LUND
UNIVERSITY

Three-Pion Scattering: From the chiral Lagrangian to the lattice

MATTIAS SJÖ | LUND UNIVERSITY | ELPAFYSIK, FYSIKDAGARNA 2023

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Some background

People & Papers

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

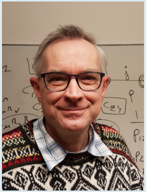
Six-meson amplitude

Bijnens & Husek, "*Six-pion amplitude*"

Phys.Rev.D, `2107.06291[hep-ph]`

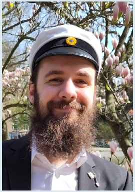
Bijnens, Husek & **Sjö**, "*Six-meson amplitude in QCD-like theories*"

Phys.Rev.D, `2206.14212[hep-ph]`



Hans Bijnens,
Lund U.

Tomáš Husek,
Lund U./Charles U.



Mattias Sjö,
Lund U.

LUND UNIVERSITY

Strengths and Weaknesses of the Lattice

Applicable to low-energy QCD

Non-perturbative

From first principles

Access to experimentally unobservable quantities

Computer-intensive

Euclidean (imaginary) time

Discrete space-time

Finite space-time volume

Often reqires heavier-than-physical masses

Strengths and Weaknesses of the Lattice

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Applicable to low-energy QCD

Non-perturbative

From first principles

Access to experimentally unobservable quantities

Synergy with ChPT

Computer-intensive

Euclidean (imaginary) time

Discrete space-time

Finite space-time volume

Often reqires heavier-than-physical masses

People & Papers – cont'd

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Three-pion K-matrix

B.-B., Bijnens, Husek, R.-L., Sharpe & **Sjö**, *"The isospin-3 three-particle K-matrix at NLO in ChPT"* *JHEP*, `2303.13206[hep-ph]`

B.-B., Bijnens, Husek, R.-L., Sharpe & **Sjö**, *"The three-particle K-matrix at NLO in ChPT for general isospin"* (in preparation, title preliminary)

Stephen Sharpe, U. of Washington

Fernando Romero-López, MIT

Jorge Baeza-Ballesteros, U. de València

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Six-Meson Diagrams

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

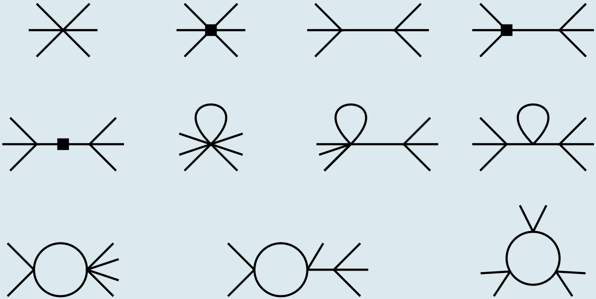
Results

Summary

Vertices

 = LO vertex  = NLO vertex

All the LO and NLO diagrams



Simplification & shortening by $\mathcal{O}(6!)$

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Flavour structures

$$\begin{aligned}
\mathcal{F}_{\{6\}}(a_1,\ldots,a_6) &= \langle t^{a_1}\cdots t^{a_6}\rangle \\
\mathcal{F}_{\{2,4\}}(a_1,\ldots,a_6) &= \langle t^{a_1}t^{a_2}\rangle\langle t^{a_3}\cdots t^{a_6}\rangle \\
\mathcal{F}_{\{3,3\}}(a_1,\ldots,a_6) &= \langle t^{a_1}t^{a_2}t^{a_3}\rangle\langle t^{a_4}t^{a_5}t^{a_6}\rangle \\
\mathcal{F}_{\{2,2,2\}}(a_1,\ldots,a_6) &= \langle t^{a_1}t^{a_2}\rangle\langle t^{a_3}t^{a_4}\rangle\langle t^{a_5}t^{a_6}\rangle
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}\big(p_1,a_1;p_2,a_2;\ldots;p_6,a_6\big) \\
&\qquad = \sum_R\sum_\sigma \mathcal{M}_R\big(\sigma[p_1,\ldots,p_6]\big)\mathcal{F}_R\big(\sigma[a_1,\ldots,a_6]\big)
\end{aligned}$$

Flavor-ordering

$\sigma \notin$ symmetries of $\mathcal{F}_R$

$\to$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$

$\to$ novel, non-unique!

