# Spectra of random sparse generators of Markovian evolution


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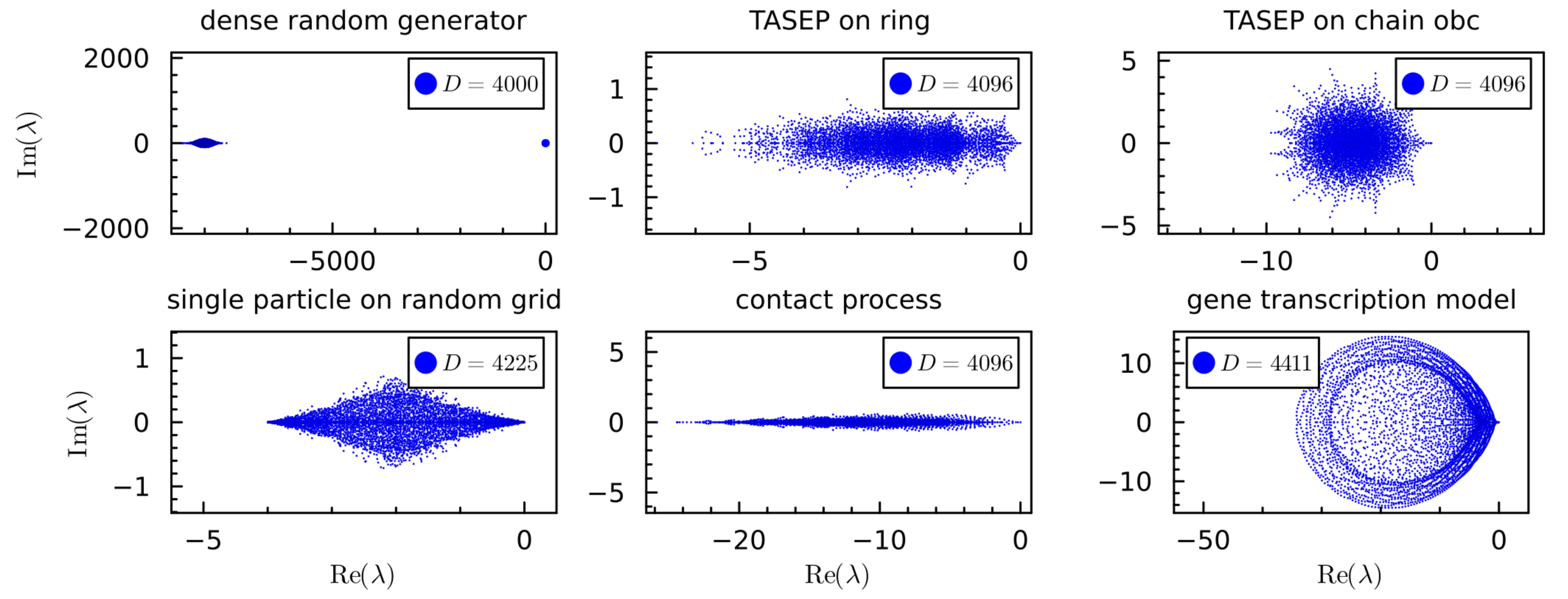
## Motivation

**Continuous time Markov evolution** of the probability vector $P(t)$,

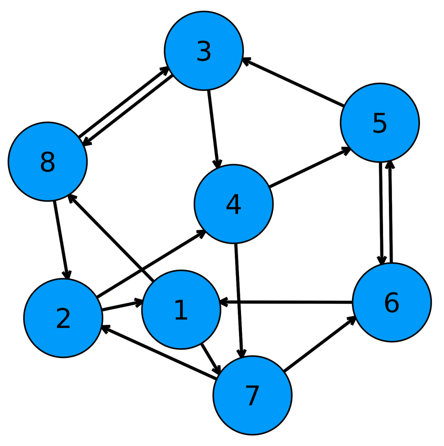
$$\frac{d}{dt}P(t) = MP(t), \qquad (1)$$

is solely determined by the **generator matrix** $M$. Generic generator matrices are modeled by **random** matrices. Spectra of **dense** random generator matrices mismatch relaxation times of physical generators (large spectral gap). We propose **sparse** random generator matrices.

