symmetry of order parameter (group theory)

Topological Superconductivity

Chiral superconductors

- Spin-singlet *d*+i*d'*-wave (spin-triplet *p*+i*p'*-wave) superconductors

Spinless superconductors

- Majorana fermions
- Engineered systems

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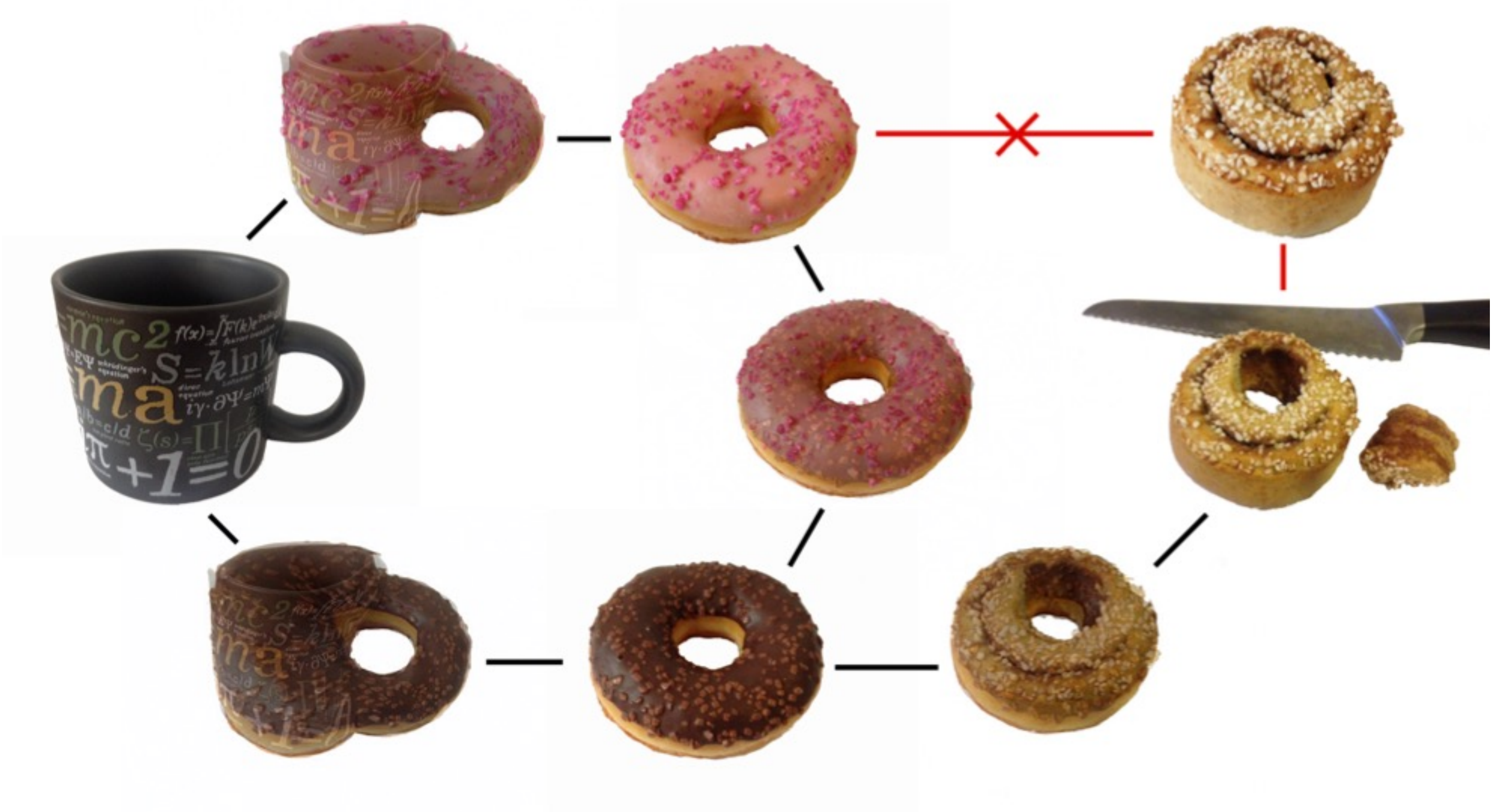
Topology



Topologically speaking: coffee cup = donut ≠ bun

1 hole 1 hole 0 hole

Classification

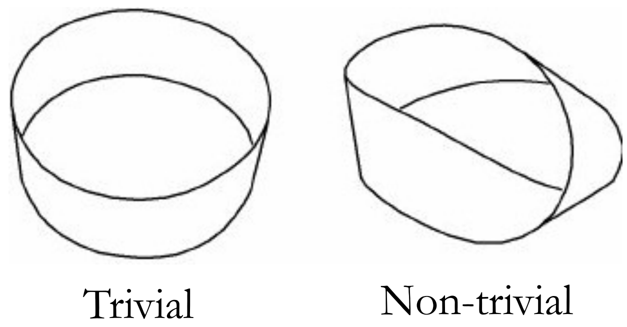
- All forms of matter can be classified according to the symmetry they break (translation, spin, gauge, time-reversal, …)

- Except **topological matter**

  Topological insulators: 2005 (quantum Hall effect: 1980)

  – Ordered but no symmetry breaking

  – Topology of the wave function



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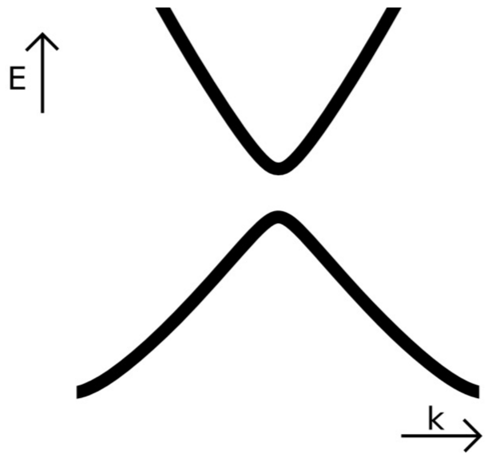
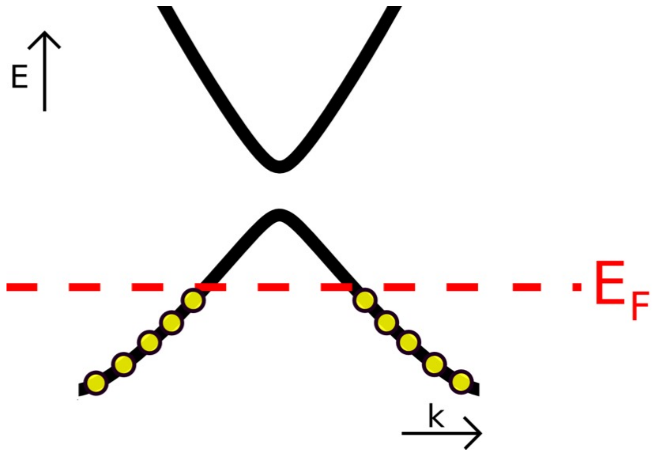
Electrons in Crystals



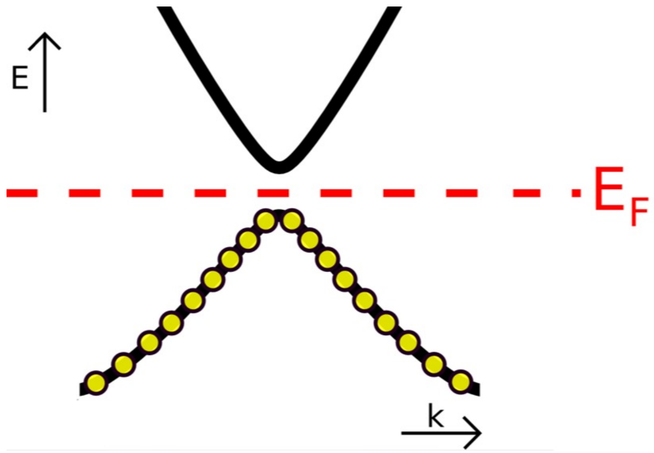
Electrons in Crystals

Metals:



← Semiconductors

Insulators:



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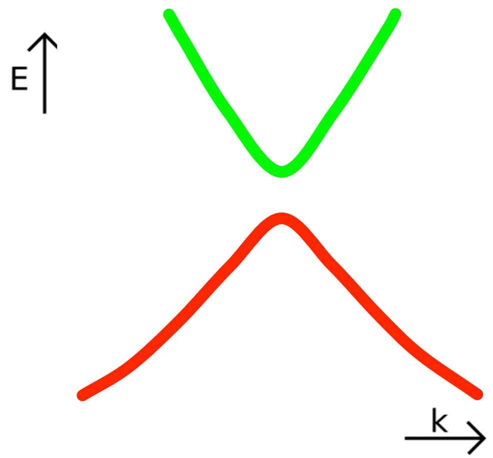
Topological Band Theory

Anything else than metals and insulators?

+ spin-orbit coupling

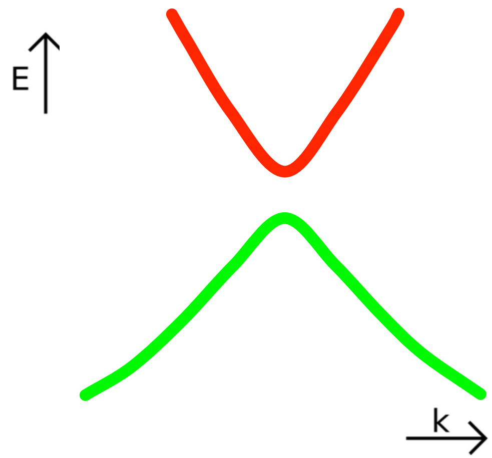
→ Band inversion

Without spin-orbit coupling



Normal band structure

With spin-orbit coupling



Topological (inverted) band structure

HgTe & CdTe Semiconductors

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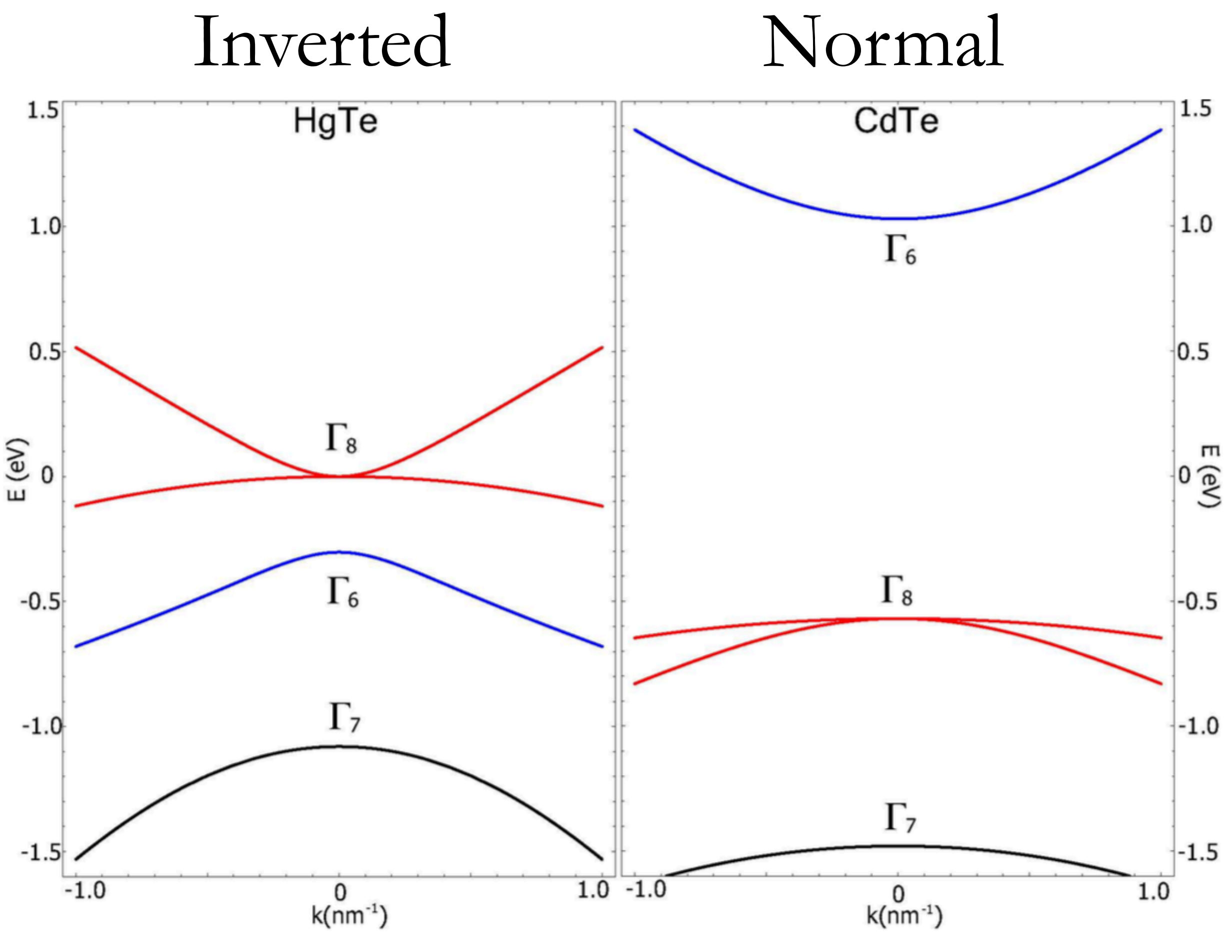

Bernevig *et al.*, Science **314**, 1757 (2006)

Topological Insulators

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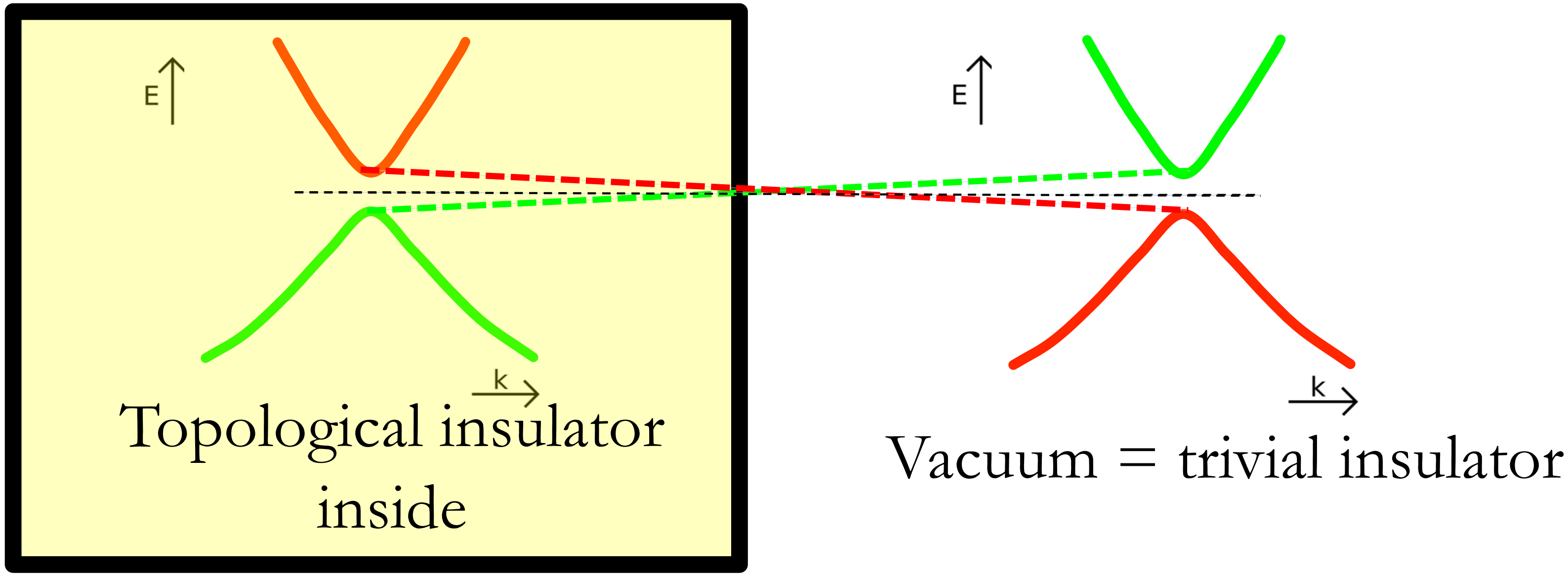

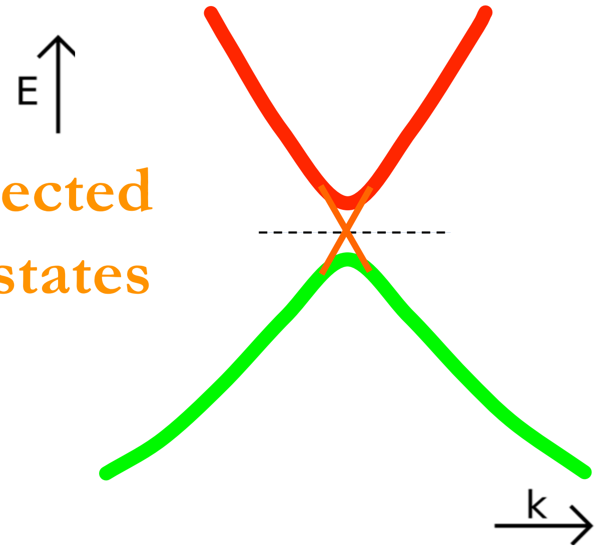

Topologically protected zero-energy edge states

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Topological Matter

Topological states of matter have

- **Bulk topological invariant**
  - Number classifying the topological class
  - Only changes with bulk gap closing
- **Protected boundary states**
  - At any boundary to other topological region (vacuum, normal metal, *s*-wave SC = trivial topological order)

Bulk-boundary correspondence

# of boundary states = change in topological invariant at boundary

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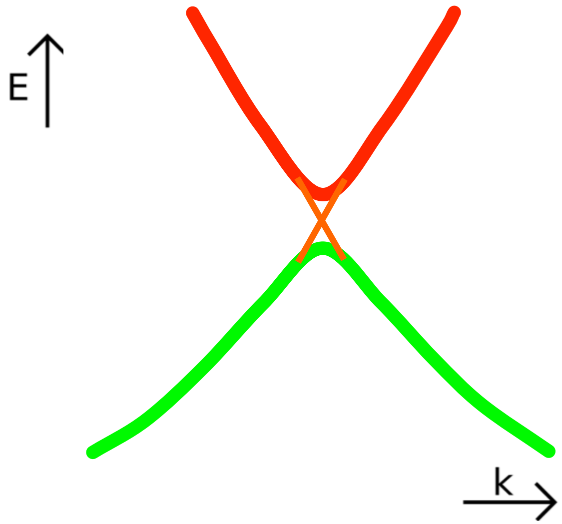
Topological Superconductors

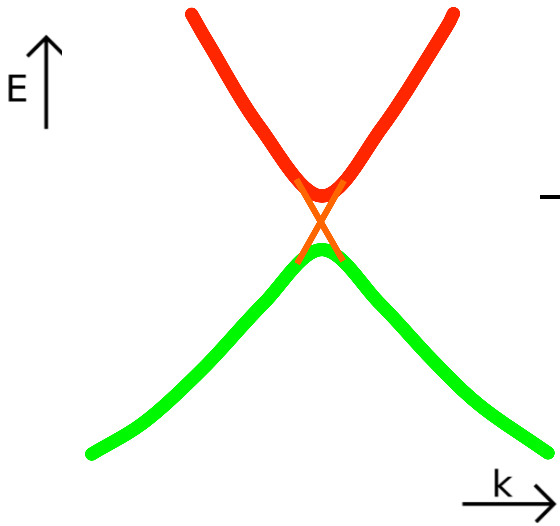
Same band theory in insulators and superconductors

Topological insulators ⟷ Topological superconductors

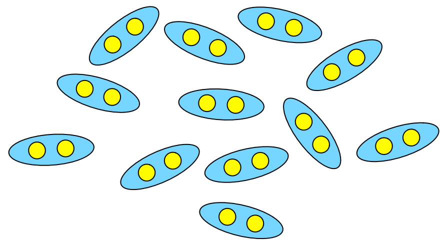
$$\left(E_{\mathbf{k}} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2} \right)$$

Same low-energy excitation spectrum





+



Condensate of
Cooper pairs

Topological Classification

Non-interacting (single-particle) insulators and superconductors: 10-fold way

		time-reversal TRS	particle-hole PHS	sublattice (chiral) SLS	Topological invariants $d=1$	$d=2$	$d=3$
Standard	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	−1	0	-	$\mathbb{Z}$	-
	DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	−1	1	-	-	$\mathbb{Z}$

Schnyder *et al.*, PRB **78**, 195125 (2008)

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Superconductors

AZ class	SU(2)	TRS	Examples in two dimensions
D	×	×	Spinless chiral $(p \pm ip)$ wave
DIII	×	○	Superposition of $(p+ip)$ and $(p-ip)$ waves
A	△	×	Spinful chiral $(p \pm ip)$ wave
AIII	△	○	Spinful p_x or p_y wave
C	○	×	$(d \pm id)$ wave
CI	○	○	$d_{x^2-y^2}$ or d_{xy} wave

		TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
Standard (Wigner-Dyson)	A (unitary)	0	0	0	-	Z	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	−1	0	0	-	Z_2	Z_2
Chiral (sublattice)	AIII (chiral unitary)	0	0	1	Z	-	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	-	-
	CII (chiral symplectic)	−1	−1	1	Z	-	Z_2
BdG	D	0	+1	0	Z_2	Z	-
	C	0	−1	0	-	Z	-
	DIII	−1	+1	1	Z_2	Z_2	Z
	CI	+1	−1	1	-	-	Z

Spinless p+ip'-wave in 1D → BDI because effective TRS

Schnyder *et al.*, PRB **78**, 195125 (2008)

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Topological Superconductivity

Chiral superconductors

Spin-singlet *d*+i*d*-wave (spin-triplet *p*+i*p*-wave) superconductors

Spinless superconductors

Topology and Majorana fermions

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d-wave SC from Strong Repulsion

Strong Coulomb repulsion, antiferromagnetic correlations

(e.g. Hubbard model near half-filling)

→ Spin-singlet pairing

→ Double electron occupation unfavorable

→ No *s*-wave pairing

→ Spin-singlet *d*-wave pairing

(best state = least number of nodes)

2D hexagonal lattice →

Spin-singlet $d(x^2-y^2)+id(xy)$ pairing

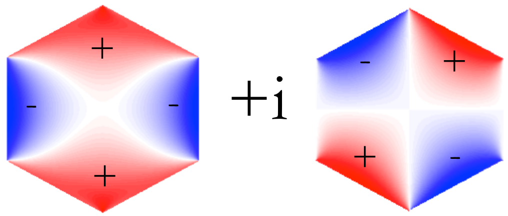
(Only combination with energy gap)

Irreducible representation Γ	Basis functions
	(a)
Γ_1^+	$\psi(\Gamma_1^+;\mathbf{k})=1,\ k_x^2+k_y^2,\ k_z^2$
Γ_2^+	$\psi(\Gamma_2^+;\mathbf{k})=k_xk_y(k_x^2-3k_y^2)(k_y^2-3k_x^2)$
Γ_3^+	$\psi(\Gamma_3^+;\mathbf{k})=k_zk_x(k_x^2-3k_y^2)$
Γ_4^+	$\psi(\Gamma_4^+;\mathbf{k})=k_zk_y(k_y^2-3k_x^2)$
Γ_5^+	$\psi(\Gamma_5^+,1;\mathbf{k})=k_xk_z$ $\psi(\Gamma_5^+,2;\mathbf{k})=k_yk_z$
Γ_6^+	$\psi(\Gamma_6^+,1;\mathbf{k})=k_x^2-k_y^2$ $\psi(\Gamma_6^+,2;\mathbf{k})=2k_xk_y$
	(b)
Γ_1^-	$\mathbf{d}(\Gamma_1^-;\mathbf{k})=\hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y,\hat{\mathbf{z}}k_z$
Γ_2^-	$\mathbf{d}(\Gamma_2^-;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$
Γ_3^-	$\mathbf{d}(\Gamma_3^-;\mathbf{k})=\hat{\mathbf{z}}k_x(k_x^2-3k_y^2),$ $k_z[(k_x^2-k_y^2)\hat{\mathbf{x}}-2k_xk_y\hat{\mathbf{y}}]$
Γ_4^-	$\mathbf{d}(\Gamma_4^-;\mathbf{k})=\hat{\mathbf{z}}k_y(k_y^2-3k_x^2),$ $k_z[(k_y^2-k_x^2)\hat{\mathbf{y}}-2k_xk_y\hat{\mathbf{x}}]$
Γ_5^-	$\mathbf{d}(\Gamma_5^-,1;\mathbf{k})=\hat{\mathbf{x}}k_z,\hat{\mathbf{z}}k_x$ $\mathbf{d}(\Gamma_5^-,2;\mathbf{k})=\hat{\mathbf{y}}k_z,\hat{\mathbf{z}}k_y$
Γ_6^-	$\mathbf{d}(\Gamma_6^-,1;\mathbf{k})=\hat{\mathbf{x}}k_x-\hat{\mathbf{y}}k_y$ $\mathbf{d}(\Gamma_6^-,2;\mathbf{k})=\hat{\mathbf{x}}k_y-\hat{\mathbf{y}}k_x$

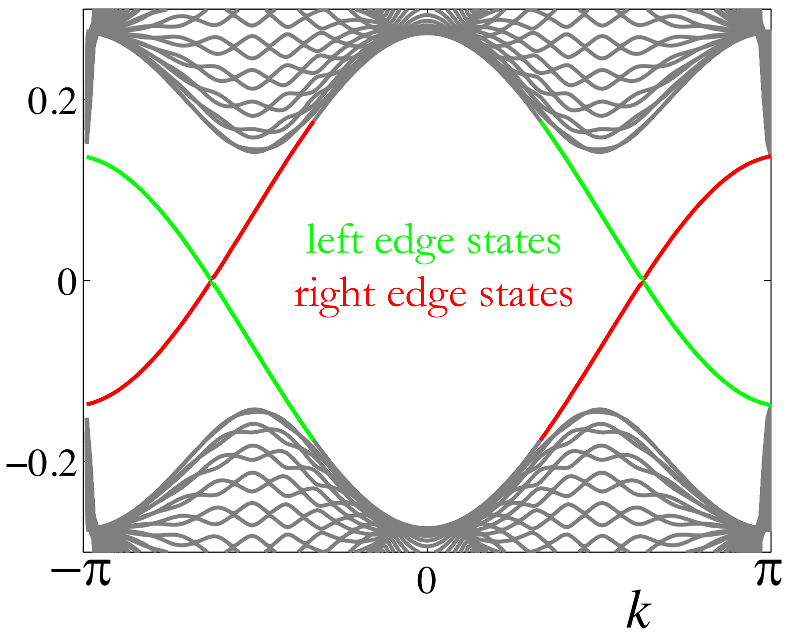
$d(x^2-y^2)+id(xy)$

Bulk and Edge Properties

- Fully gapped bulk $E_{\mathrm{QP}}(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}$



- Two chiral (co-propagating) edge states per edge



ABS, PRL **109**, 197001 (2012)

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Topological Invariant

d+id'-wave SC breaks TRS → Chern number invariant

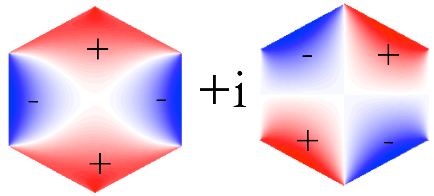
$$\mathcal{N} = \frac{1}{4\pi}\int_{\mathrm{BZ}} \mathrm{d}^2k\, \hat{\mathbf{m}} \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial k_y}\right)$$

Skyrmion number
$$\hat{\mathbf{m}} = \frac{1}{\sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}} \begin{pmatrix} \mathrm{Re}\,\Delta(\mathbf{k}) \\ \mathrm{Im}\,\Delta(\mathbf{k}) \\ \varepsilon(\mathbf{k}) \end{pmatrix}$$

Counts unit sphere area spanned by **m** as **k** covers the BZ

Bottom of band: **m** ~ -**z**
Top of band: **m** ~ **z**
→ Non-zero $\mathcal{N}$ iff Δ has finite winding along lines of constant ε

d+id'-wave winds twice around Γ → $|\mathcal{N}| = 2$
→ **2 chiral edge states**



ABS and Honerkamp JPCM **26**, 423201 (2014)

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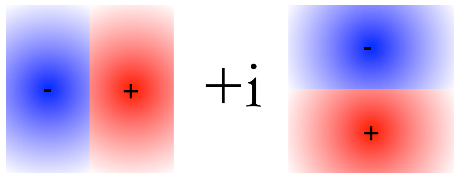
Chiral *p*+i*p* SC Properties

Spin-triplet *p*(x)+i*p*(y)-wave spin-triplet, $\mathbf{d} = (0, 0, k_x + ik_y)$

- Fully gapped in the bulk $E_{QP}(\mathbf{k}) = \sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}$



p(x) +i *p*(y)

- Break TRS → finite Chern number/Skyrmion winding

$$\mathcal{N} = \frac{1}{4\pi} \int_{\mathrm{BZ}} \mathrm{d}^2k \, \hat{\mathbf{m}} \cdot \left(\frac{\partial \hat{\mathbf{m}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{m}}}{\partial k_y} \right)$$

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{\varepsilon(\mathbf{k})^2 + |\Delta(\mathbf{k})|^2}} \begin{pmatrix} \mathrm{Re}\, \Delta(\mathbf{k}) \\ \mathrm{Im}\, \Delta(\mathbf{k}) \\ \varepsilon(\mathbf{k}) \end{pmatrix}$$

p+i*p*'-wave winds once around Γ → $|\mathcal{N}| = 1$

→ **One chiral edge state per edge**

Doped Graphene, *d*+i*d'* SC?

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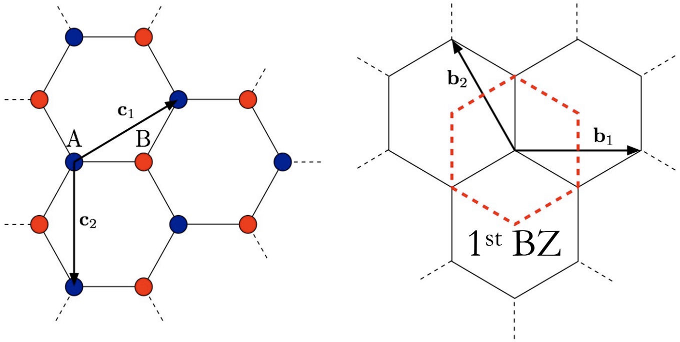
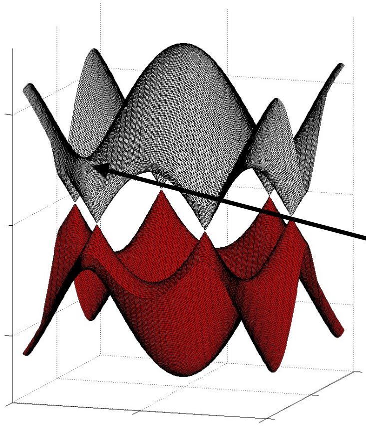
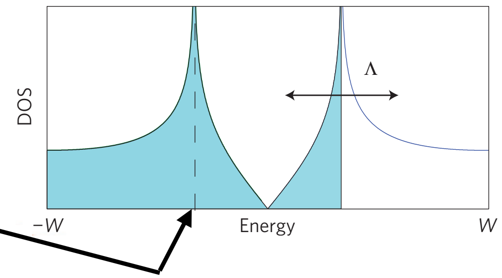
Honeycomb lattice



Band structure with van Hove singularities



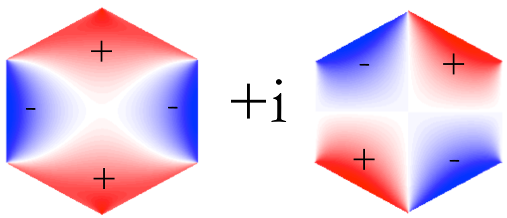


Diverging DOS

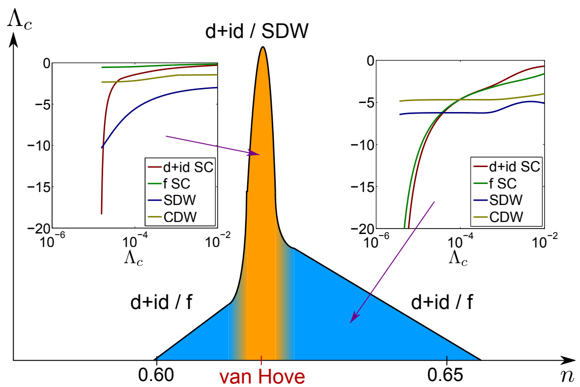
→ favorable for SC

Pairing from repulsive interactions

- Strong interactions [1]
- Perturbative RG [2]
- Functional RG [3]



$d(x^2\text{-}y^2)$ +i $d(xy)$



[1]: ABS and Doniach, PRB **75**, 134512 (2007), [2]: Nandkishore *et al.*, Nat. Phys. **8**, 158 (2012), [3]: Kiesel *et al.*, PRB **86**, 020507 (2012)

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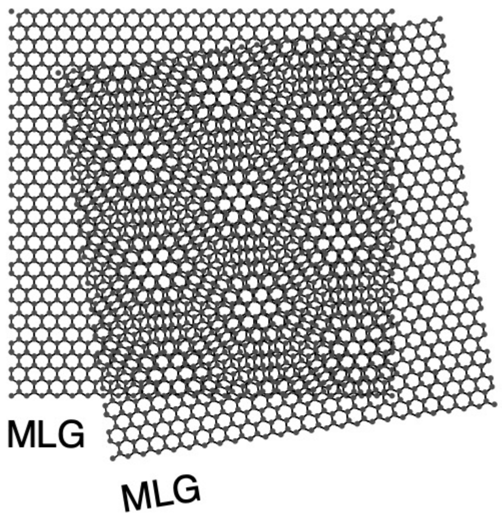
Other Chiral *d+id'* SCs?

- SrPtAs
- $Na_xCoO_2 \bullet yH_2O$
- β-MNCl
- κ-$(BEDT\text{-}TTF)_2X$
- (111) bilayer $SrIrO_3$
- $In_3Cu_2VO_9$
- Twisted (~45°) cuprate bilayers
- …

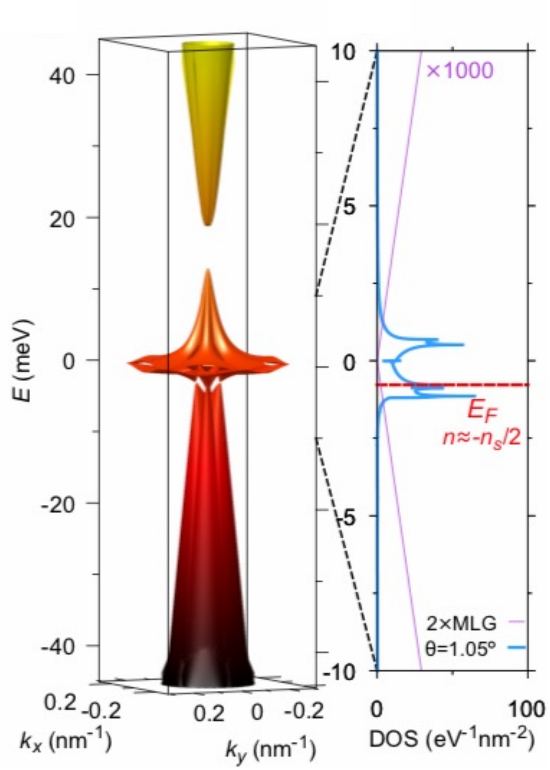
See e.g. review: ABS and Honerkamp JPCM **26**, 423201 (2014)

Twisted Bilayer Graphene

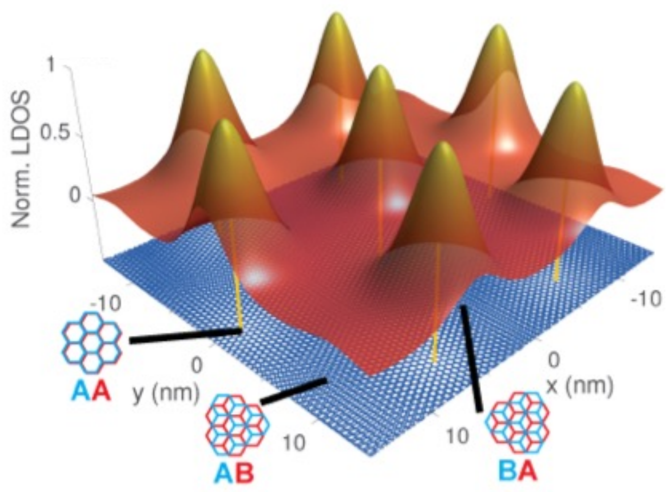
Supercell moiré pattern



Very small "magic" angles
→ **low energy flat bands**



Varying LDOS in moiré cell



Bistritzer&MacDonald, PNAS 108, 12233 (2011); Cao et al, Nature 556, 43 (2018); Nature 556, 80 (2018)

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Why are Flat Bands Interesting?

Locally or regionally flat bands → divergent DOS

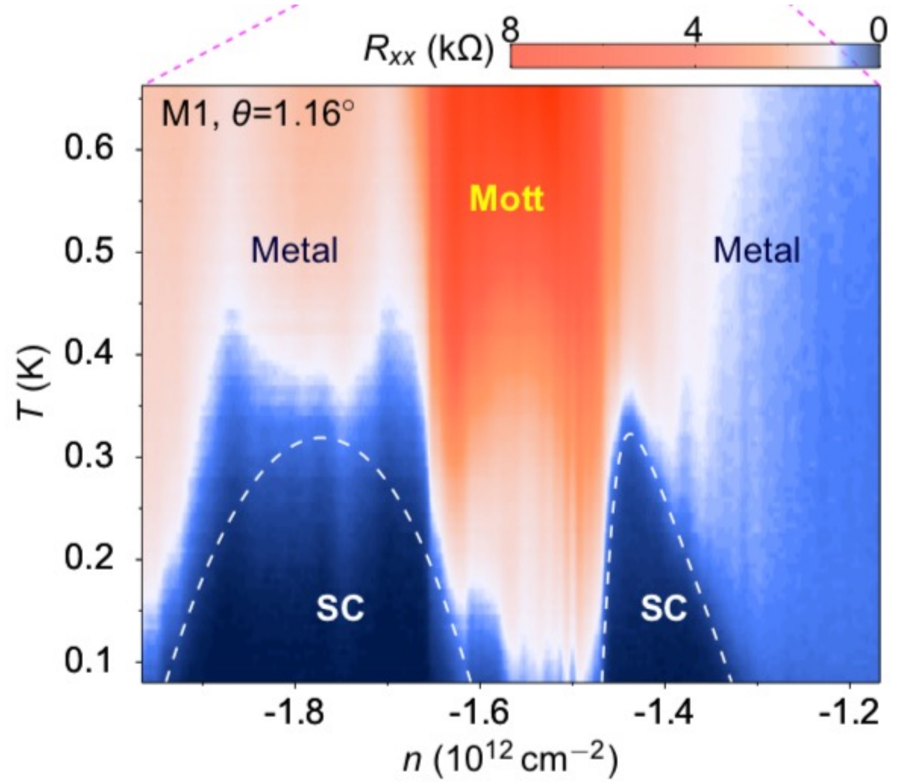
Electronic ordering, even with weak interactions

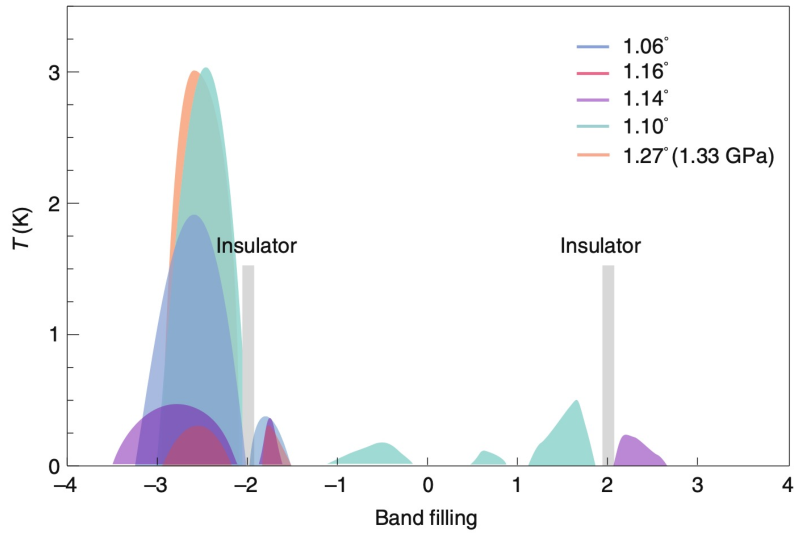
Magnetism (Stoner criterion): $\mathrm{DOS}(E_F)U > 1$

Superconductivity (BCS): $T_c \propto e^{-\frac{1}{\mathrm{DOS}(E_F)V}}$

Superconductivity in TBG

Superconducting domes throughout moiré flat band





Similarities with cuprates:

- Strong coupling, Mott + SC domes, pseudogap state, strange metal phase, …

→ *d*-wave pairing on honeycomb lattice!?

Cao et al, Nature 556, 43 (2018); Balents et al, Nat. Phys. 16, 725 (2020)

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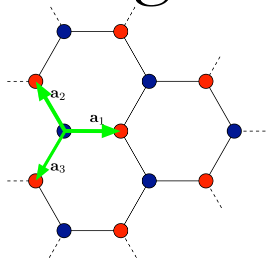
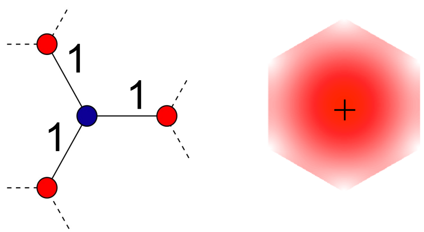
"Cuprate SC" in Graphene?