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

The $\pi\pi\pi$ K-matrix

Finite-volume energy shift

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Energy spectrum: n particles, box of size L

$$\det\left[F_n^{-1}(E, \boldsymbol{P}, L) + \mathcal{K}_n(E^*)\right] = 0$$

$n = 2$ particles

Lüscher, "*Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories*" *Commun. Math. Phys.* (1986)

$n = 3$ particles

Hansen & Sharpe, "*Relativistic, model-independent, three-particle quantization condition*" *Phys. Rev. D*, `1408.5933[hep-lat]`

(and several other approaches)

Properties of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Lorentz invariant,
Scheme dependent,
On-shell only

NLO in ChPT
$\to$ reduces to
algebraic relation

Related to $\mathcal{M}_3^{\mathrm{NLO}}$
through integral
equation

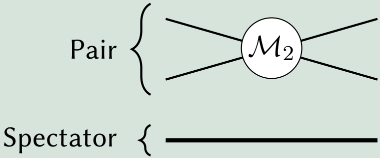
$\to \mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

Divergence-free
(no poles or cuts)

LUND UNIVERSITY

Anatomy of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

3 particles, 2 scattering



Form of $\mathcal{M}_2$

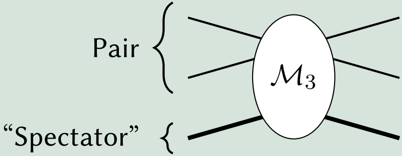
$\to \mathcal{M}_2(\boldsymbol{p})_{lm,l'm'}$

$$\boldsymbol{p} \; \text{— spectator momentum}$$

$$Y_m^l(\theta, \phi) \; \text{— pair angular momentum}$$

Anatomy of $\mathcal{K}_{\text{df},3}^{\text{NLO}}$

3 particles, 3 scattering



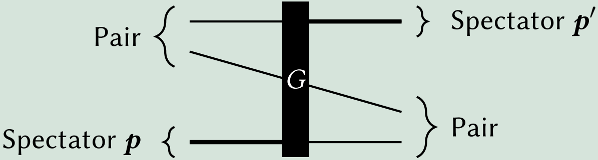
Form of $\mathcal{M}_3$

$\to \mathcal{M}_3(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'}$

Particle exchange $\Leftrightarrow$ Spectator choice

Anatomy of $\mathcal{K}_{\mathbf{df},3}^{\mathbf{NLO}}$

On-shell propagator substitute



Properties of G

On-shell only — propagator-like near pole:

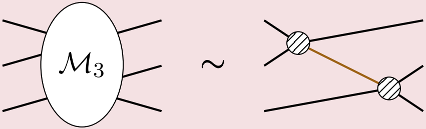
$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M_\pi + i\epsilon}$$

Smooth cutoff far from pole — non-analytic

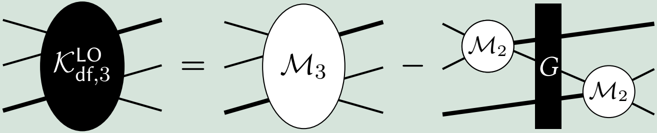
$\mathcal{K}_{\text{df},3}$ at leading order