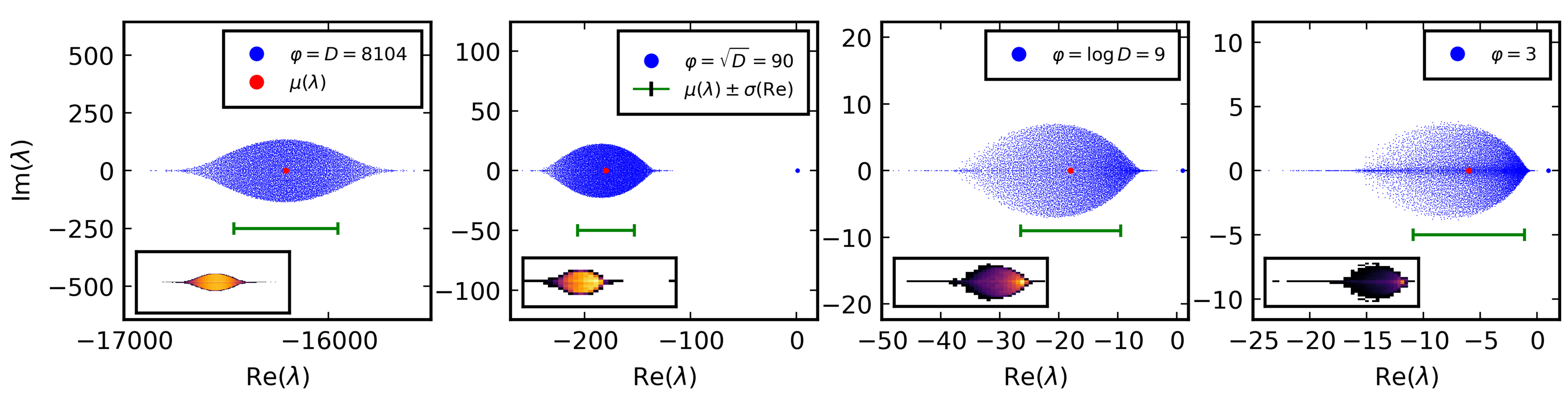



## Sparse random generator matrix ensemble

- **generator matrix** $M = K - J$ is $D \times D$-matrix, where
  - $K$ is adjacency matrix of **random graph**
  - $J$ is diagonal matrix with $J_{ii} = \sum_{i \neq j} K_{ji}$
  - $M$ is negative (combinatorial) Laplacian on graph
- Random graph is **uniformly sampled** from ensemble of all graphs with $D$ vertices and **fixed vertex degree** $\varphi$ (degree = in-degree = out-degree).
- $\varphi$ controls the **sparsity** of $M$. Sparsity $\varphi$ can be
  - independent of $D$: single particles
  - dependent on $D$: many-body systems with $\varphi \sim \log D$
- Random graph is highly likely **strongly connected** if $\varphi \geq 2$.
- **Edge weights** $(K_{ij})$ are positive, iid distributed, e.g $\chi^2$, exponential, uniform, etc.