Cuprates → order parameter on nearest neighbor bonds

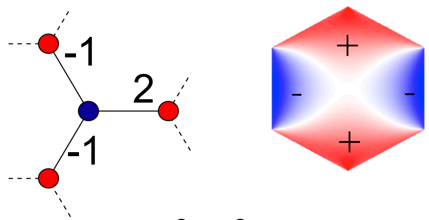
$$\Delta_{ij} = -J\langle s_{ij}\rangle = -J\langle c_{i\downarrow}c_{j\uparrow} - c_{i\uparrow}c_{j\downarrow}\rangle$$



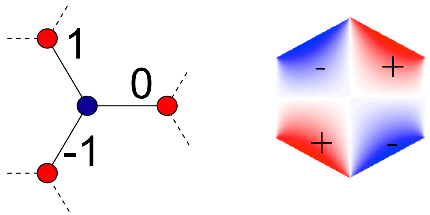
Order parameter symmetries



extended *s*-wave



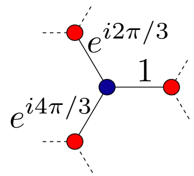
d(x^2-y^2)-wave



d(xy)-wave

Chiral

d(x^2-y^2)+i*d*(xy)



Cp. Cuprates:

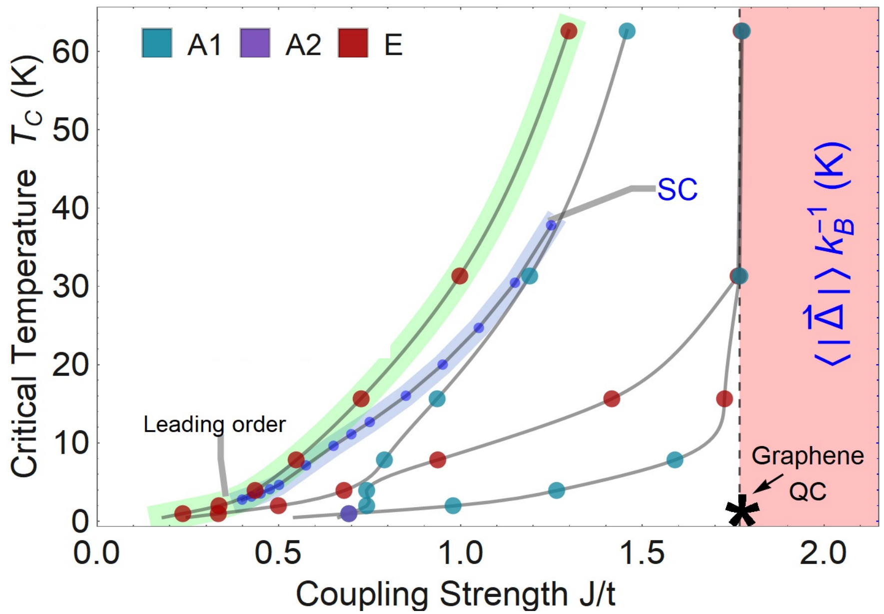
d(x^2-y^2)

on square lattice

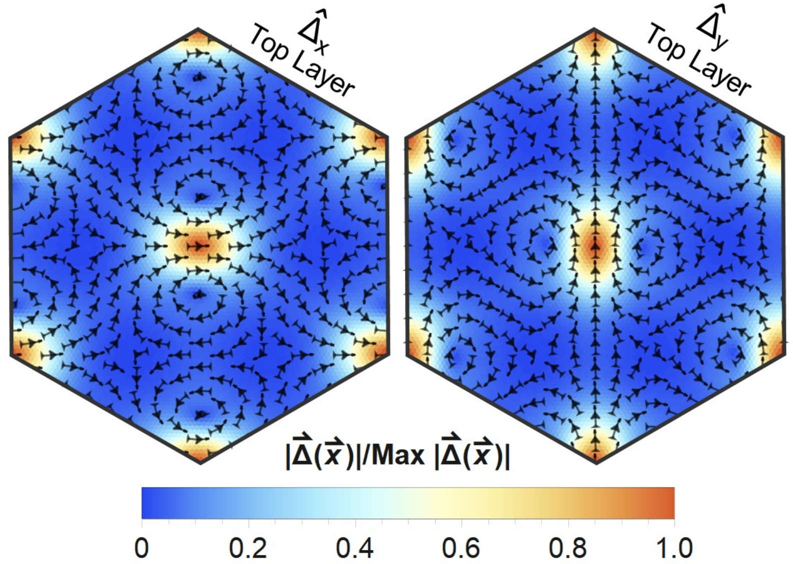
ABS and Doniach PRB 75, 134512 (2007)

Superconduting State in TBG

T_c at magic angle with doping at DOS peak



Two-fold solution: $\hat{\Delta}_x, \hat{\Delta}_y$



- Highest T_c for two-fold degenerate solution
- Very low J for realistic T_c at magic angle

- Peaks in AA regions
- **Moiré-scale nematicity**
  (breaks C_3 rotation)

Löthman, Schmidt, Parhizgar, ABS, Commun. Phys. 5, 92 (2022)

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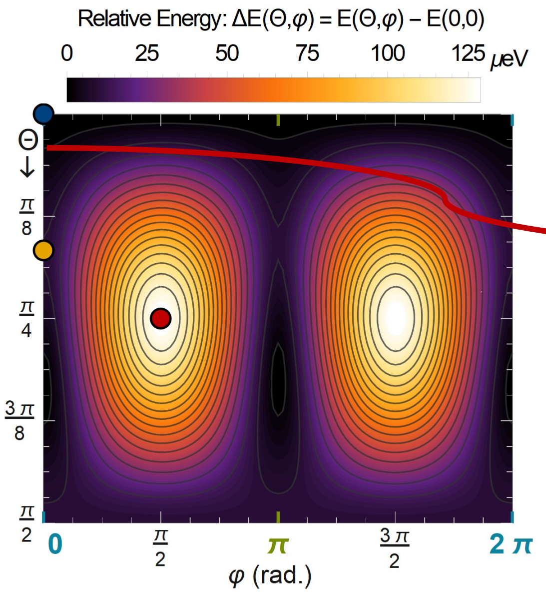
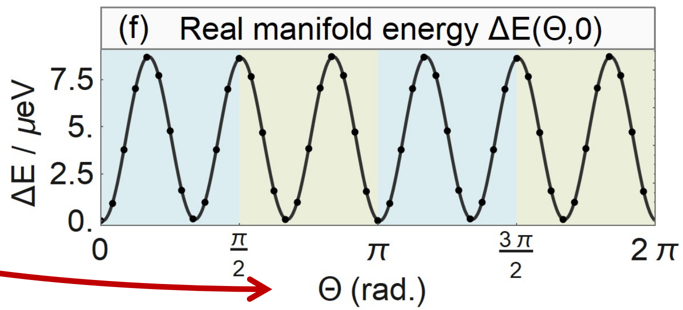
Moiré-scale Nematicity

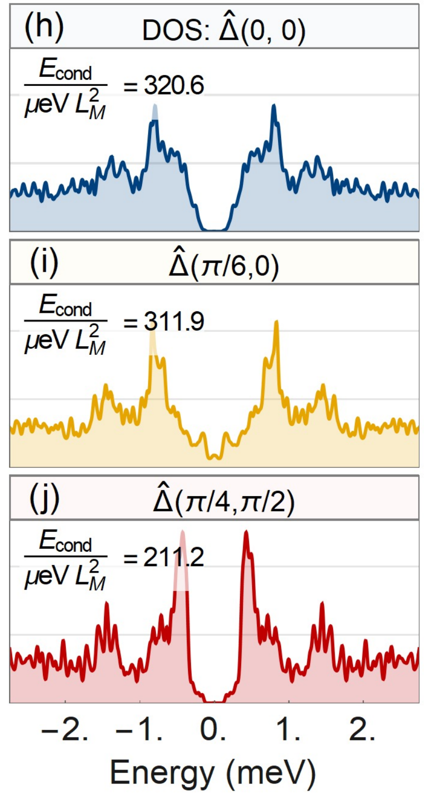
At T_c all linear combinations are solutions: $\hat{\Delta}(\Theta,\varphi) = \|\hat{\Delta}\| \left(\cos\Theta\hat{\Delta}_x + e^{i\varphi}\sin\Theta\hat{\Delta}_y\right)$

At T = 0:





- **3-fold degenerate nematic ground state**
- Real valued
- Chiral solution worst!



Nematic with full gap

Cp. gapped chiral *d*-wave in graphene & nodal *d*-wave in cuprates

Löthman, Schmidt, Parhizgar, ABS, Commun. Phys. 5, 92 (2022)

Atomic-scale *d*-wave Nematicity

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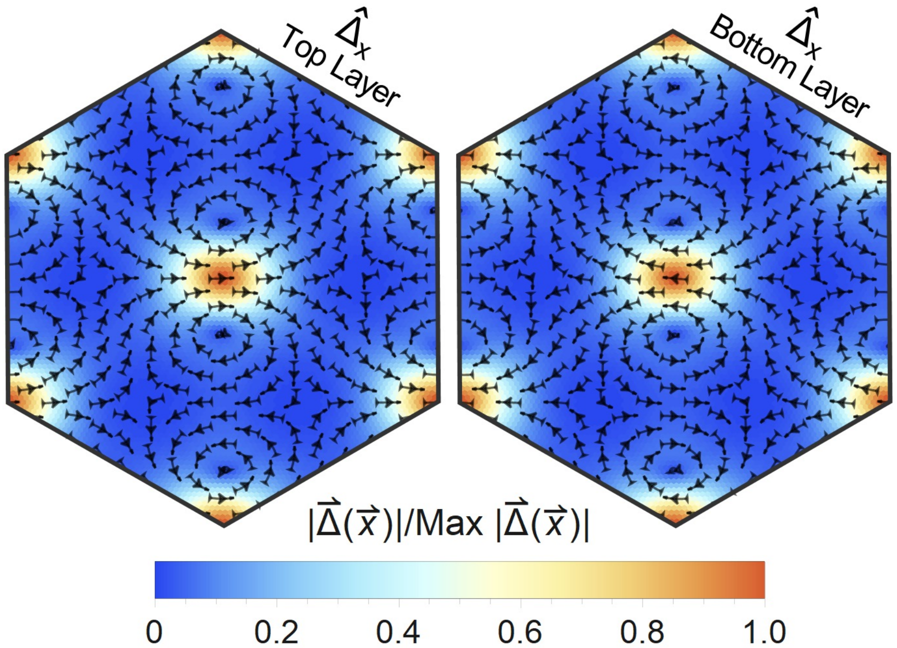
Decompose order on bonds: $\Delta(\vec{x}_i) = |\Delta(\vec{x}_i)| \left(\cos\tau(\vec{x}_i) f_{d_{x^2-y^2}} + \sin\tau(\vec{x}_i) f_{d_{xy}} \right)$

Vector field for *d*-wave order: $\vec{\chi}(\vec{x}_i) = \cos\tau(\vec{x}_i)\hat{x} + \sin\tau(\vec{x}_i)\hat{y}$

d-wave form factors



- **Atomic-scale *d*-wave nematicity**
- Aligned with moiré-scale nematicity in AA regions
- Vortex structure outside AA regions

Cuprate → Nodal *d*-wave

Graphene → Gapped chiral *d*+i*d*'-wave

Twisted bilayer graphene →
Gapped inhomogeneous (nematic) *d*-wave

Löthman, Schmidt, Parhizgar, ABS, Commun. Phys. 5, 92 (2022)

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Topological Superconductivity

Chiral superconductors

Spin-singlet d+id'-wave (spin-triplet p+ip'-wave) superconductors

Appears often for 2D irreps
Fully gapped bulk
Finite Chern number $\mathcal{N}$, set by phase winding of Δ
Chiral edge states crossing bulk gap, # = $\mathcal{N}$
Breaks TRS, preserves at least S_z symmetry

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Topological Superconductivity

Chiral superconductors

Spin-singlet *d*+i*d'*-wave (spin-triplet *p*+i*p'*-wave) superconductors

Spinless superconductors

Majorana fermions

Engineered systems

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“Spinless” *p+ip*’ Superconductor

- Spinless superconductor → *p*-wave pairing
- No known intrinsic “spinless” SC
- Multiple proposals for engineered “spinless” *p+ip*’ superconductors last ~ 10 years
  - 1D spinless $\Delta \sim k$ (class BDI)
  - 2D spinless $\Delta \sim k_x + ik_y$ (class D)

Can be **topological superconductors**

Topological boundary states are **Majorana Fermions (MFs)**

1D: Localized zero-energy end states

2D: dispersive edge modes or localized zero-energy vortex states

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Schrödinger, Dirac, and Majorana

Schrödinger (1925)

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

relativistically

correct

Dirac (1928)

$$\sum_{\mu=0}^{3} i\hbar\gamma^\mu\partial_\mu\psi = mc\psi$$

4x4 complex matrices

real equation

Majorana (1937)

$$\sum_{\mu=0}^{4} i\hbar\tilde{\gamma}^\mu\partial_\mu\psi = mc\psi$$

4x4 imaginary matrices

- Spin-1/2
- Electron & positron (hole)

$$c^\dagger \neq c$$

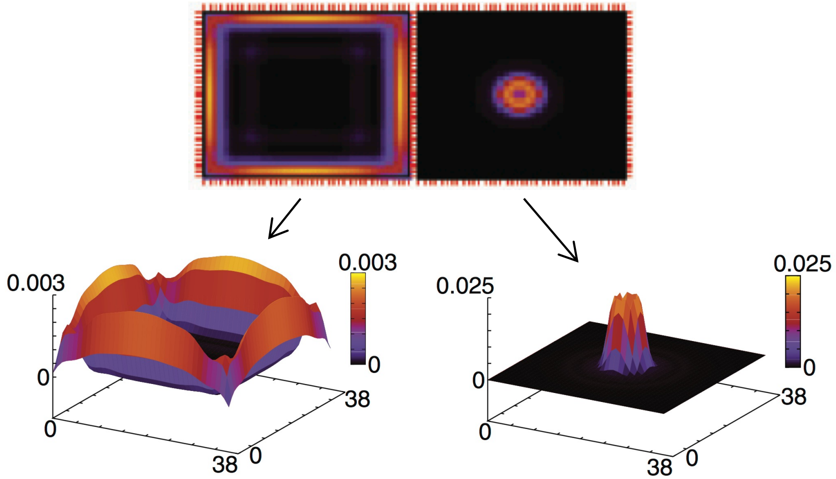
- Particle = Antiparticle: $\gamma^\dagger = \gamma$
- Electron "=" 2 Majorana fermions: $c^\dagger = \gamma_1 + i\gamma_2$

Majorana Fermions

New particle ~ ½ electron

- Emergent particle
- Appears in pairs

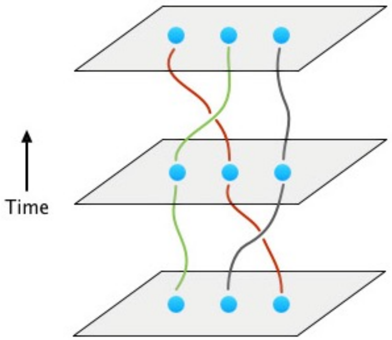
Condensed matter systems

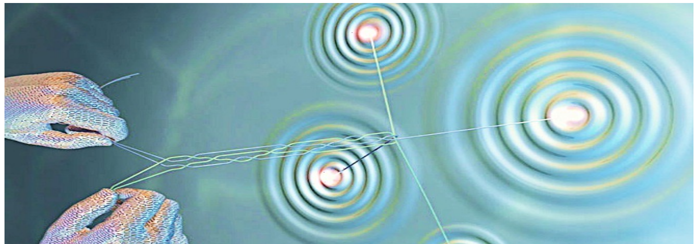


MF on sample edge MF in vortex core

Non-Abelian statistics in 2D

→ **Robust quantum computation by braiding**





Sci. Am. **294**, 56 (2013)

Quantum gate operation
= particle braiding

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Excitations in Superconductors

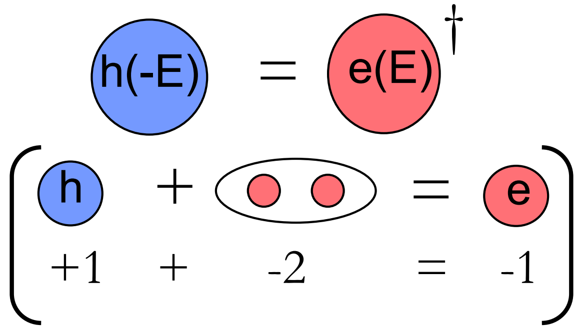
Quasiparticles in a superconductor:

- Part electron and part hole
- Mixed spin-up and spin-down

$$\begin{cases} a = uc_\uparrow^\dagger + vc_\downarrow \\ a^\dagger = u^* c_\uparrow + v^* c_\downarrow^\dagger \end{cases}$$

$$|v_\mathbf{k}|^2 = 1 - |u_\mathbf{k}|^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_\mathbf{k}}{E_\mathbf{k}}\right)$$

→ E = 0 states are Majorana fermions: $\gamma^\dagger = \gamma$ (if we ignore spin)



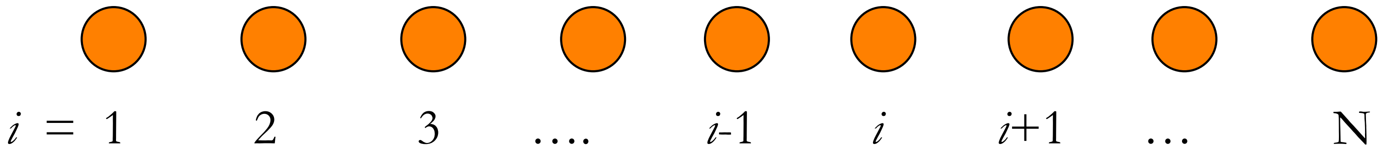
But …

- Superconductors often have an energy gap
  - Topological SCs have E = 0 boundary states
- E = 0 states are often spin-degenerate (2 Majorana → 1 electron)

→ "Spinless" topological superconductor

Kitaev's 1D Toy Model

1D chain of spinless electrons with superconducting pairing



$$H = -\mu \sum_i c_i^\dagger c_i - \frac{1}{2} \sum_i t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \mathrm{H.c.}$$

Chemical potential — Nearest neighbor hopping — Spinless p-wave pairing

Kitaev, arXiv:cond-mat/0010440 (2001)

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Majorana Basis

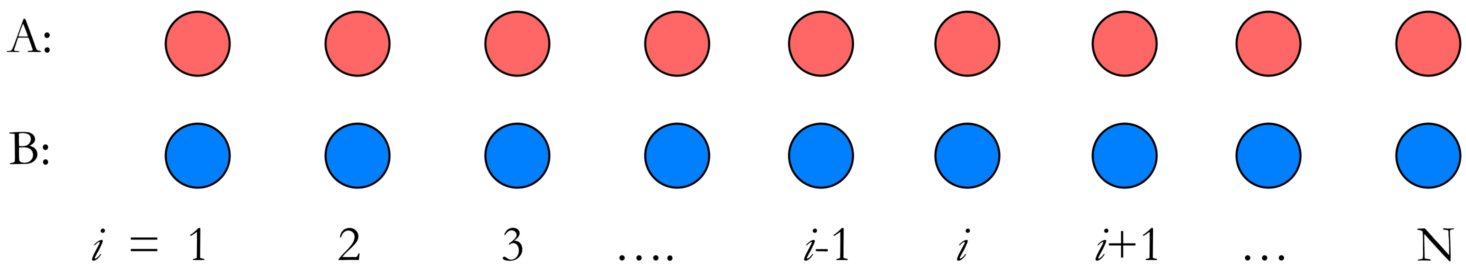
$$H = -\mu \sum_i c_i^\dagger c_i - \frac{1}{2} \sum_i t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \text{H.c.}$$

$$c_i = \frac{1}{2}(\gamma_i^B + i\gamma_i^A) \quad \Rightarrow \quad \begin{aligned} (\gamma_i^\alpha)^\dagger &= \gamma_i^\alpha \\ \{\gamma_i^\alpha, \gamma_j^\beta\} &= 2\delta_{ij}\delta_{\alpha\beta} \end{aligned}$$

Majorana fermions

Change basis

$$H = -\frac{\mu}{2} \sum_{i=1}^{N} (1 + i\gamma_i^B \gamma_i^A) - \frac{i}{4} \sum_{i=1}^{N-1} \left[(\Delta + t)\gamma_i^B \gamma_{i+1}^A + (\Delta - t)\gamma_i^A \gamma_{i+1}^B \right]$$