LUND UNIVERSITY

One-particle exchange pole



One-particle exchange subtraction



$\mathcal{K}_{\mathrm{df},3}$ at next-to-leading order

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

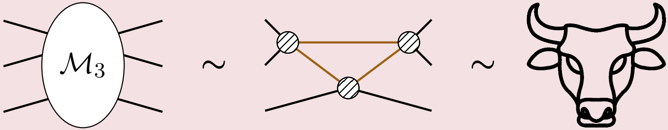
Leading order

Next-to-leading order

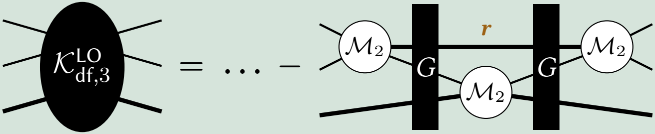
Results

Summary

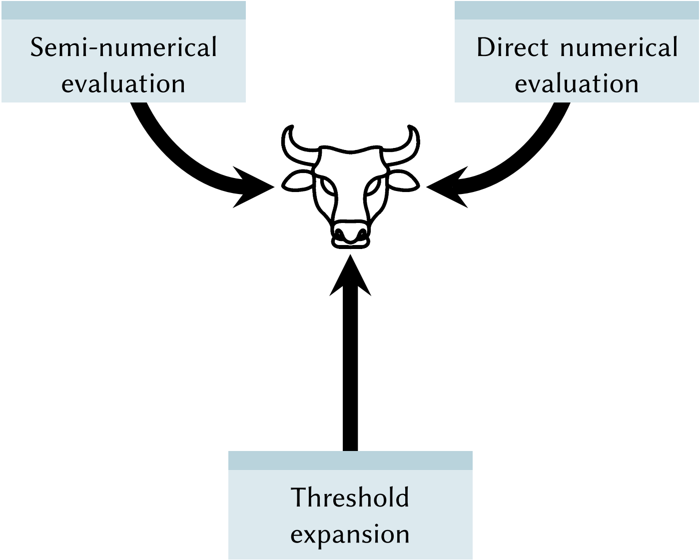
Bull's head cut



Bull's head subtraction



Different approaches



LUND UNIVERSITY

Threshold expansion

Background
The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification
The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results
Summary

Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2 \qquad \text{(system above-thr'ness)}$$

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2 \qquad \text{(pair above-thr'ness)}$$

$$\tilde{t}_{ij} \propto (p_i - p_j')^2 \qquad \text{(spectator above-thr'ness)}$$

Compound parameters

$$\Delta_{\mathrm{A}} = \sum_i (\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_{\mathrm{B}} = \sum_{ij} \tilde{t}_{ij}^2 - \Delta^2$$

Threshold expansion

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_{\mathrm{A}}\Delta_{\mathrm{A}} + \mathcal{K}_{\mathrm{B}}\Delta_{\mathrm{B}} + \mathcal{O}(\Delta^3)$$