## Bulk spectrum: Mean and width




Eigenvalues $\lambda_i$ of $M$. **Mean** of bulk is

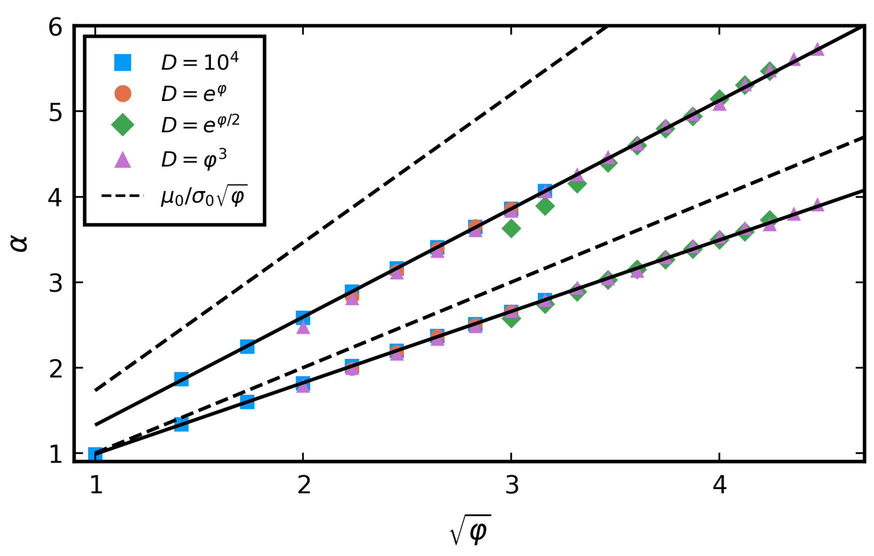
$$\mu(\lambda) := \left\langle \frac{1}{D}\sum_{i=1}^{D} \lambda_i \right\rangle = -\varphi\mu_0$$

and **horizontal width** of bulk

$$\sigma(\mathrm{Re}\,\lambda) := \left\langle \frac{1}{D}\sum_{i=1}^{D} (\mathrm{Re}\,\lambda_i - \mu(\lambda))^2 \right\rangle^{1/2}$$

$$\geq \sigma_0\sqrt{\varphi} + O\left(\sqrt{\frac{\varphi}{D}}\right).$$

| | | |
|---|---|---|
| bulk close to 0 | $\Longleftrightarrow$ | $\alpha$ small |
| bulk distant from 0 | $\Longleftrightarrow$ | $\alpha$ large |
| $\varphi$ constant in $D$ | $\Longleftrightarrow$ | $\alpha$ constant in $D$ |
| $\varphi$ grows with $D$ | $\Longleftrightarrow$ | $\alpha$ grows with $D$ |




- Average $\langle \dots \rangle$ is over matrix ensemble.
- $\mu_0 = \langle M_{ij} \rangle$, $\sigma_0^2 = \mathrm{var}(M_{ij})$, $i \neq j$.

**Ratio of mean and width** $\alpha$ is given by

$$\alpha := \frac{|\mu(\lambda)|}{\sigma(\mathrm{Re}\,\lambda)} \lesssim \sqrt{\varphi}$$

**Numerically** $(\varphi \ll D)$ : $\alpha \sim \sqrt{\varphi}$

**Connectivity $\varphi$ increases with $D$ $\Longrightarrow$ bulk diverges from equilibrium eigenvalue**

## Conclusion

- Bulk spectrum of sparse random Markov generators $M$ diverges from equilibrium eigenvalue 0 iff connectivity $\varphi$ increases with state space size $D$.
- Spectral edges are determined by extreme values of diagonal of $M$ $\Longrightarrow$
  - left tail of $M_{ij}$ decreases at most as power-law: spectral gap closes as power in $D$ for constant $\varphi$ and crossover to increasing gap at $\varphi \sim \log D$.
  - Largest eigenvalue investigated for exponential ($\chi^2$) and power-law right tail (uniform).
- Spacing ratios agree with random matrices for $\varphi \geq 2$.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## Spectral edges