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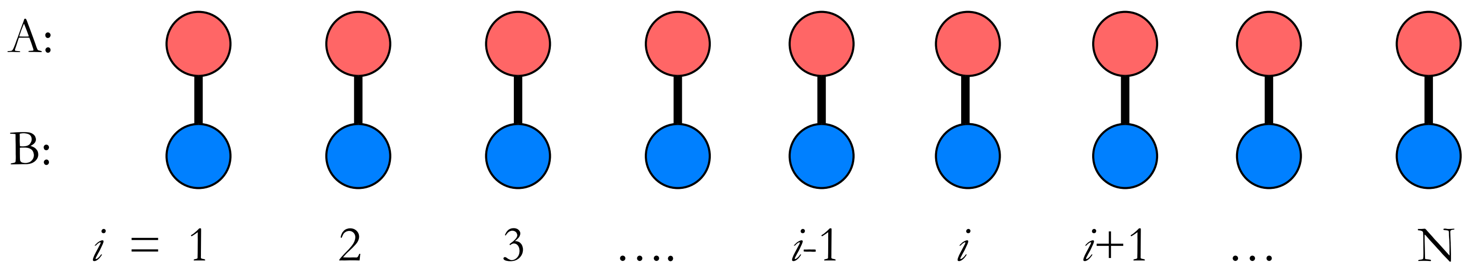
Trivial Phase

$$H = -\frac{\mu}{2}\sum_{i=1}^{N}(1 + i\gamma_i^B\gamma_i^A) - \frac{i}{4}\sum_{i=1}^{N-1}\left[(\Delta + t)\gamma_i^B\gamma_{i+1}^A + (\Delta - t)\gamma_i^A\gamma_{i+1}^B\right]$$

Topological trivial phase: $\Delta = t = 0$, $\mu < 0$



Unique ground state

- Vacuum state for electrons
- Bulk gap (|μ| lowest excitation energy)

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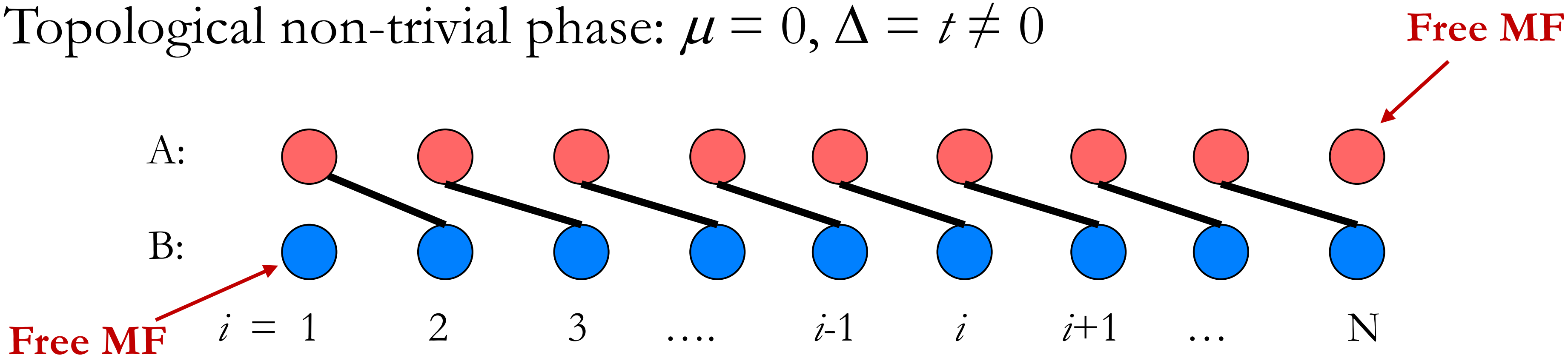
Non-Trivial Phase

$$H = -\frac{\mu}{2}\sum_{i=1}^{N}(1 + i\gamma_i^B\gamma_i^A) - \frac{i}{4}\sum_{i=1}^{N-1}\left[(\Delta + t)\gamma_i^B\gamma_{i+1}^A + (\Delta - t)\gamma_i^A\gamma_{i+1}^B\right]$$

Topological non-trivial phase: $\mu = 0, \Delta = t \neq 0$



$$\begin{cases} d_i = \frac{1}{2}(\gamma_{i+1}^A + i\gamma_i^B) \\ f = \frac{1}{2}(\gamma_1^A + i\gamma_N^B) \end{cases} \Longrightarrow H = t\sum_{i}^{N-1}(d_i^\dagger d_i - \frac{1}{2})$$

Degenerate ground state

- Bulk gap (t)
- Zero-energy MFs at end points

Majorana Fermions in BdG

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How can we get "½ electron" in the BdG formalism?

Never if $\epsilon_{\mathbf{k}} \sum_{\sigma} c^{\dagger}{}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \left[\bar{\Delta} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^{\dagger}{}_{\mathbf{k}\uparrow} c^{\dagger}{}_{-\mathbf{k}\downarrow} \Delta\right] = \left(c^{\dagger}{}_{\mathbf{k}\uparrow},\ c_{-\mathbf{k}\downarrow}\right) \begin{bmatrix} \epsilon_{\mathbf{k}} & \Delta \\ \bar{\Delta} & -\epsilon_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{\dagger}{}_{-\mathbf{k}\downarrow} \end{pmatrix}$

→ Not in spindegenerate (e.g. chiral *p*+i*p'* or *d*+i*d'*) superconductors

But if $\tilde{\mathcal{H}} = \mathbf{a}_{\mathbf{k}}^{\dagger} \begin{pmatrix} \varepsilon(\mathbf{k})\sigma_0 & \hat{\Delta}(\mathbf{k}) \\ \hat{\Delta}^{\dagger}(\mathbf{k}) & -\varepsilon(\mathbf{k})\sigma_0 \end{pmatrix} \mathbf{a}_{\mathbf{k}}$ $\quad \left[\mathbf{a}_{\mathbf{k}} = (a_{\mathbf{k}\uparrow}, a_{\mathbf{k}\downarrow}, a^{\dagger}_{-\mathbf{k}\uparrow}, a^{\dagger}_{-\mathbf{k}\downarrow})\right]$

1 electron represented by 2 vector components

→ MF if E=0 eigenstate has no spatial overlap with other states

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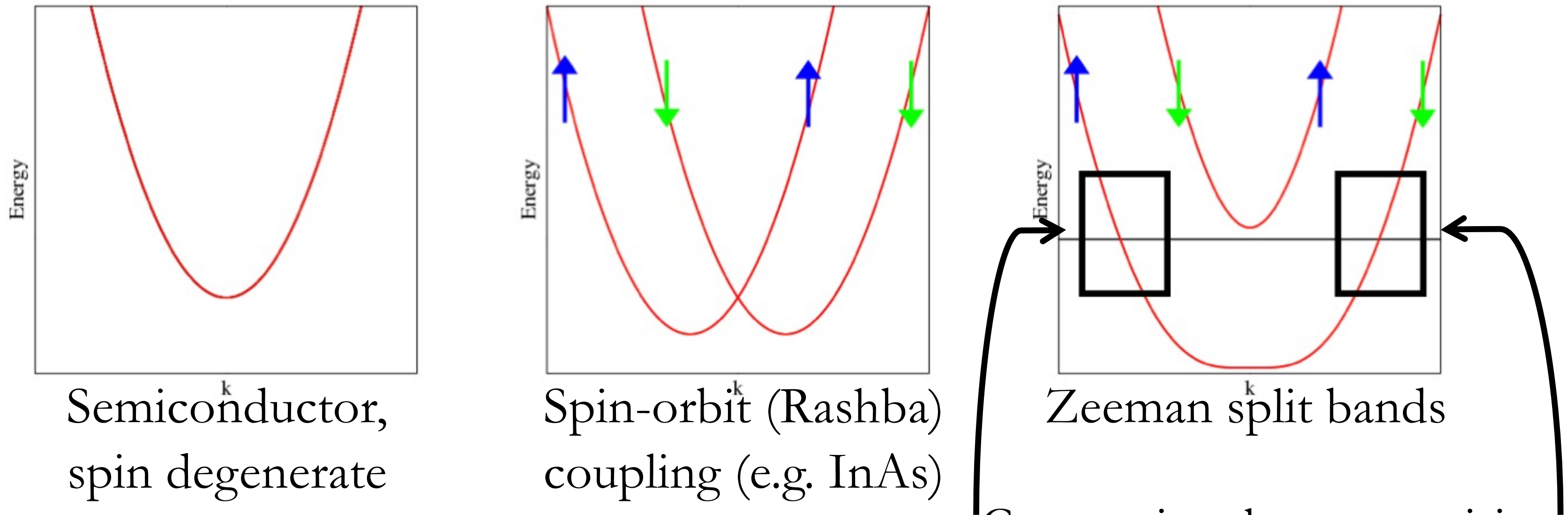
SOC Semiconductors

Spin-orbit coupled (SOC) semiconductor + magnetic field



Semiconductor, spin degenerate

Spin-orbit (Rashba) coupling (e.g. InAs)

Zeeman split bands

Conventional *s*-wave pairing

4x4 BdG description needed due to SOC + Zeeman field

Spinless '*p*+i*p*' superconductor with MFs if $|V_z| > |\Delta|$

2D: Sau *et al.* PRL **104**, 040502 (2010). 1D: Lutchyn *et al.* PRL **105**, 077001 (2010), Oreg *et al.* PRL **104**, 077002 (2010)

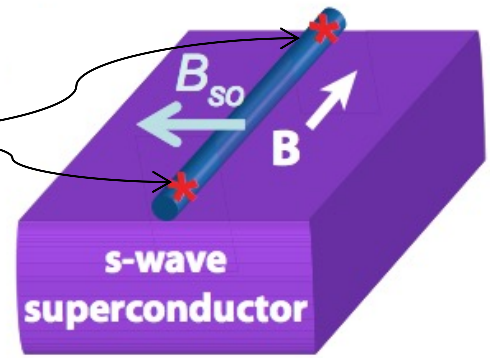
Experimental Hunt in Nanowires

1D InSb nanowire
(Semiconductor with strong SOC)

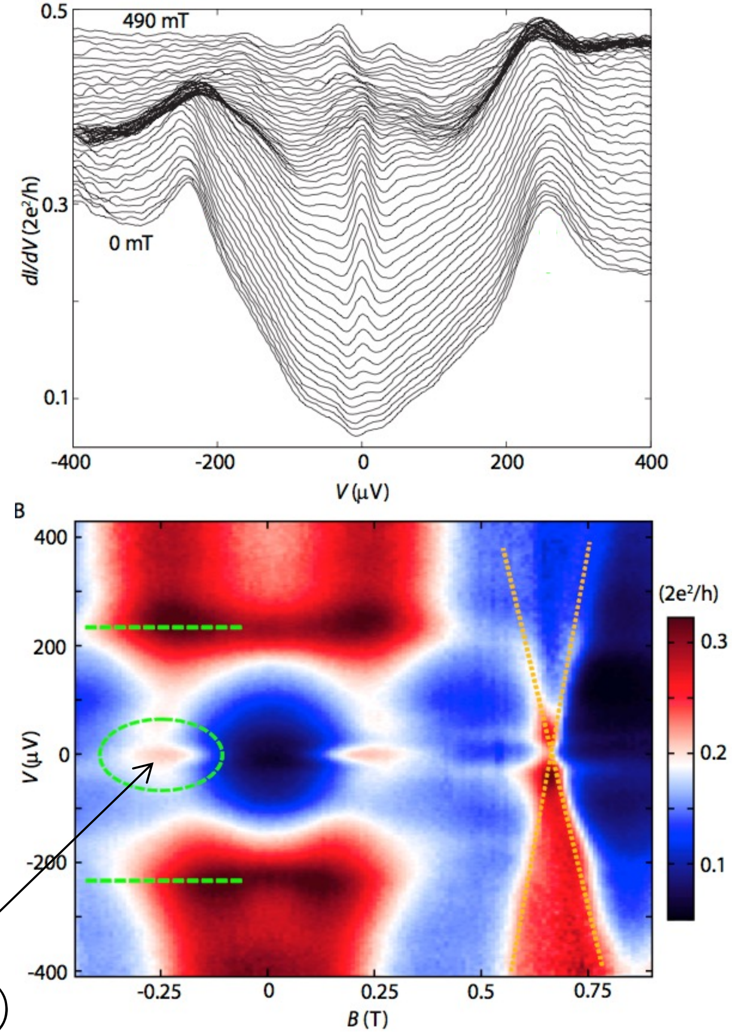
\+ *s*-wave superconductor

\+ Magnetic field

MFs?



Conductance through wire



MFs (?)

Mourik *et al.*, Science **336**, 1003 (2012)

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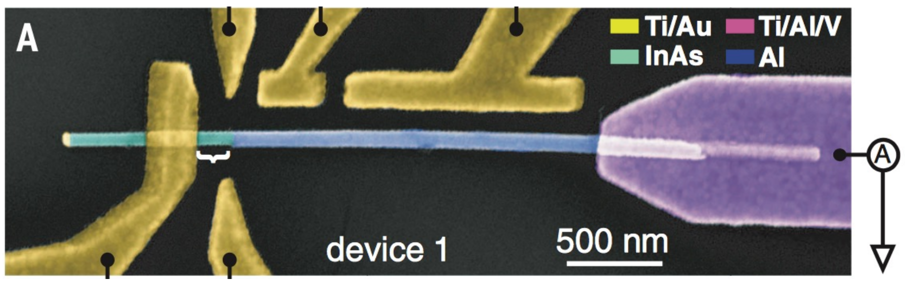
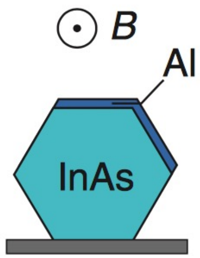
Nanowires with Hard Gaps

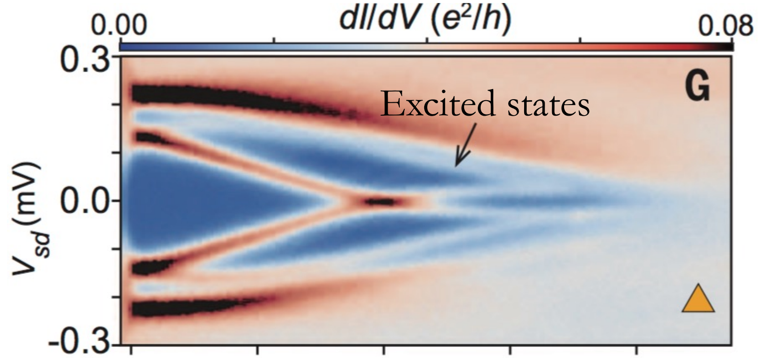
1D InAs nanowire
+ Al superconductor
+ Magnetic field

} Hard SC gap

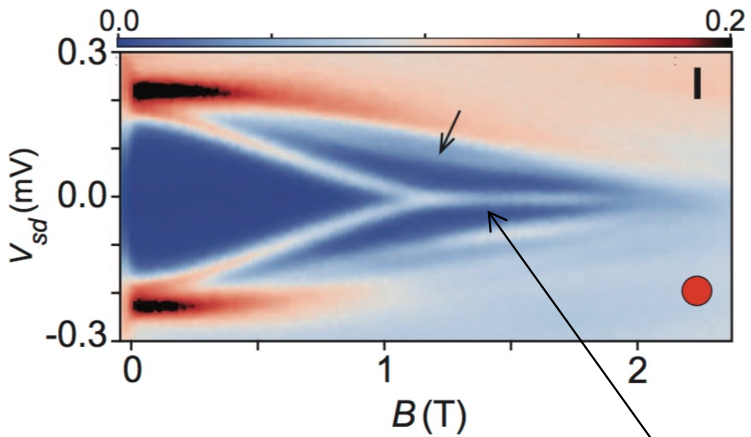




Conductance at different gate biases



Only Andreev bound states



Topological phase with MF (?)

Deng *et al.*, Science **354**, 1557 (2016)

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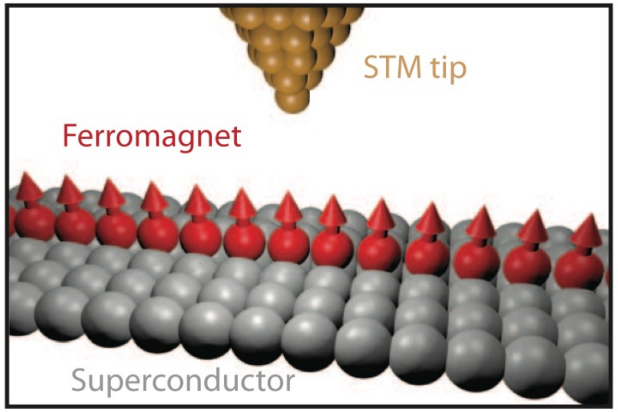
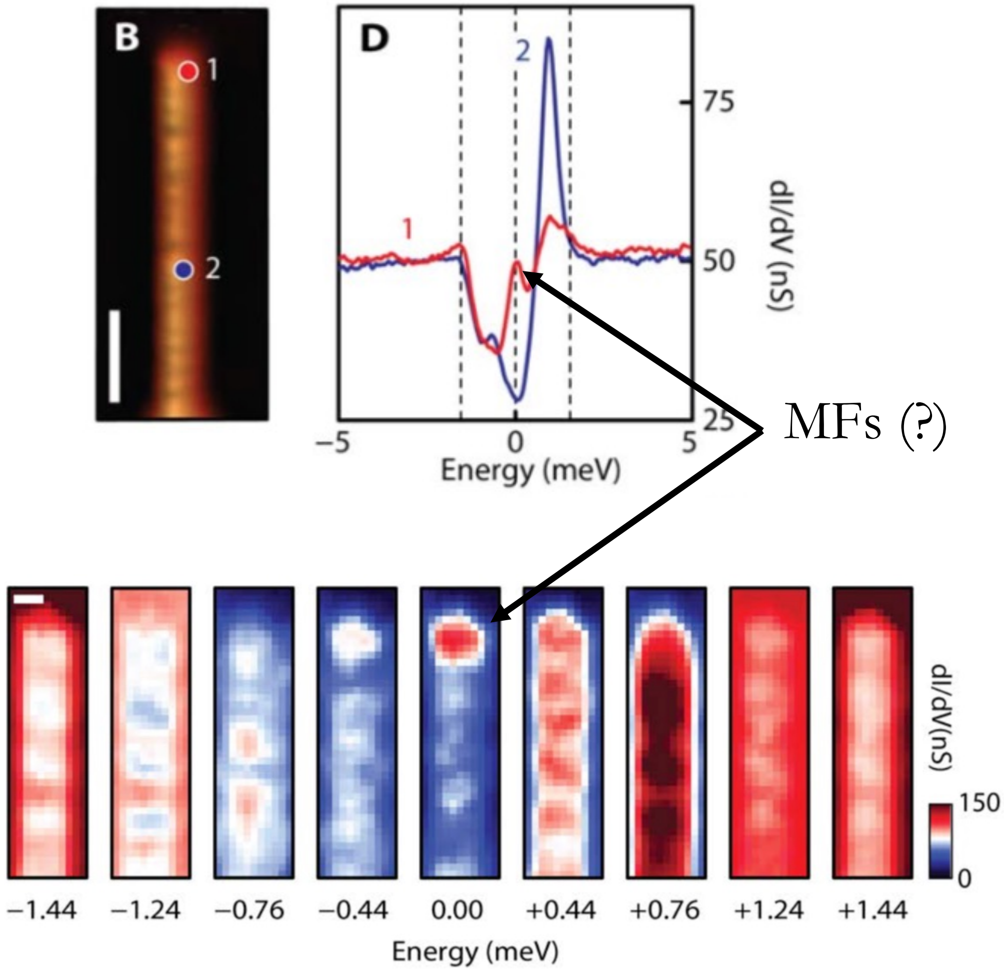
Hunt with Magnetic Atoms

Pb substrate
(SC with strong SOC)
+ Fe ad-atoms





Also: MFs with predicted spin-polarization

Nadj-Perge *et al.*, Science **346**, 602 (2014), Jeon *et al.*, Science **358**, 772 (2017)

Magnetic Atoms on Superconductors

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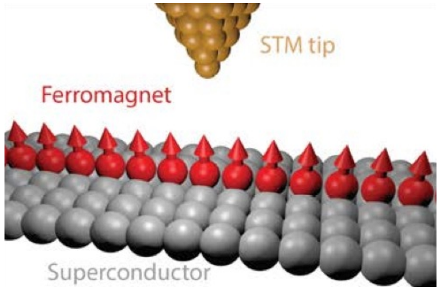
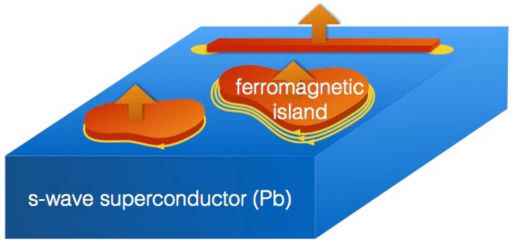
Magnetic atoms on a SOC superconductor





$$\mathcal{H} = \mathcal{H}_{kin} + \mathcal{H}_{SO} + \mathcal{H}_{sc} + \mathcal{H}_{V_z}$$