Different approaches

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

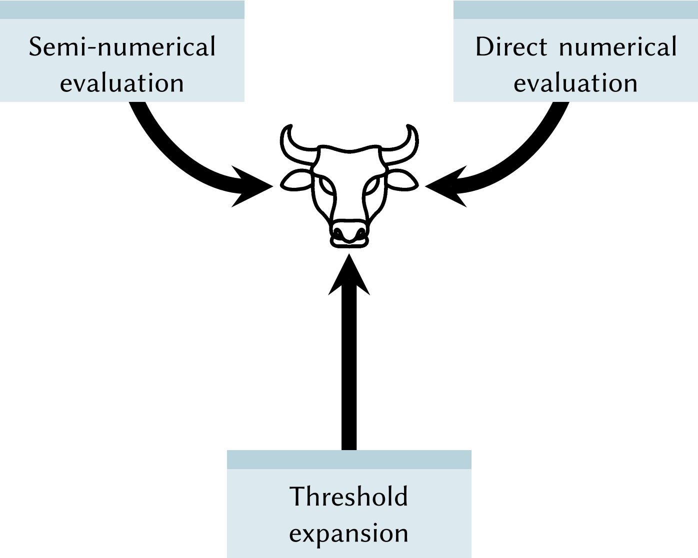
Anatomy

Leading order

Next-to-leading order

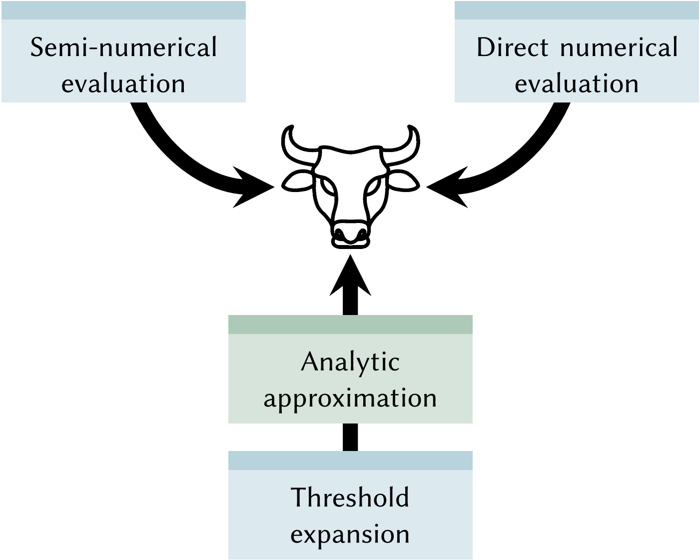
Results

Summary



LUND UNIVERSITY

Different approaches



Results!

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

$$
\begin{aligned}
\mathcal{K}_0 &= \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\mathrm{r}}_{(0)}\right], \\
\mathcal{K}_1 &= \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}(1999 + 1920\log 3) - \mathcal{D}_1 + 384L + \ell^{\mathrm{r}}_{(1)}\right], \\
\mathcal{K}_2 &= \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\mathrm{r}}_{(2)}\right], \\
\mathcal{K}_{\mathrm{A}} &= \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_{\mathrm{A}} - 9L + \ell^{\mathrm{r}}_{(\mathrm{A})}\right], \\
\mathcal{K}_{\mathrm{B}} &= \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_{\mathrm{B}} + 54L + \ell^{\mathrm{r}}_{(\mathrm{B})}\right].
\end{aligned}
$$

$$
\mathcal{D}_0 \approx -0.0563476589\,, \qquad \mathcal{D}_1 \approx 0.129589681\,, \qquad \mathcal{D}_2 \approx 0.432202370\,,
$$
$$
\mathcal{D}_{\mathrm{A}} \approx 9.07273890 \cdot 10^{-4}\,, \qquad \mathcal{D}_{\mathrm{B}} \approx 1.62394747 \cdot 10^{-4}\,,
$$