**Spectral gap**: $\gamma_* = \min_{\mathrm{Re}\,\lambda_i < 0} |\mathrm{Re}\,\lambda_i|$

**horizontal extent**: (largest eigenvalue) $\tilde{\gamma} = \max_{1 \leq i \leq D} |\mathrm{Re}\,\lambda_i|$

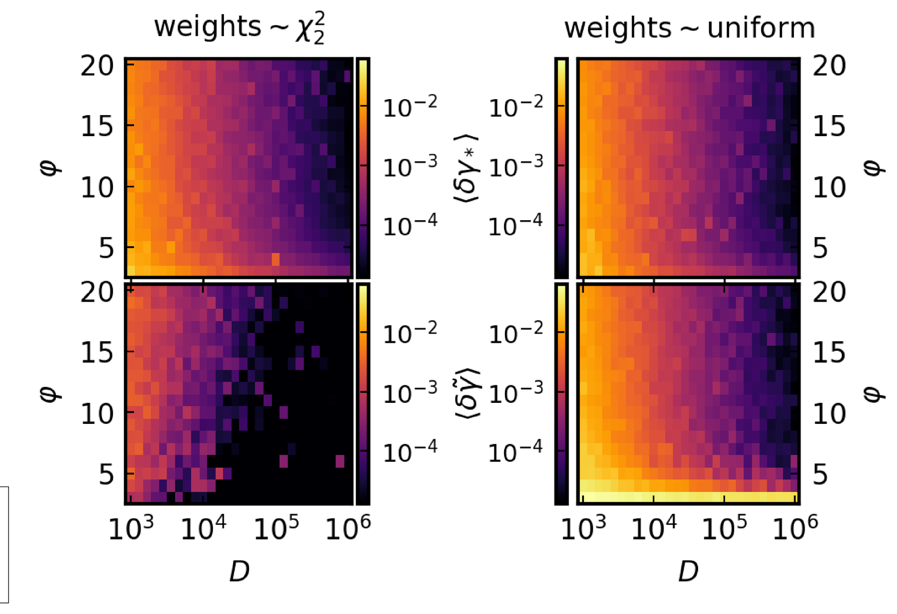
For symmetric generators:

$$\tilde{\gamma} \geq \max_{1\leq i \leq D} J_{ii} \text{ and } \min_{1\leq i \leq D} J_{ii} \geq \gamma_*.$$

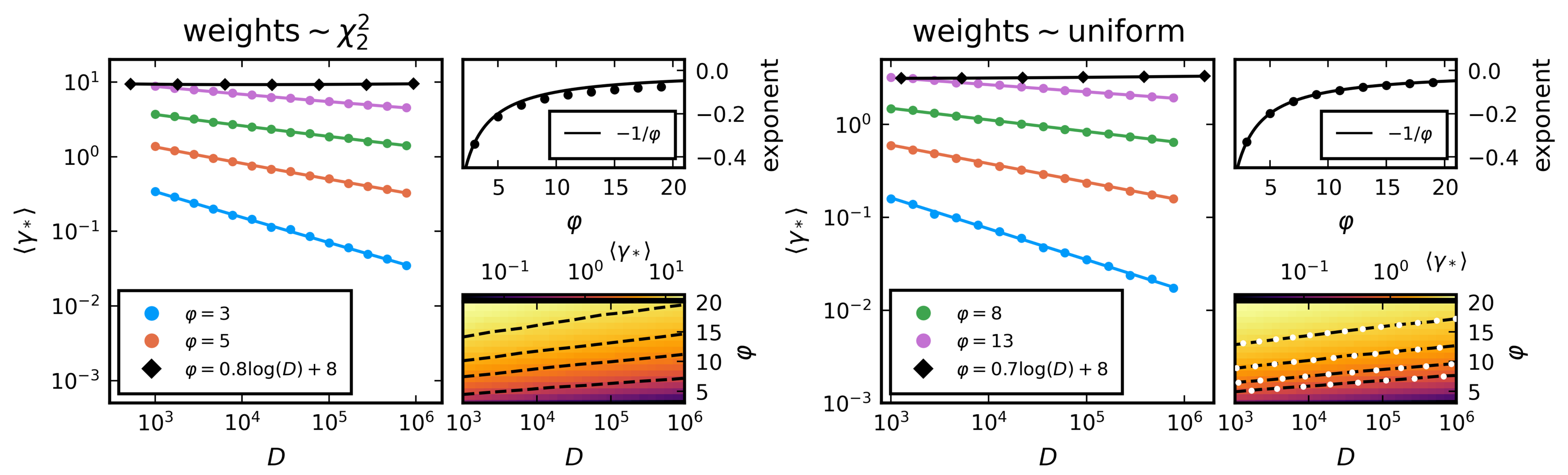
**Numerically** $(\varphi \ll D)$ for non-symmetric:

$$\tilde{\gamma} \approx \max_{1\leq i \leq D} J_{ii} \text{ and } \min_{1\leq i \leq D} J_{ii} \approx \gamma_*.$$