SOC superconductor

$$\mathcal{H}_{kin} = -t \sum_{\langle \mathbf{i},\mathbf{j}\rangle,\sigma} c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} - \mu \sum_{\mathbf{i},\sigma} c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma}$$

$$\mathcal{H}_{SO} = -\frac{\lambda}{2} \sum_{\mathbf{i}} \Big[(c^\dagger_{\mathbf{i}-\hat{x}\downarrow} c_{\mathbf{i}\uparrow} - c^\dagger_{\mathbf{i}+\hat{x}\downarrow} c_{\mathbf{i}\uparrow}) + i(c^\dagger_{\mathbf{i}-\hat{y}\downarrow} c_{\mathbf{i}\uparrow} - c^\dagger_{\mathbf{i}+\hat{y}\downarrow} c_{\mathbf{i}\uparrow}) + \text{H.c.} \Big]$$

$$\mathcal{H}_{sc} = \sum_{\mathbf{i}} \Delta_{\mathbf{i}} (c^\dagger_{\mathbf{i}\uparrow} c^\dagger_{\mathbf{i}\downarrow} + \text{H.c.})$$

Magnetic atoms on sites **a**

(to 1st approximation)

$$\mathcal{H}_{V_z} = - \sum_{\mathbf{a},\sigma,\sigma'} (V_z(\mathbf{a}) \hat{\mathbf{n}} \cdot \boldsymbol{\sigma})_{\sigma\sigma'} c^\dagger_{\mathbf{a}\sigma} c_{\mathbf{a}\sigma'}$$

Nadj-Perge *et al.*, Science **346**, 6209 (2014), Li *et al.*, Nat. Commun. **7**, 12297 (2016)

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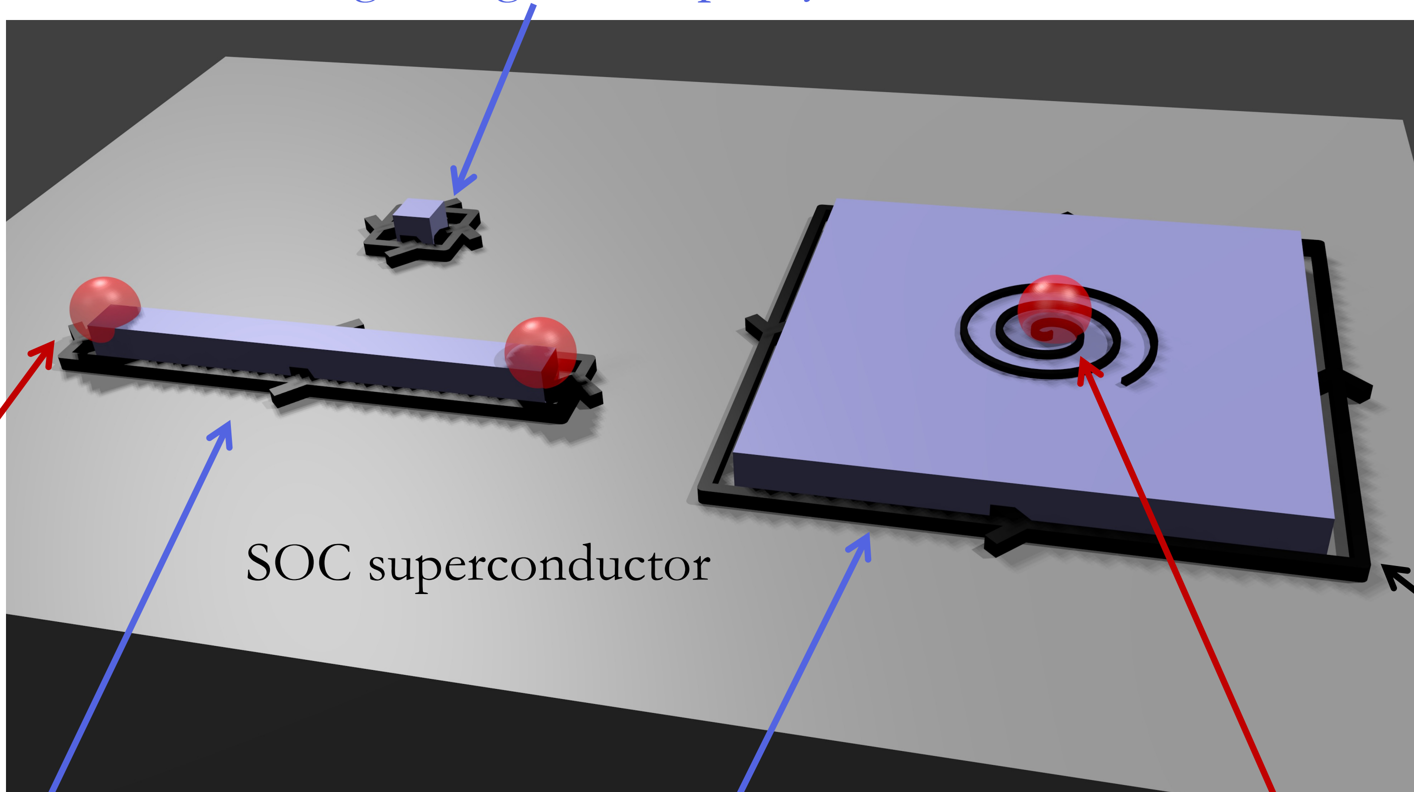
Flexible Setup

Self-consistent solution for the superconducting order parameter

$\left(\Delta_{\mathbf{i}} = -V_{sc}\langle c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow}\rangle\right)$

Single magnetic impurity



MF wire end state

1D ferromagnetic wire

2D ferromagnetic island

MF vortex core state

Circulating currents

PRL 115, 116602 (2015)

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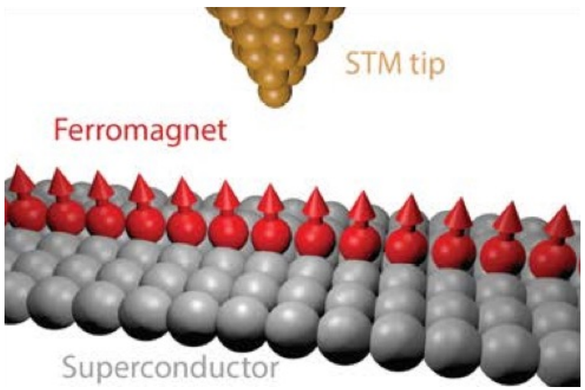
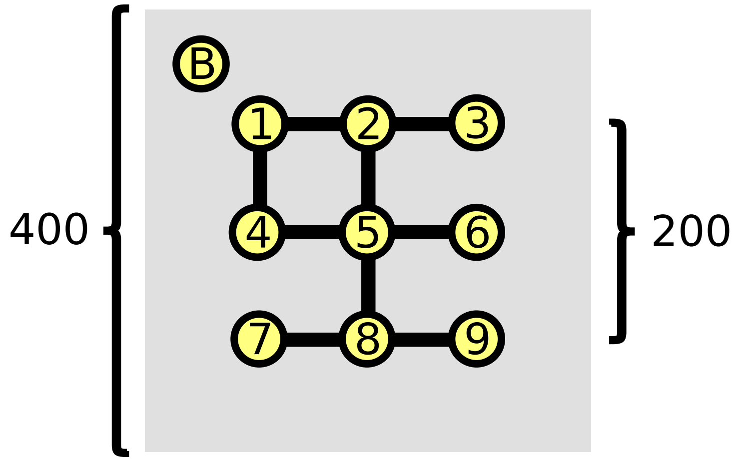
Magnetic Impurity Wire Networks

Are there simple, but unique, signals of MFs?





Ferromagnetic Atom Wire Networks

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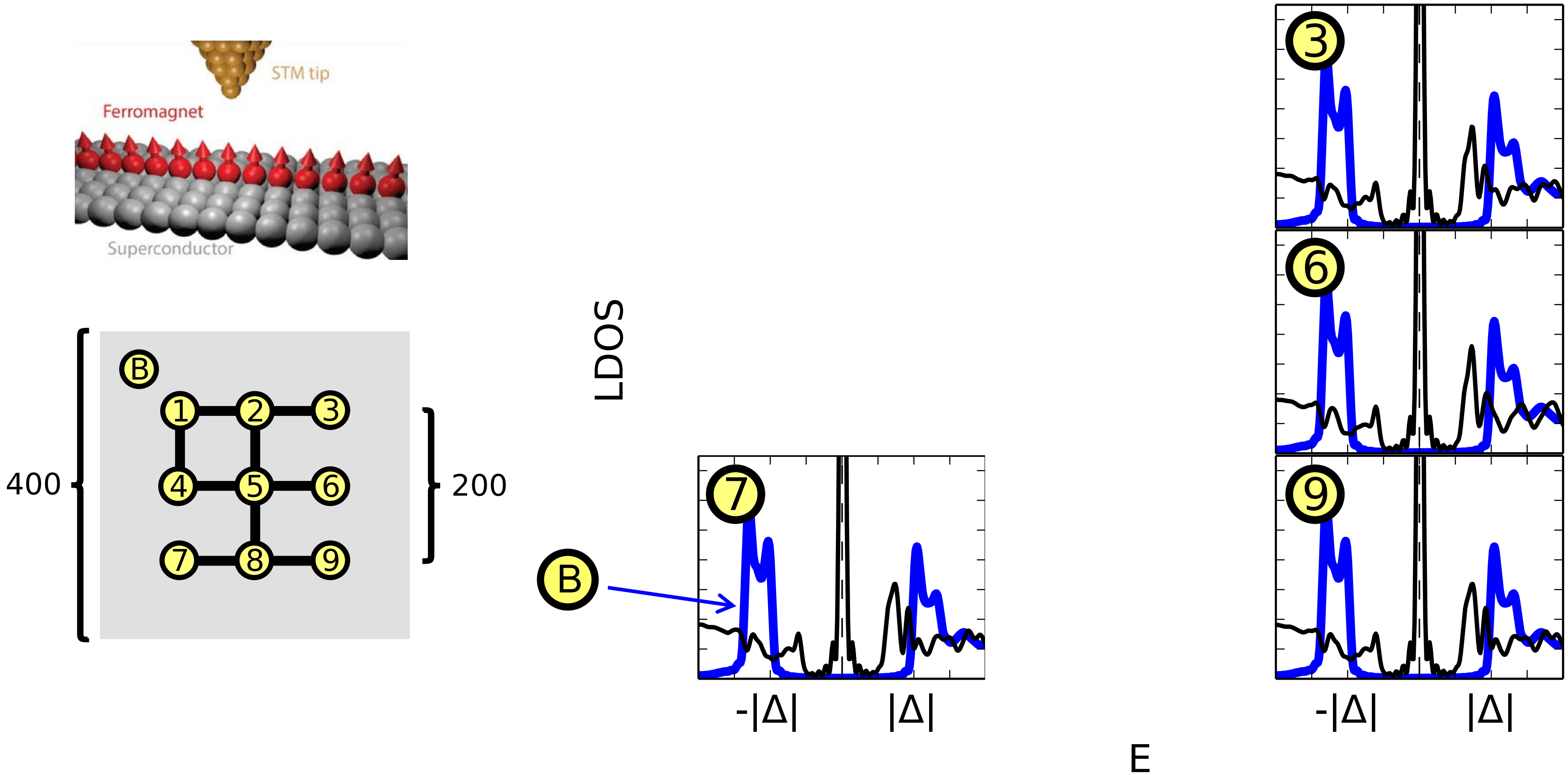
Wire network of ferromagnetic atoms on SOC superconductor

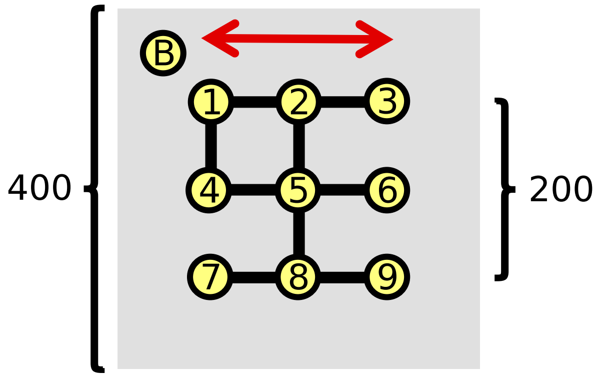


Zero-energy MFs at odd-wire junctions (black)

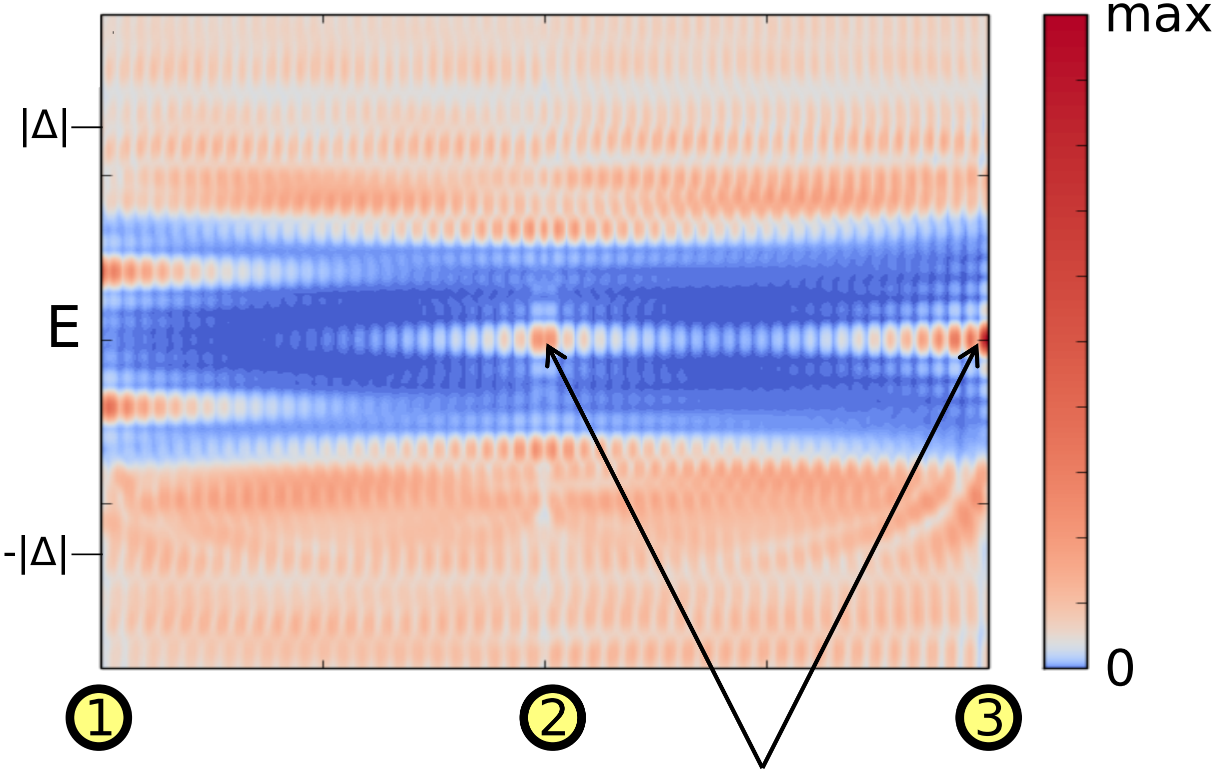
No subgap states at even-wire junctions (red)

Björnson and ABS, PRB **94**, 100501(R) (2016)

Odd- and Even-Wire Junctions



LDOS along upper wire segment



Only clear subgap states are MFs at odd-wire junctions

MF wire end states

Björnson and ABS, PRB **94**, 100501(R) (2016)

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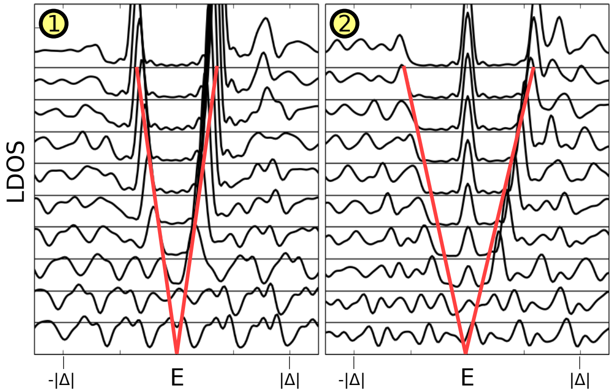
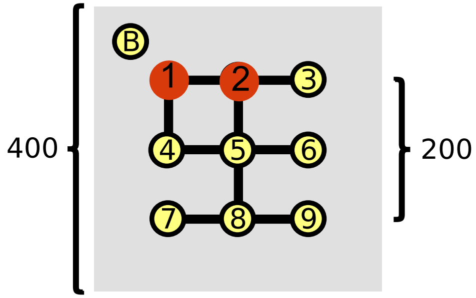
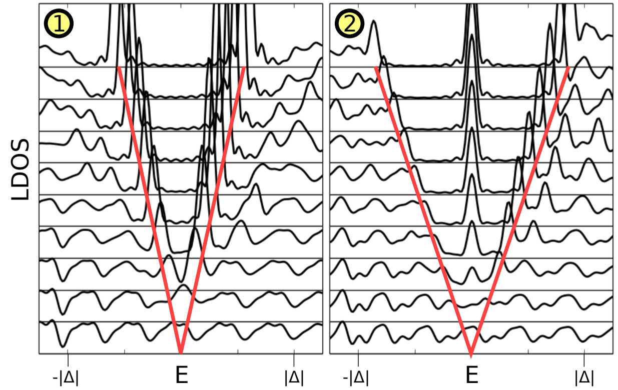
Parameter Dependencies

SOC: increased gap

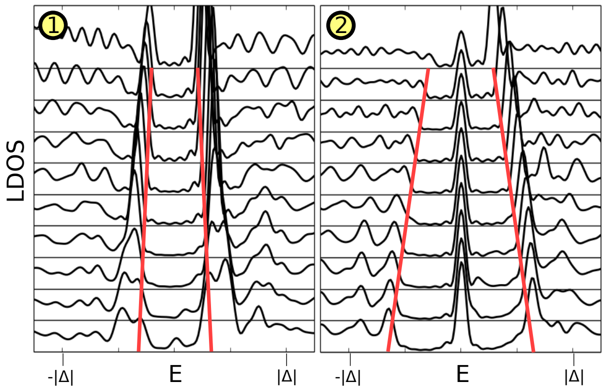




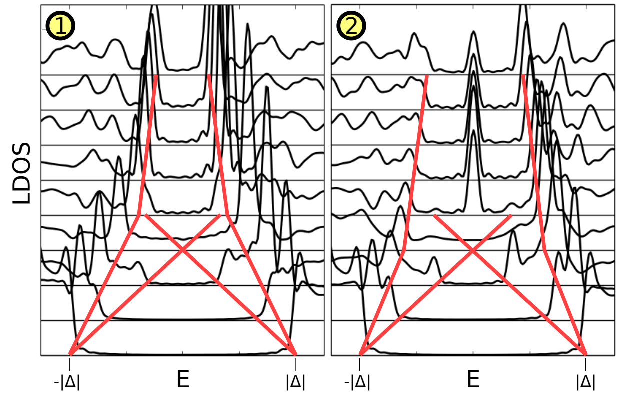
SC order parameter: increased gap



Chemical potential



Magnetic moment: TPT



Large LDOS difference between even- and odd-wire junctions for all parameters

Björnson and ABS, PRB **94**, 100501(R) (2016)

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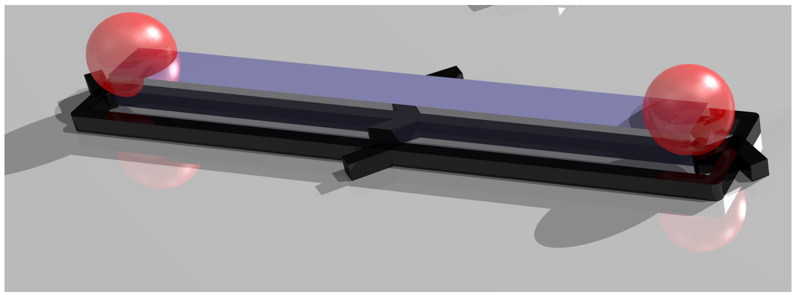
Majorana Oscillations and Localization

How are MFs interacting with other states?