Reconciliation with the lattice

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

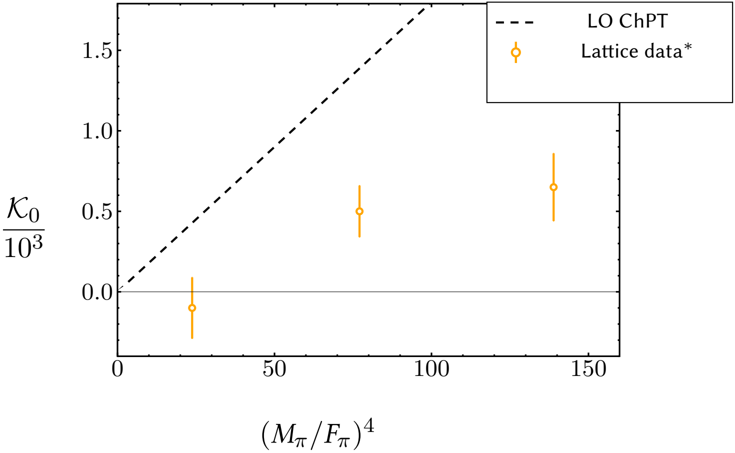
Anatomy

Leading order

Next-to-leading order

Results

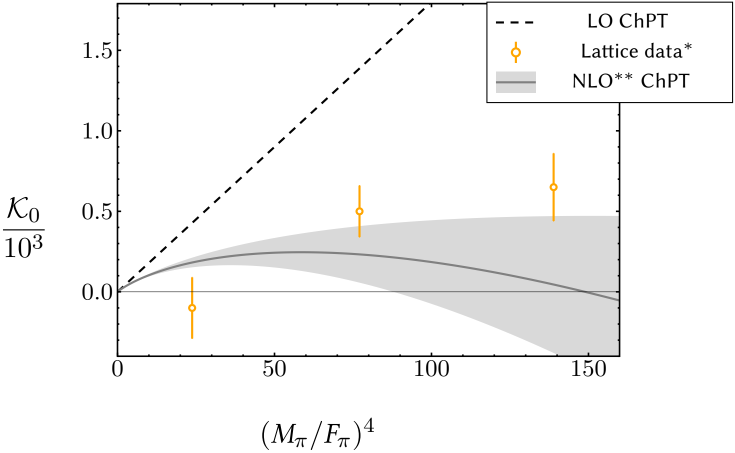
Summary



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, `2021.06144[hep-lat]`

Reconciliation with the lattice

Background
The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification
The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results
Summary



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144[hep-lat]

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

Reconciliation with the lattice

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

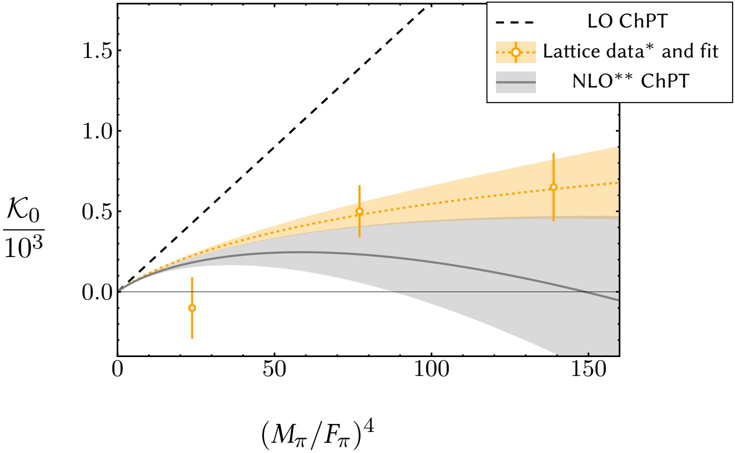
Anatomy

Leading order

Next-to-leading order

Results

Summary



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, `2021.06144[hep-lat]`

** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, `hep-ph/0103088`

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

LUND UNIVERSITY

Summary and next steps

Complete

$\pi\pi\pi \to \pi\pi\pi$

Complete

$\pi\pi\pi$ K-matrix

Summary and next steps

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

Complete

$\pi\pi\pi \to \pi\pi\pi$

Complete

$\pi\pi\pi$ K-matrix

Next: Remaining isospin channels

$I = 3$	Singlet
$I = 2$	Doublet
$I = 1$	Singlet, Doublet
$I = 0$	Antisymmetric singlet

Summary and next steps

Complete

$\pi\pi\pi \to \pi\pi\pi$ ...

Complete

... $\pi\pi$ K-matrix

Next?

$\pi\pi K$

πKK

Next: Remaining ... ls

$I = 3$	Singlet
$I = 2$	Doublet
$I = 1$	Singlet, Doublet
$I = 0$	Antisymmetric singlet

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification

The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results

Summary

Backup slides

Six-Meson Diagrams

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

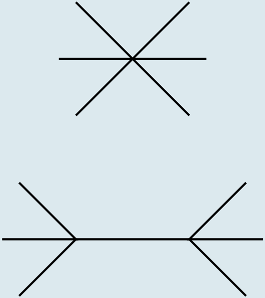
Leading order

Next-to-leading order

Results

Summary

Leading order



Osborn (1969)

Susskind & Frye (1970)

Six-Meson Diagrams

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

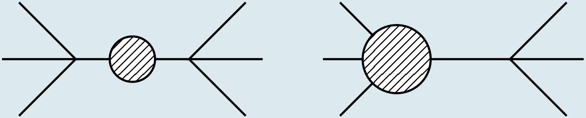
Leading order

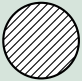
Next-to-leading order

Results

Summary

Reducible to smaller results

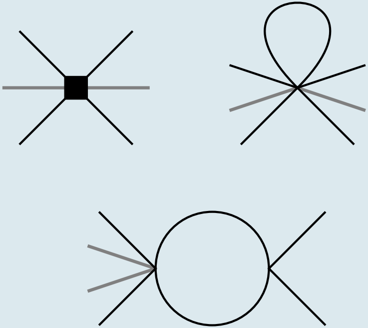


 = all 1-loop diagrams (known from before*)