$\Longrightarrow$ **Spectral edges are given by extreme values of $J_{ll}$ distribution.**




## Average spectral gap




| $K_{ij} = M_{ij}$ | $J_{jj}$ $(F)$ |
|---|---|
| $\chi^2$, exponential | Gamma |
| uniform | Irwin-Hall |

$$\langle \gamma_* \rangle \approx D \int dx x f(x)(1 - F(x))^{D-1}$$

weights $\sim$ uniform: ($J_{ii}$ power-law left tail)

$$\text{EVT:} \quad \langle \gamma_* \rangle \approx \Gamma\left(1 + \frac{1}{\varphi}\right)(\varphi!)^{1/\varphi} D^{-1/\varphi}$$

**$\varphi$ const**: $\langle \gamma_* \rangle \sim D^{-1/\varphi}$

$1 \ll \varphi \ll D$ : $\langle \gamma_* \rangle \approx \varphi D^{-1/\varphi}$

**$\langle \gamma_* \rangle$ const**: $\varphi \sim \log D \quad (1 \ll \varphi \ll D)$

(Approximate) power-law left tail $J_{ii}$, $\varphi \ll D$:

- **$\varphi$ constant** $\Longrightarrow$ power-law decay of average spectral gap
- **$\varphi \sim \log D$** $\Longrightarrow$ average spectral gap $\approx O(1)$

## Average horizontal extent

$$\langle \tilde{\gamma} \rangle \approx D \int dx x f(x) F(x)^{D-1}$$

weights $\sim \chi^2_2$: ($J_{ii}$ exponential right tail)

$$\text{EVT:} \quad \langle \tilde{\gamma} \rangle \approx 2\log D + 2(\varphi - 1)\log\log D + 2\gamma - 2\log\Gamma(\varphi)$$

**$\varphi$ const**: $\langle \tilde{\gamma} \rangle \sim \log D$

**$\varphi \sim \log D$** : $\langle \tilde{\gamma} \rangle \sim \log D$

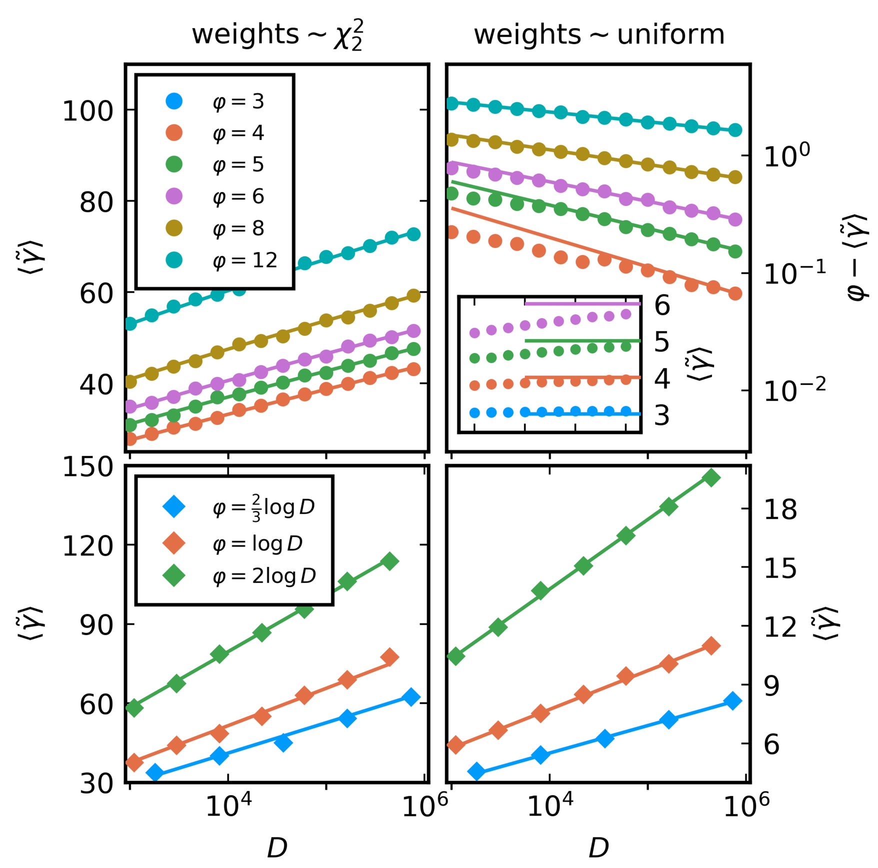
weights $\sim$ uniform: ($J_{ii}$ power-law right tail)