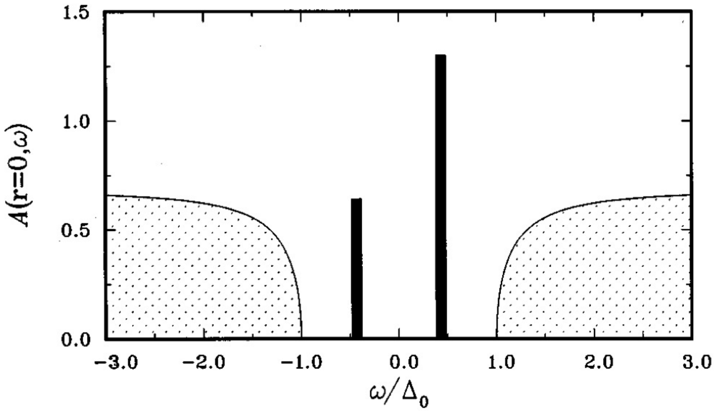


YSR Subgap States

Magnetic impurity (classical spin S = V_z) in *s*-wave SC

- Yu-Shiba-Rusinov (YSR) subgap states

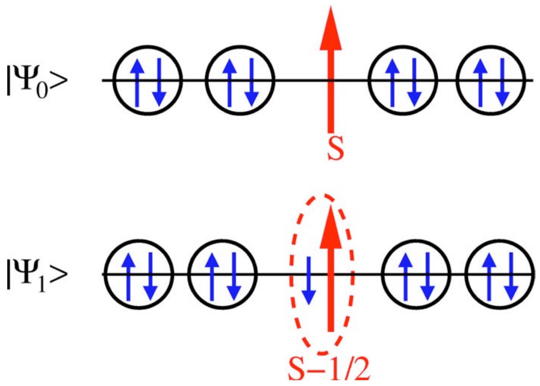
$$E_{\rm YSR}^{\pm} = \pm\Delta \left[1-\left(\frac{\pi\rho S}{2}\right)^2\right] \Big/ \left[1+\left(\frac{\pi\rho S}{2}\right)^2\right]$$



No change in energy levels with SOC

$$E_{\rm YSR}^{\pm} = 0$$

Quantum phase transition (QPT)



Balatsky et al., RMP **78**, 373 (2006)

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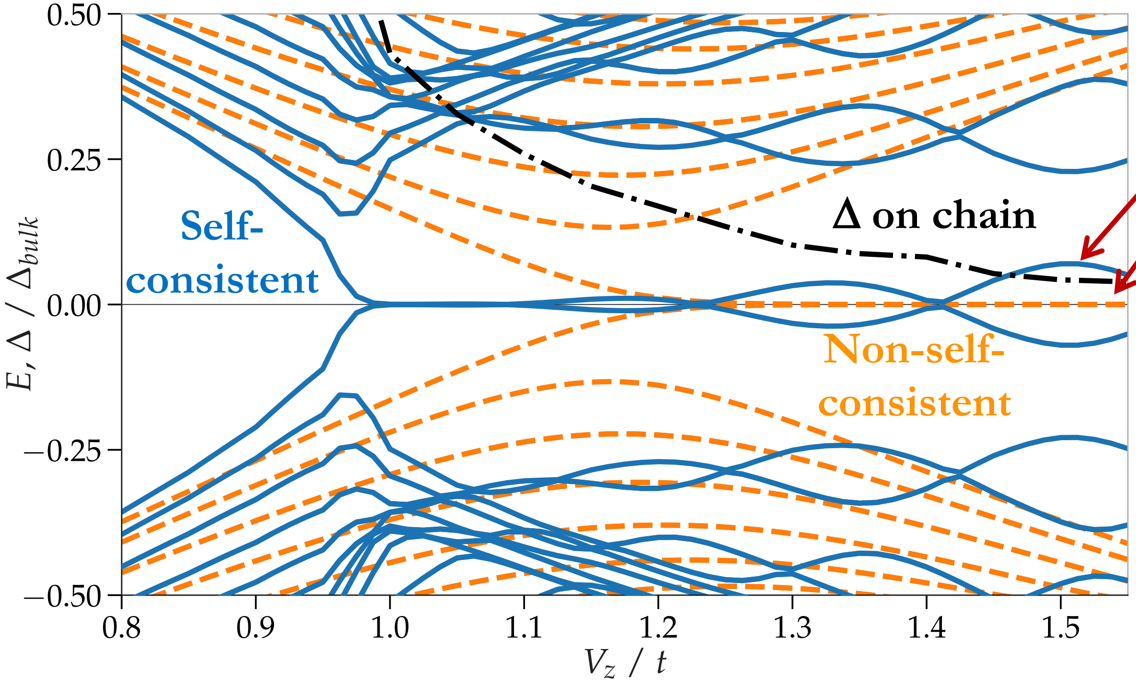
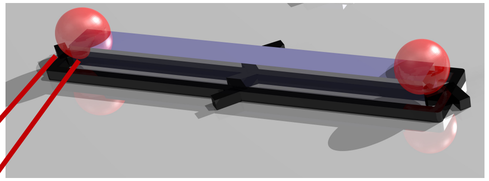
Spectrum for Impurity Chain





MFs?

Self-consistency:

$(\Delta_{\mathbf{i}} = -V_{sc}\langle c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow}\rangle)$

- Δ suppressed on chain sites
- Phase transition at lower V_z
- Energy oscillations
- YSR states lower energies

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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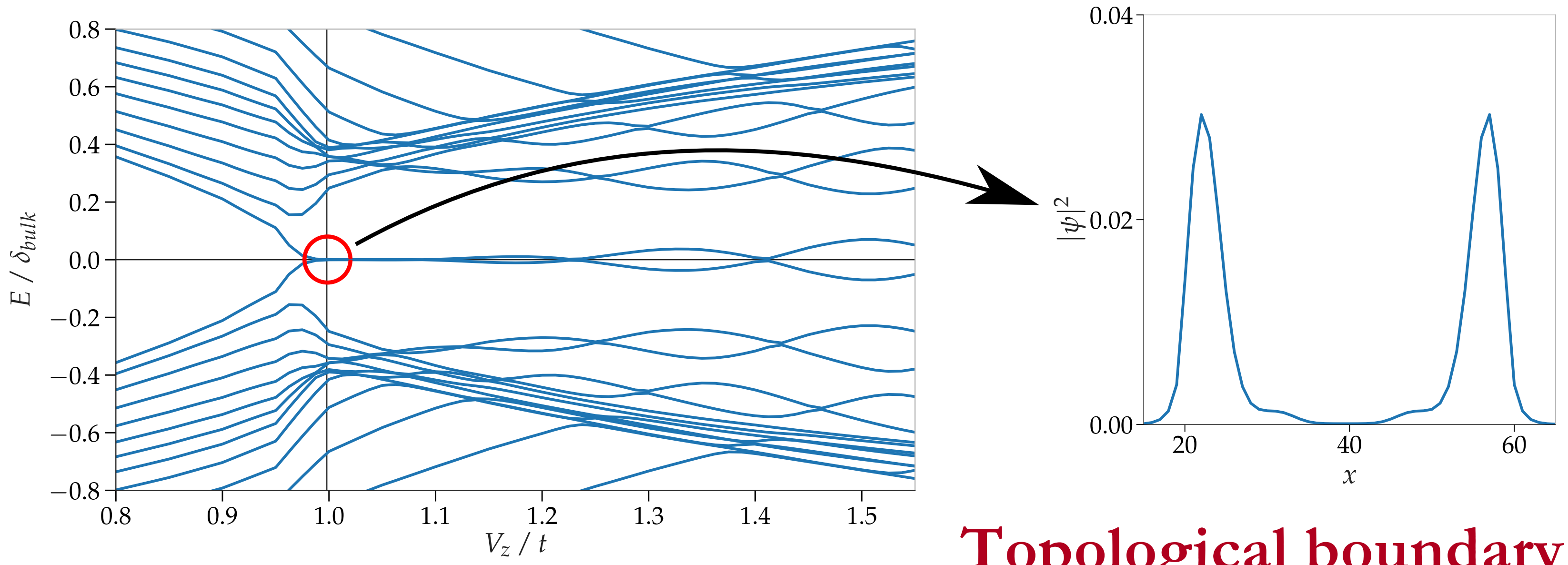
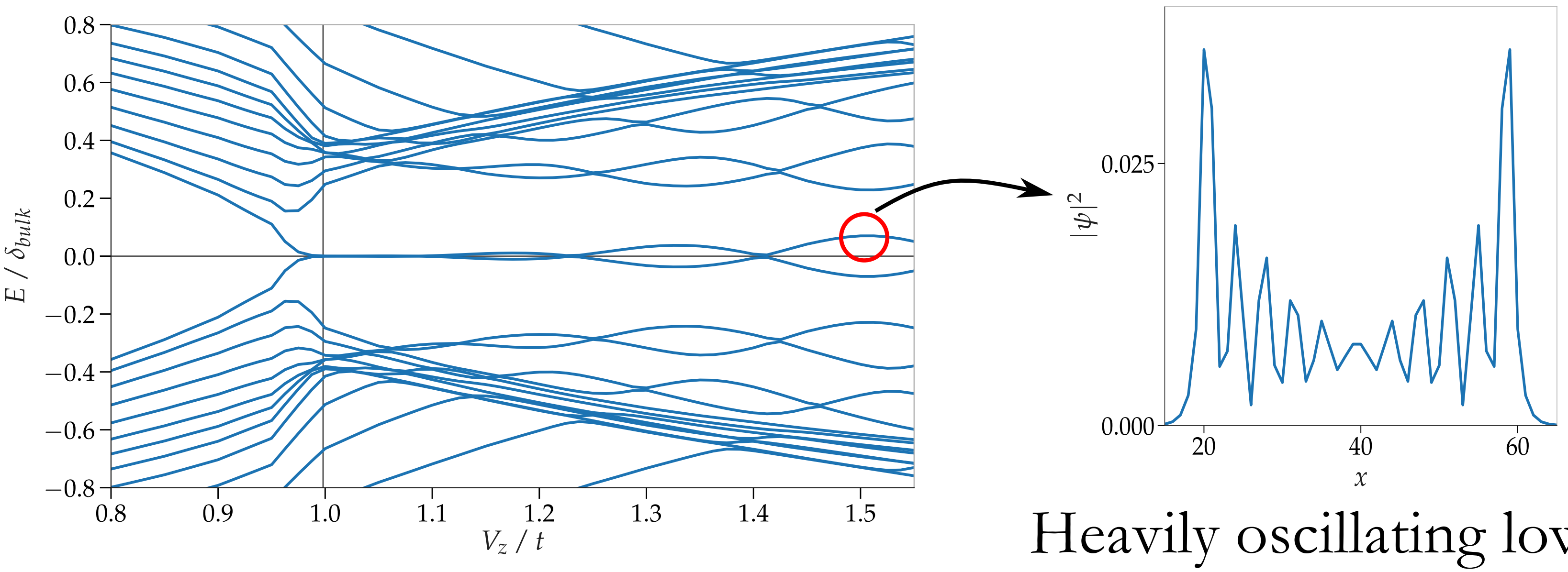
Lowest Energy State



Topological boundary mode
= “MF”

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

Lowest Energy State at High V_z



Heavily oscillating lowest energy state = MF ?

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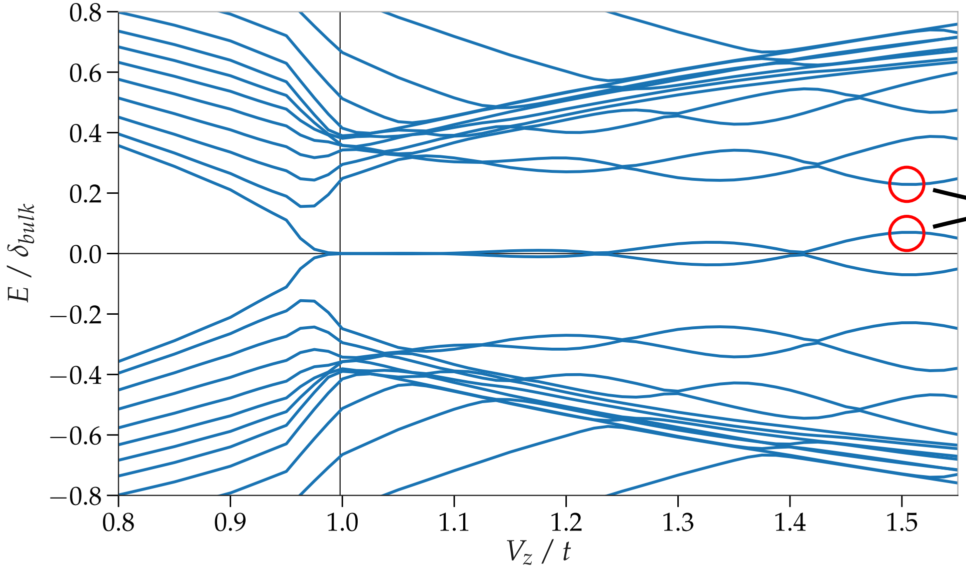
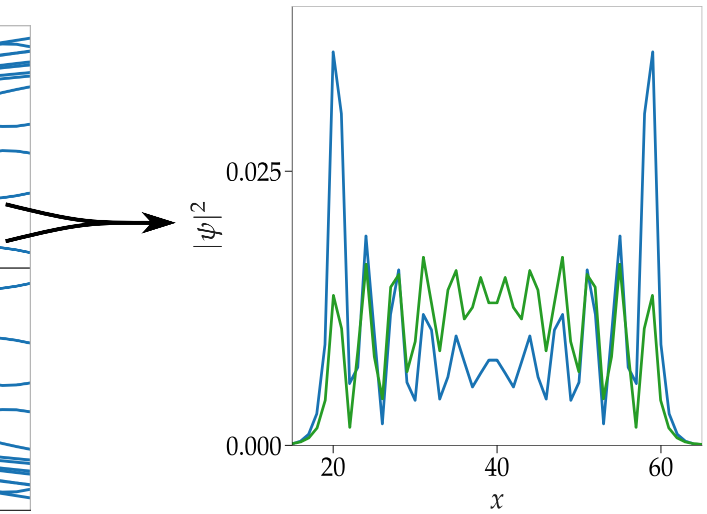
Higher Energy State at High V_z





YSR state also oscillates

Oscillations from MF or YSR?

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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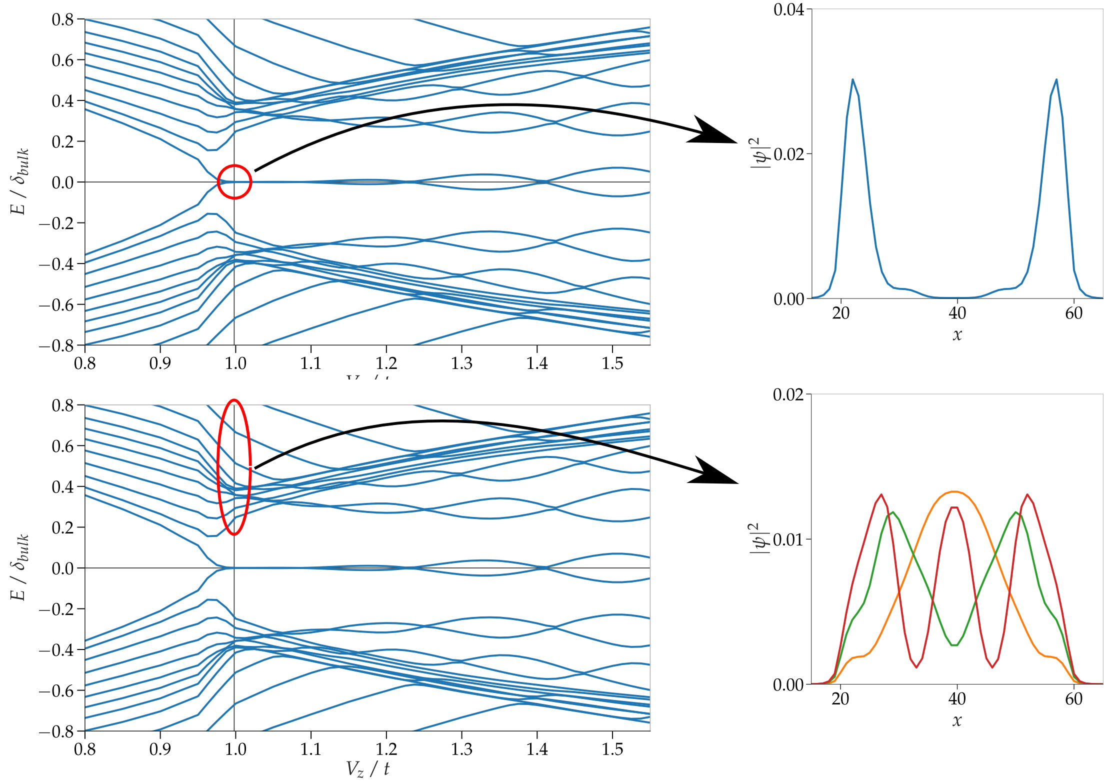
Clean MF and YSR States at TPT

At TPT, well-behaved states → Basis states



MF = topological
boundary mode

YSR states ~
Particle-in-a-box states
along chain

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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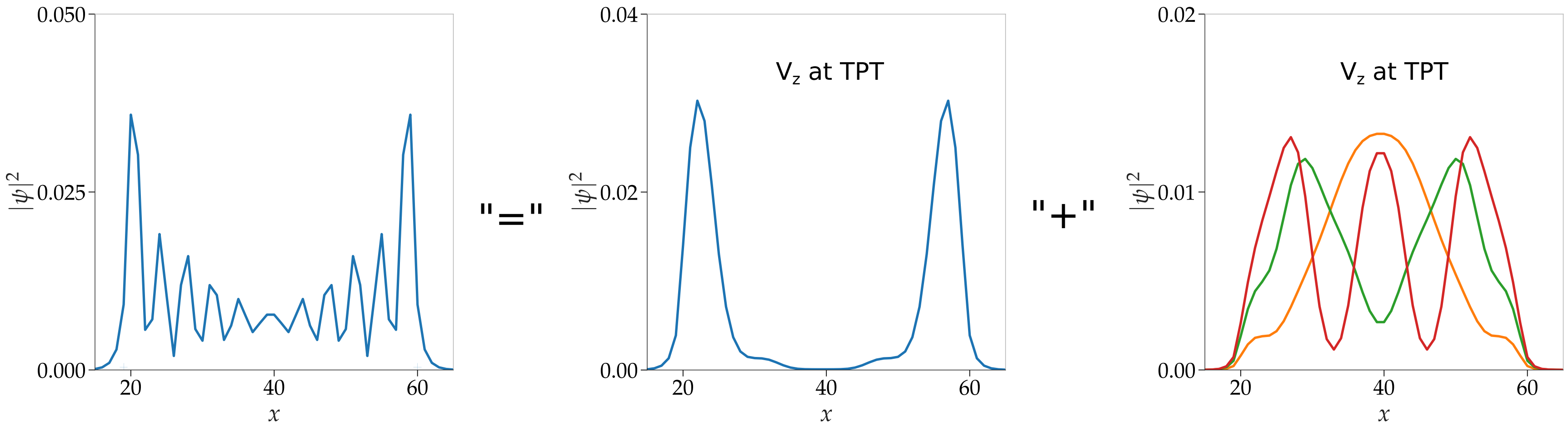
Basis Decomposition

Using states at TPT as a basis:

$$|\psi_0(V_z)\rangle \approx \Gamma_0(V_z)\,|\tilde{\psi}_0\rangle + \sum_{i\neq 0}\Gamma_i(V_z)\,|\tilde{\psi}_i\rangle$$



Basis states

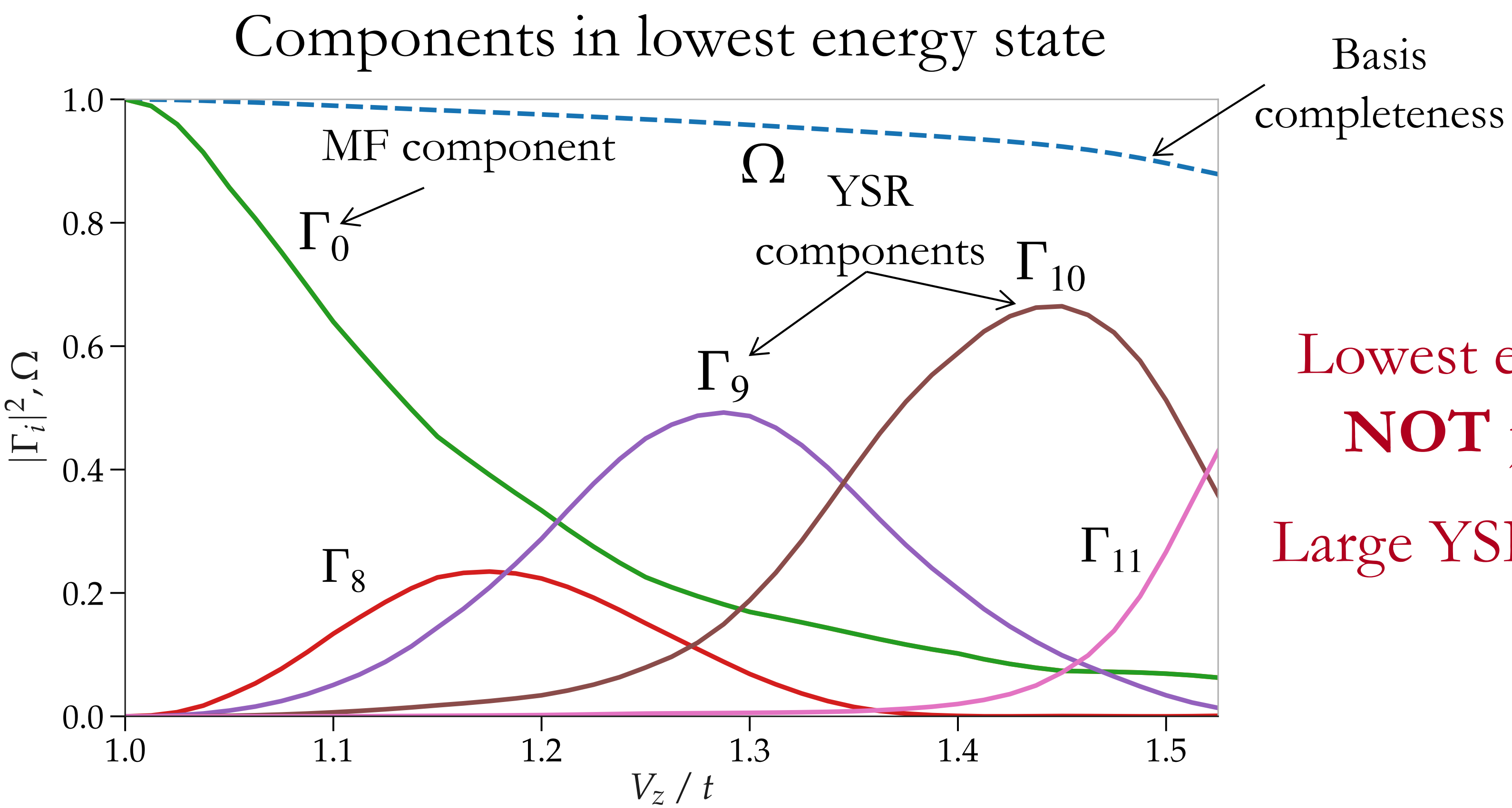
$$|\tilde{\psi}_i\rangle = |\psi_i(V_z = V_{z,TPT})\rangle$$

