**Spectra of random sparse generators of Markovian evolution**

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**Motivation**

Continuous time Markov evolution of the probability vector \( P(t) \)

\[
\frac{d}{dt} P(t) = MP(t),
\]

(1)

is solely determined by the generator matrix \( M \). Generic generator matrices are modeled by random matrices. Spectra of dense random generator matrices mismatch relaxation times of physical generators (large spectral gap). We propose sparse random generator matrices.

**Sparse random generator matrix ensemble**

- generator matrix \( M = K + i = D \times D \)-matrix, where
  - \( K \) is adjacency matrix of random graph
  - \( i \) is diagonal matrix with \( J = \sum_i K_{ii} \)
  - \( M \) is negative (combinatorial) Laplacian on graph

- Random graph is uniformly sampled from ensemble of all graphs with \( D \) vertices and fixed vertex degree \( \varphi \) (degree = in-degree = out-degree).

- \( \varphi \) controls the sparsity of \( M \). Sparsity \( \varphi \) can be
  - independent of \( D \): single particles
  - dependent on \( D \): many-body systems with \( \varphi \sim \log D \)

- Random graph is highly likely strongly connected if \( \varphi \geq 2 \).

- Edge weights \( (K_{ij}) \) are positive, i.i.d distributed, e.g. \( \chi^2 \), exponential, uniform, etc.

**Bulk spectrum: Mean and width**

\[
\lambda_i = \min \{ \lambda_i, \lambda_{i+1} \} \quad |\Re \lambda_i| > \lambda_B
\]

Eigenvectors \( \lambda_i \). Mean of bulk is

\[
\mu(\lambda) = \frac{1}{D} \sum_{i=1}^{D} \lambda_i = -\varphi \mu_{ij}
\]

and horizontal width of bulk

\[
\sigma(\Re \lambda_j) = \left( \sum_{i=1}^{D} (\Re \lambda_i - \mu(\lambda))^2 \right)^{1/2}
\]

\[
\geq \sqrt{\frac{\varphi}{D}} + \frac{\sqrt{\varphi}}{\sqrt{D}}
\]

- Average \( \langle \ldots \rangle \) is over matrix ensemble.

- \( \mu_{ij} = \langle M_{ij} \rangle \)

- \( \sigma(\Re \lambda) \) is given by

\[
\sigma = \sqrt{\frac{\varphi}{D}} + \frac{1}{\sqrt{\sigma(\Re \lambda)}}
\]

- Numerically \( \varphi \ll D \): \( \alpha \sim \sqrt{D} \)

**Conclusion**

- Bulk spectrum of sparse random Markov generators \( M \) diverges from equilibrium eigenvalue \( 0 \) if connectivity \( \varphi \) increases with state space size \( D \)

- Spectral edges are determined by extreme values of diagonal of \( M \)

- Left tail of \( M_{ij} \) decreases at most as power-law: spectral gap closes as power in \( D \) for constant \( \varphi \) and crossover to increasing gap at \( \varphi \sim \log D \)

- Largest eigenvalue investigated for exponential (\( \chi^2 \)) and power-law right tail (uniform).

- Spacing ratios agree with random matrices for \( \varphi \geq 2 \).

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**Spectral edges**

Spectral gap \( \gamma = \max_{|\Re \lambda| < D} |\Re \lambda| \)

(largest eigenvalue) \( \tilde{\gamma} = \max_{|\Re \lambda| < D} |\Re \lambda| \)

For symmetric generators:

\( \tilde{\gamma} \geq \max \lambda_i \) and \( \min \lambda_i \geq \varphi \).

Numerically \( \varphi \ll D \) for non-symmetric:

\( \tilde{\gamma} \approx \max \lambda_i \) and \( \min \lambda_i \approx \varphi \).

Spectral edges are given by extreme values of \( D \) distribution.

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**Average spectral gap**

\[
\langle \gamma \rangle \approx D \int dx f(x)(1 - F(x))^{D-1}/D
\]

weights ~ uniform: \( (J, \text{power-law left tail}) \)

\[
\varphi \text{ constant } \Rightarrow \text{power-law decay of average spectral gap}
\]

\[
\varphi \sim \log D \Rightarrow \text{average spectral gap } \sim O(1)
\]

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**Complex spacing ratios**

\[
\gamma_j = \min_{i < j} \{ |\lambda_i - \lambda_j| \}
\]

\[
\lambda_i (\lambda_j) \text{ nearest (next nearest) to } \lambda_i
\]

Complex spacing ratios agree with non-Hermitian Gaussian random matrices for \( \varphi \geq 2 \).