$$\text{EVT:} \quad \langle \tilde{\gamma} \rangle \approx \varphi - \Gamma\left(1 + \frac{1}{\varphi}\right)(\varphi!)^{1/\varphi} D^{-1/\varphi}$$

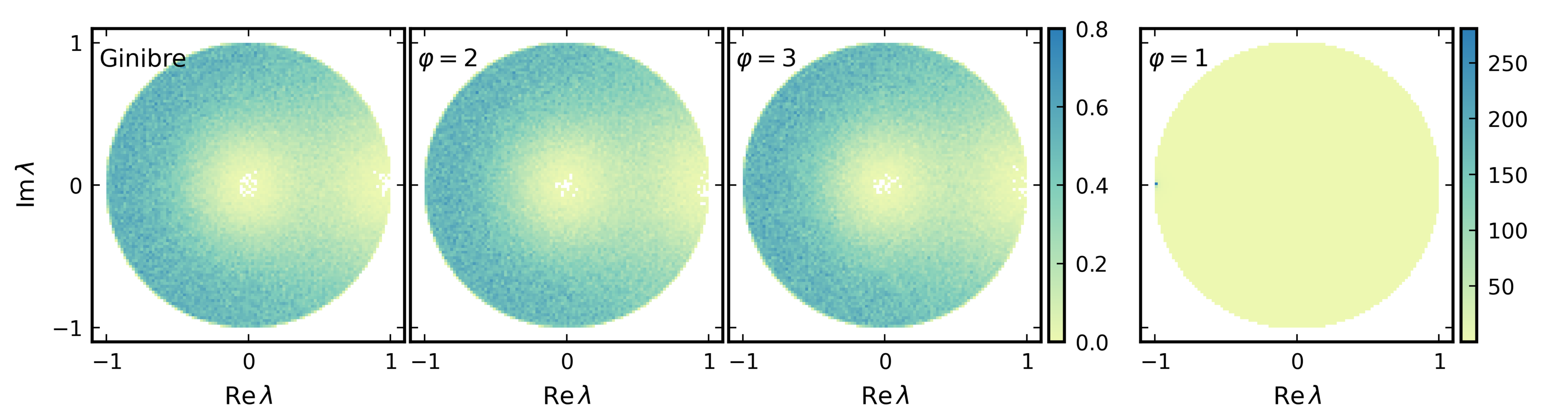
**$\varphi$ const**: $\langle \tilde{\gamma} \rangle \sim \varphi - D^{-1/\varphi}$

$1 \ll \varphi \ll D$ : $\langle \tilde{\gamma} \rangle \approx \varphi(1 - D^{-1/\varphi}) \sim \varphi$

**$\varphi \sim \log D$** : $\langle \tilde{\gamma} \rangle \sim \log D$




## Complex spacing ratios




**Spacing ratios**: $z_j = \dfrac{\lambda_j^N - \lambda_j}{\lambda_j^{NN} - \lambda_j} = re^{i\theta}$

$\lambda_i^N$ $(\lambda_i^{NN})$ nearest (next nearest) to $\lambda_i$

| | GinOE | $\varphi = 2$ | $\varphi = 3$ | $\varphi = 1$ |
|---|---|---|---|---|
| $-\langle \cos\theta \rangle$ | 0.7379 | 0.7359 | 0.7372 | 0.7871 |
| $\langle r \rangle$ | 0.2347 | 0.2225 | 0.2284 | 0.3516 |

**Complex spacing ratios agree with non-Hermitian Gaussian random matrices for $\varphi \geq 2$**