Overlap coefficients

$$\Gamma_i(V_z) = \langle\psi_0(V_z)|\tilde{\psi}_i\rangle$$

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

MF-YSR Hybridization



Lowest energy state is **NOT** just the MF

Large YSR components

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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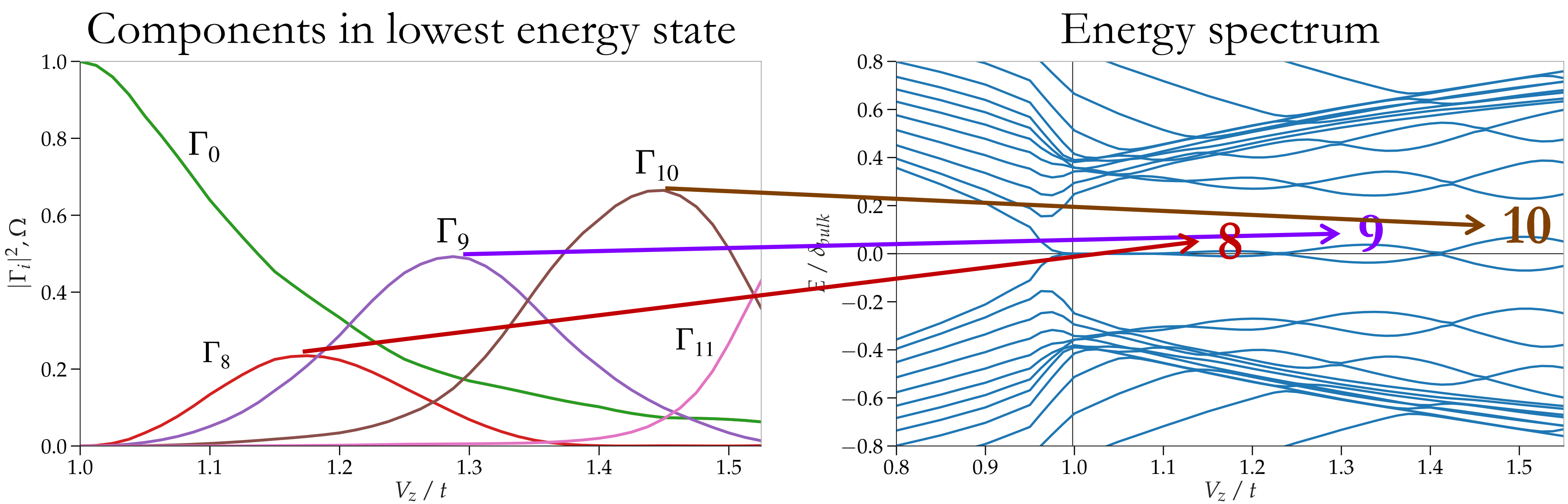
MF-YSR Hybridization



Lowest energy state = MF + lowest YSR

→ Energy oscillations

→ Amplitude oscillations in wave functions

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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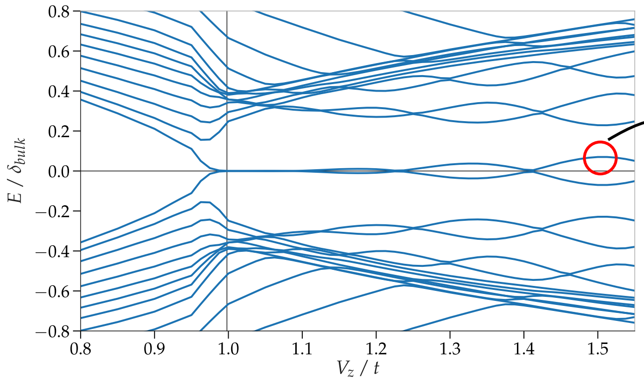
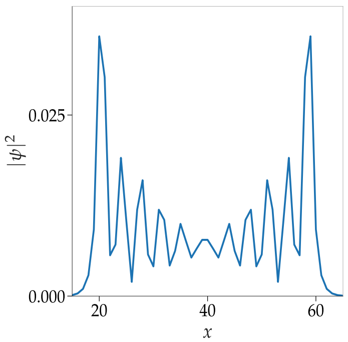
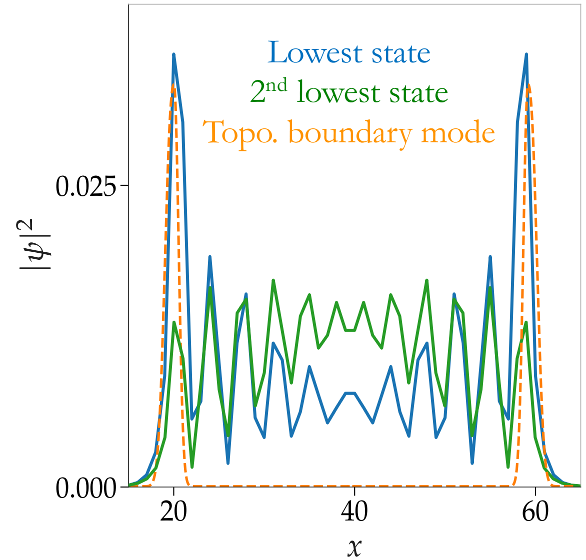
Extracting the MF





Lowest state =
MF + YSRs



How to extract the MF = topo. boundary mode ?

- FWHM of first peak
- Corrected MF state (subtract YSR states): $\underbrace{|\psi_M(V_z)\rangle}_{\text{MF}} \approx \underbrace{|\psi_0(V_z)\rangle}_{\text{Lowest state}} - \underbrace{\sum_{i\neq 0}\Gamma_i(V_z)\,|\tilde{\psi}_i\rangle}_{\text{YSRs}}$
- Topological boundary mode: $\varphi_M(x) = C e^{\frac{1}{\alpha}\int_0^x M(x')dx'}$
  - Effective mass gap: $M(x) = |\Delta(x)| - \beta|V_z(x)|$

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

MF Localization Length

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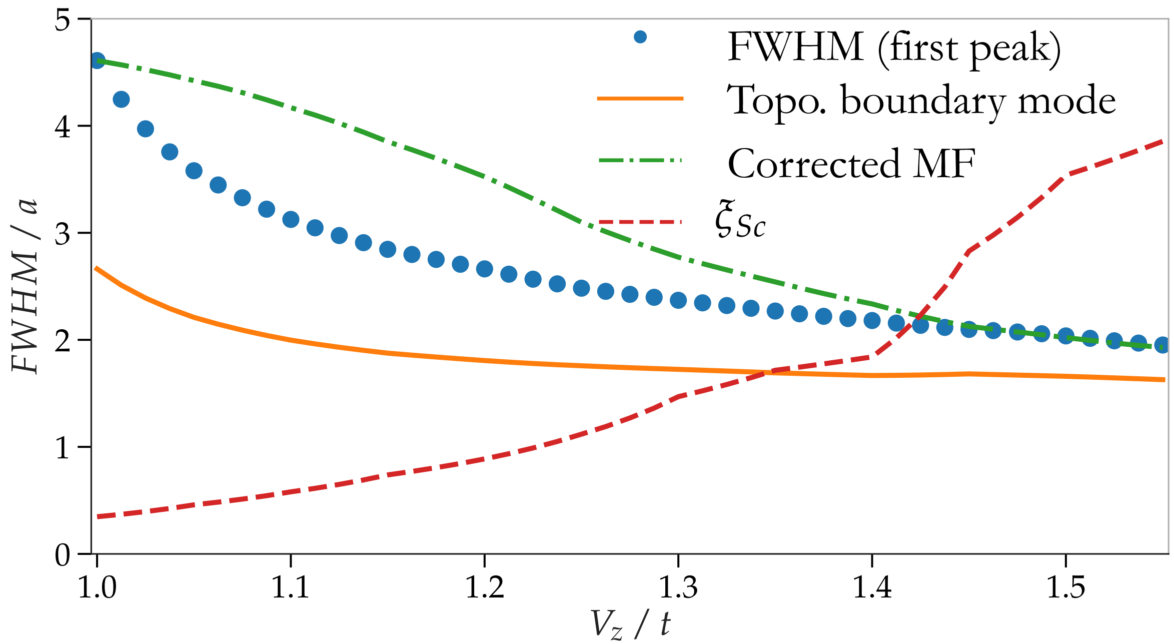

Topological boundary mode = **MF** ~ first peak in lowest state

Localization increases with V_z (opposite to SC coherence length)

Theiler, Björnson, and ABS, PRB **100**, 214504 (2019)

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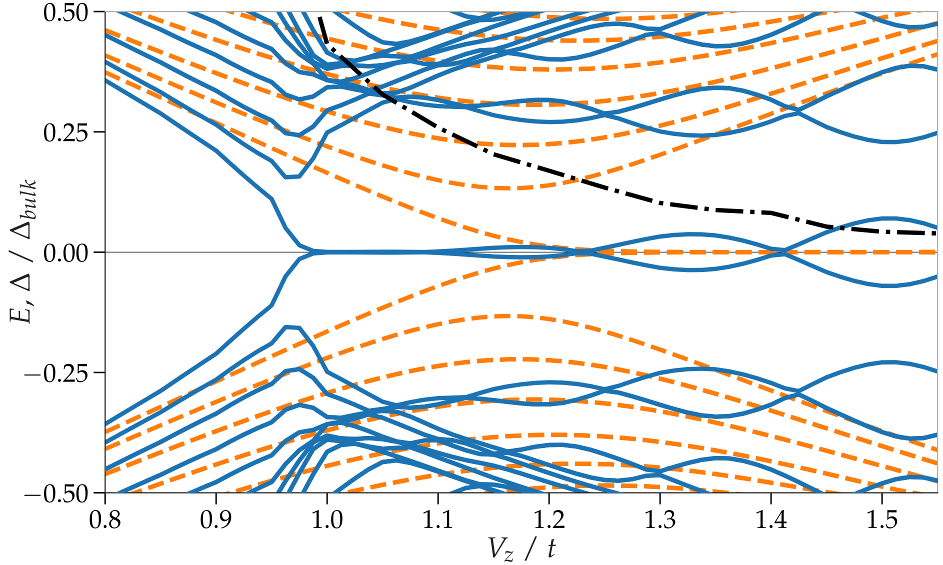
Summary

Lowest energy state = MF (topo. boundary mode) + YSRs

- Energy and amplitude oscillations due to YSR contributions
- MF ~ first peak in lowest state
- Enhanced effects by self-consistent treatment of superconductivity



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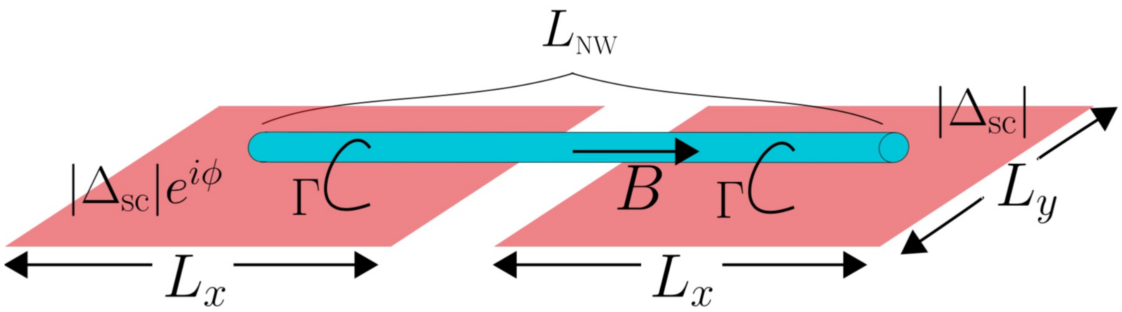
Nanowire SNS Junctions

When do false MFs appear and how to detect them?



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Are zero-Energy states MFs?

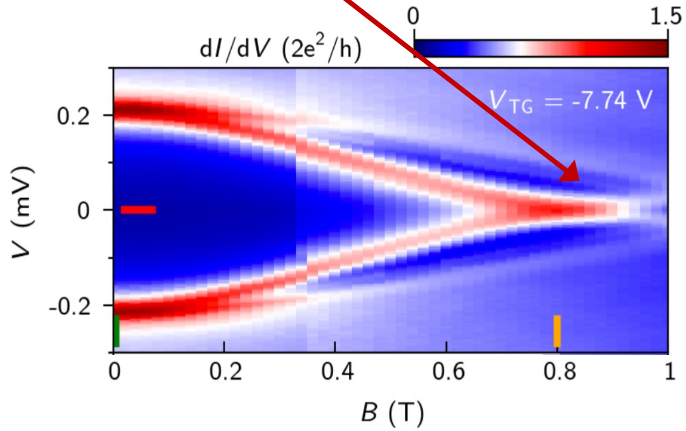
Interfaces/edges/impurities often host trivial zero-energy Andreev bound states (ABS)

How to distinguish MFs?

- Stable zero-energy peak
- Quantized conductance
- Bulk gap closing
- …

Not unique to MBS

MFs (?)



Zhang *et al.*, arXiv:2101.11456, see also Prada *et al.*, Nat. Rev. Phys. 2, 575 (2020)

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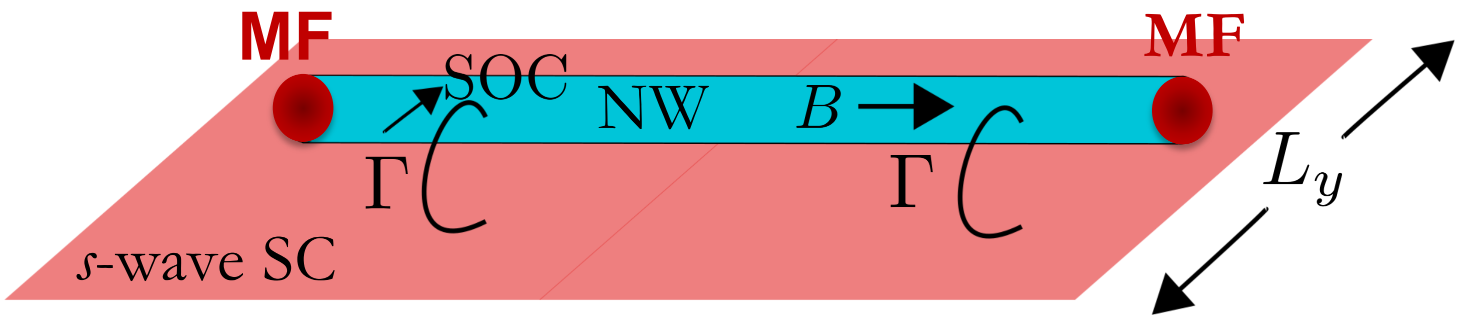
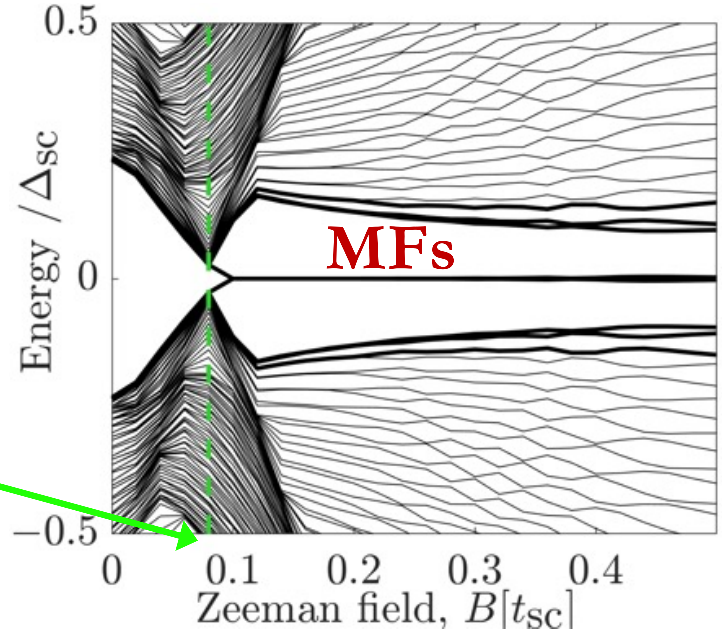
Nanowire + Superconductor