*Bijnens & Lu, "*Meson-meson scattering in QCD-like theories*" *JHEP*, `1102.0172[hep-ph]`

Six-Meson Diagrams

4-meson but with more legs



Six-Meson Diagrams

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

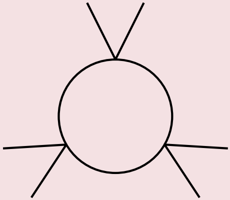
Leading order

Next-to-leading order

Results

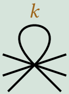
Summary

New and difficult



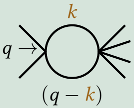
One-Loop Integrals

One-propagator integral



$$\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - M^2} \sim \frac{1}{4-d} + \text{(finite)}$$

Two-propagator integral



$$\sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\{1, k^\mu, k^\mu k^\nu\}}{(k^2 - M^2)\left[(k-q)^2 - M^2\right]}$$

$$\sim \frac{1}{4-d} + \bar{J}(q^2) + \text{(finite)}$$

One-Loop Integrals

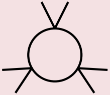
Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification

The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results

Summary

Three-propagator integral

 $\sim$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\{1, k^\mu, k^\mu k^\nu, k^\mu k^\nu k^\rho\}}{(k^2 - M^2)\left[(k-q_1)^2 - M^2\right]\left[(k+q_2)^2 - M^2\right]}$$

In principle reducible to $\bar{J}$ but impractical.
Instead: elegant but redundant basis

$$\{C_0, C_1, C_2, C_3\}(p_1, \ldots, p_6)$$

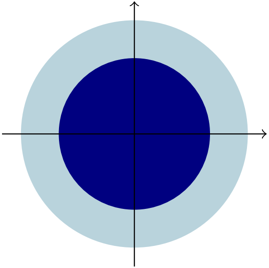
where $C_3 \sim \frac{1}{4-d} + \bar{C}_3$ (others are finite)

Background
The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification
The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results
Summary

The bull's head

How to integrate?

$$\int \frac{\mathrm{d}^3\boldsymbol{r}}{2\omega_r} \frac{[\text{Non-analytic}]}{[\text{Complicated}]}$$



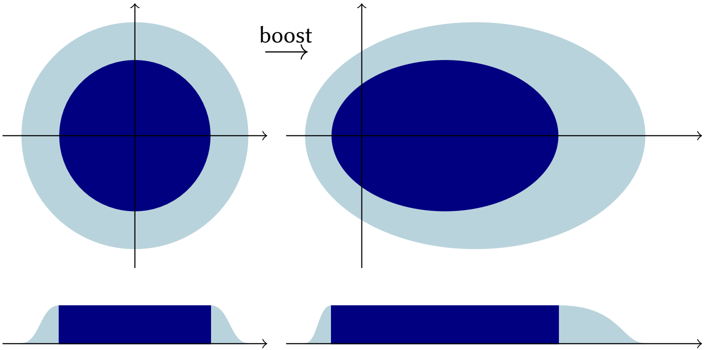
analytic, has poles
non-analytic, smooth

Background
The $\pi\pi\pi \to \pi\pi\pi$ amplitude
Diagrams
Simplification
The $\pi\pi\pi$ K-matrix
Anatomy
Leading order
Next-to-leading order
Results
Summary

The bull's head

How to integrate?

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{[\text{Complicated angular dependence}]}{[\text{Much simpler}]}$$



The bull's head

Background

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Diagrams

Simplification

The $\pi\pi\pi$ K-matrix

Anatomy

Leading order

Next-to-leading order

Results

Summary

How to integrate?

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{\text{[Simple] - [Smooth]}}{\text{[Much simpler function]}}$$