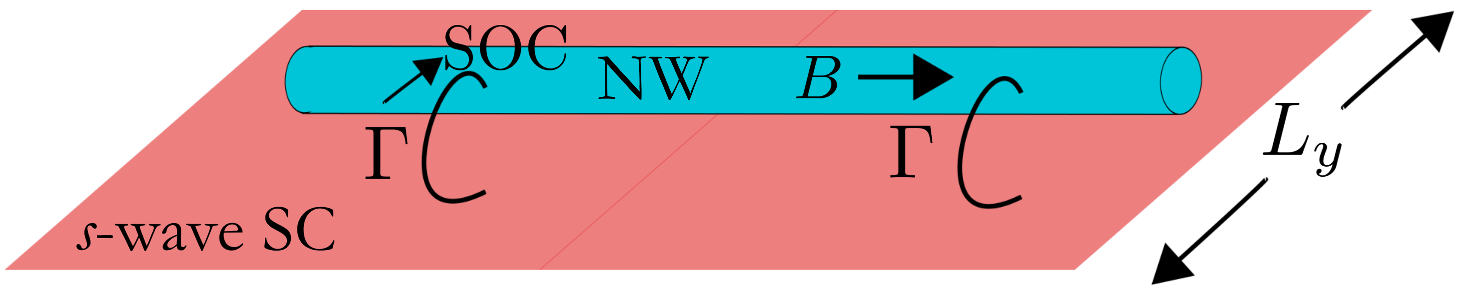


$$H = H_{\mathrm{SC}} + H_{\mathrm{NW}} + H^{\Gamma}_{\mathrm{SC-NW}}$$

MBS beyond topological phase transition (TPT) at B_c

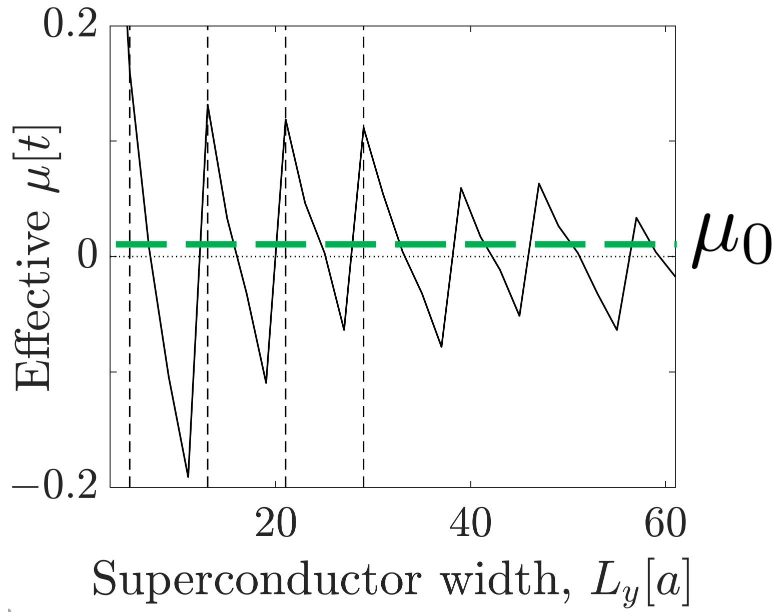


Nanowire + Superconductor



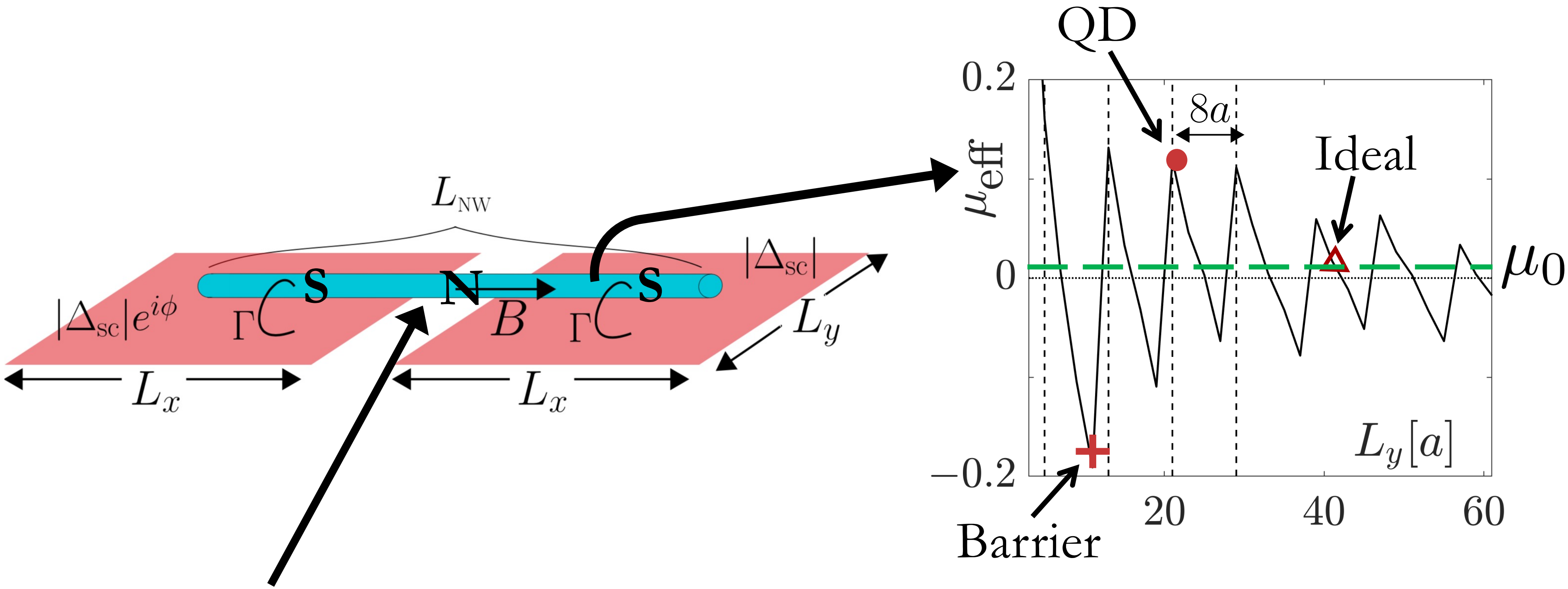
$$H = H_{\mathrm{SC}} + H_{\mathrm{NW}} + H^{\Gamma}_{\mathrm{SC-NW}}$$

Heavily modified effective chemical potential (and SOC) in NW



Awoga, Cayao, and ABS, PRL **123**, 117001 (2019), see also Reeg et al. PRB 97, 165425 (2018)

Short SNS Junction



Quantum dot (QD) or barrier emerges spontaneously in short NW junctions

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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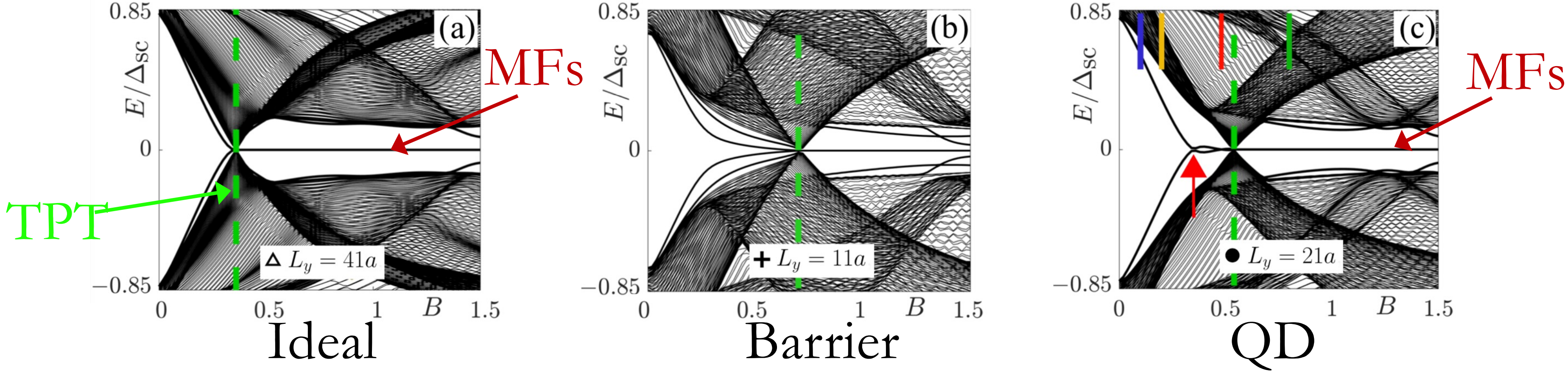
False MFs in QD Regime

Energy spectrum of junction



Zero-energy QD states before TPT

→ **false MFs**

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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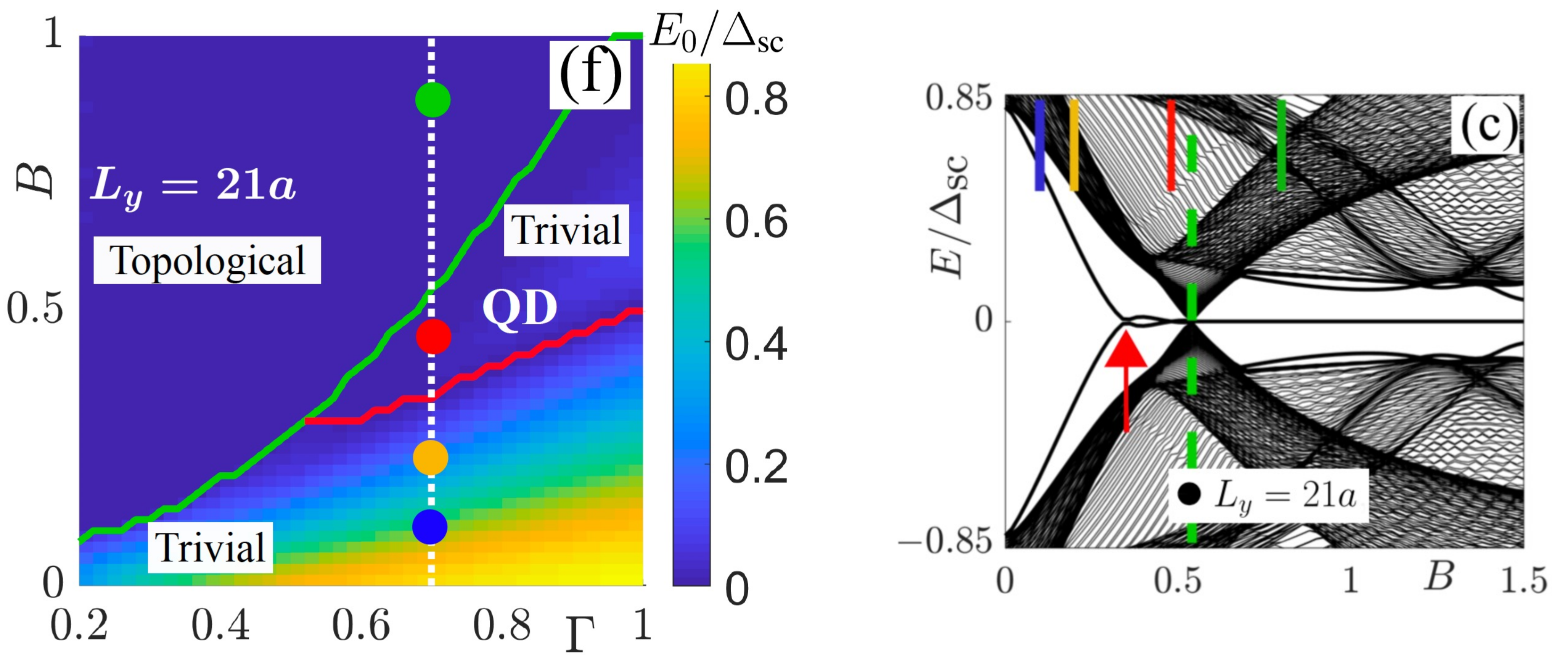
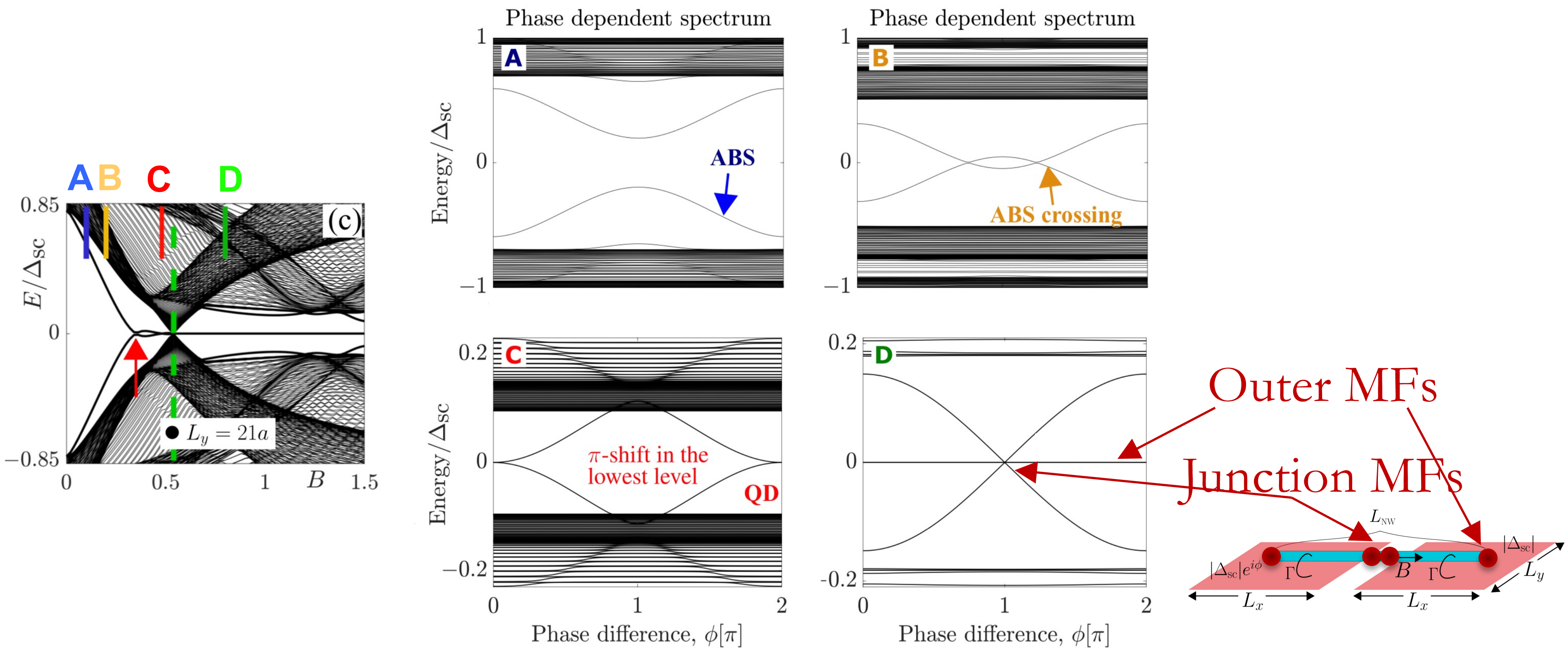
False MFs in QD Regime



Zero-energy (trivial) QD states
always in strong coupling regime

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

Phase Dependent Spectrum

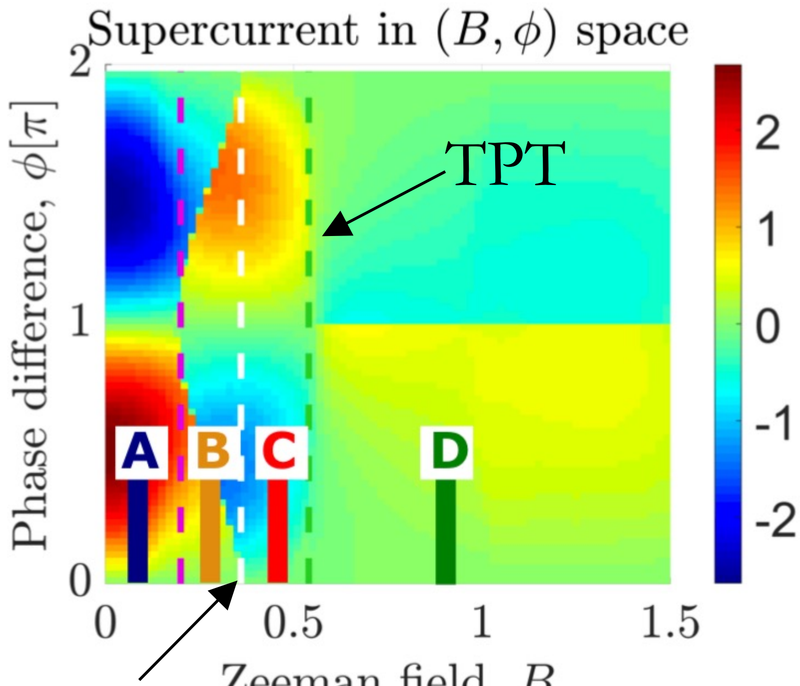


π-shifted phase energy spectrum in QD regime

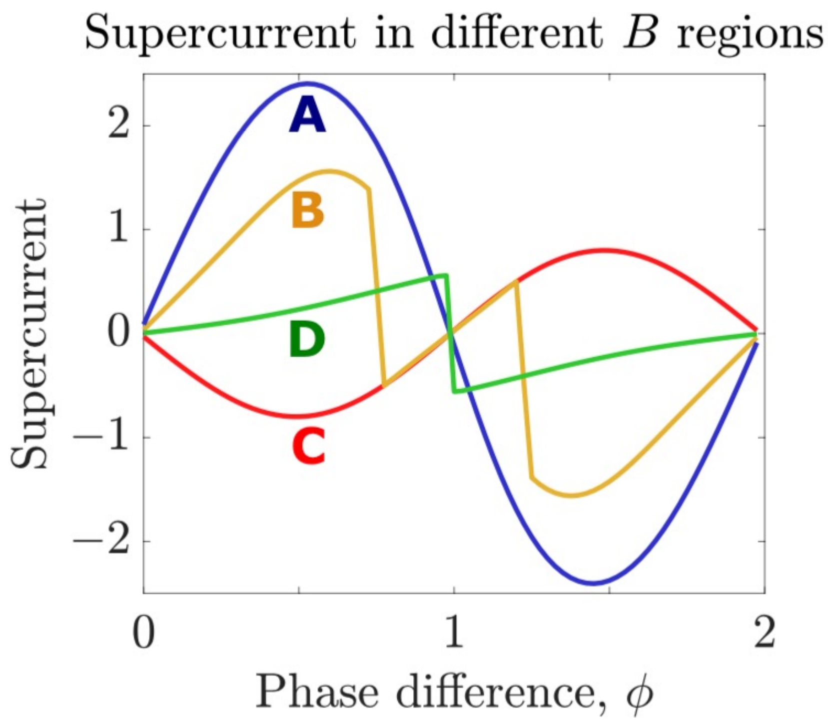
(due to spin flip in occupied state)

Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

π-Shifted Supercurrent



Zero-energy states appears



False MFs ←→ π-shifted supercurrent

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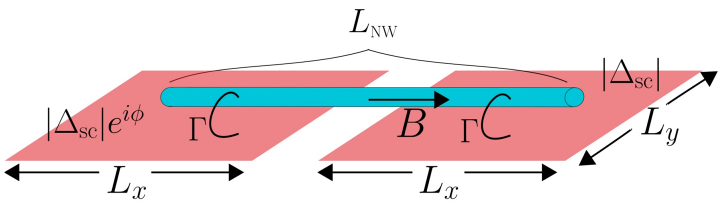
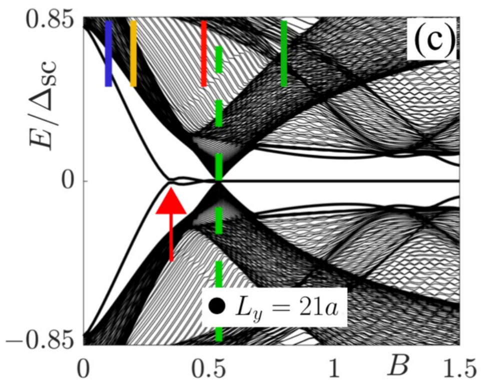
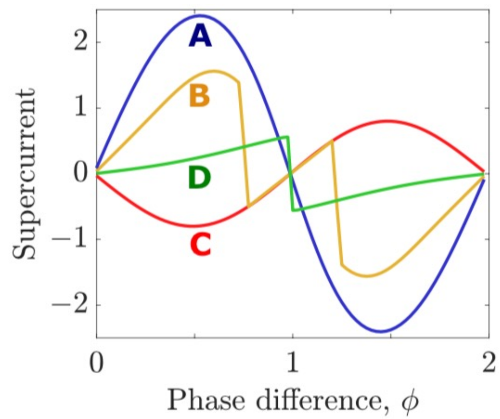
Summary

Nanowire SNS junctions



- False MFs common in short junctions
  - Due to spontaneous QD formation
  - Distinguishable by π-shifted supercurrent





Awoga, Cayao, and ABS, PRL **123**, 117001 (2019)

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Topological Superconductivity

Spinless superconductors

- Majorana fermions
- Engineered systems

Prototype: Kitaev model for 1D spinless SC
Materials: SOC + magnetism + *s*-wave SC
Majorana fermion:

- Non-local, "½ electron"
- Topological boundary state in spinless SCs
- Topological quantum computation

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Summary

- Introduction to superconductivity
  - BCS, BdG, group theory
- Topological superconductivity
  - Chiral superconductors: '*p*+i*p*' and '*d*+i*d*' superconductors
    - Appears often in 2D irreps
    - Topology set by Chern/winding number of order parameter
    - Chiral edge states
  - "Spinless" superconductors → Majorana fermions
    - SOC + magnetism + *s*-wave superconductivity
    - Topological edge state = Majorana fermion ~ non-local "½ electron"

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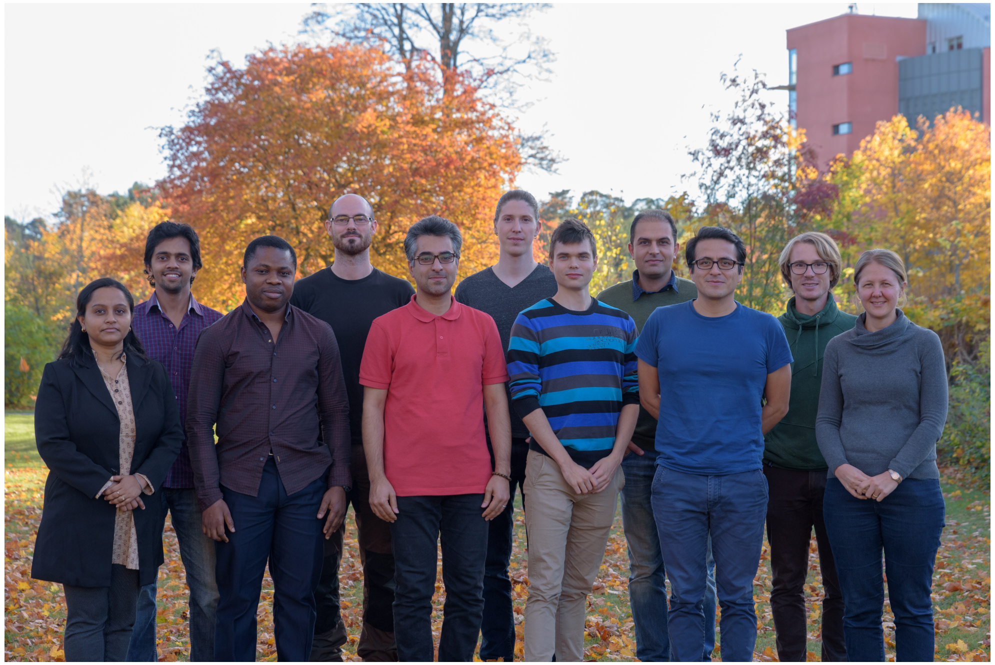
Acknowledgements



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