RG boundaries and Cardy's variational ansatz for multiple perturbations

Anatoly Konechny
Heriot-Watt University

August 1, 2023
NORDITA workshop "New Perspectives on Quantum Field Theory with Boundaries, Impurities, and Defects"

based on arXiv:2306.13719

Plan of the talk

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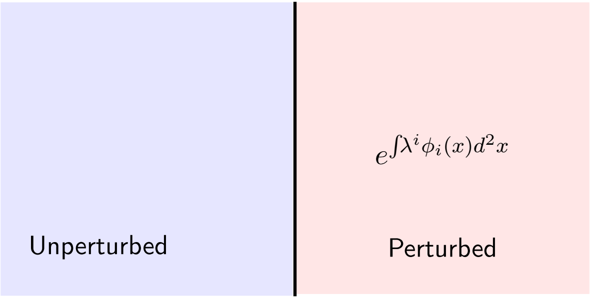
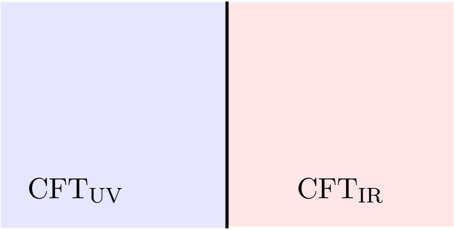
- RG boundaries and gapped phases
- Cardy's variational principle
- look at the examples:
  Ising model,
  Tricritical Ising model
- Reflect on the findings

RG interfaces

We are interested in RG flows of 2D Euclidean QFTs. We would like to have a tool for establishing a global picture of flows and phases that start from a given UV fixed point. An interesting object called RG interface was proposed by I. Brunner and D. Roggenkamp (2007). If the flow is triggered by a perturbation $\Delta S = \int d^2x\, \lambda^i \phi_i(x)$ of the UV fixed point one can consider putting this perturbation on a half plane and letting it flow with the renormalization group flow.



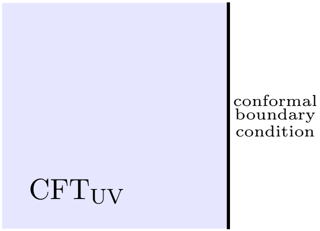
In the far infrared, if the RG flow ends up in a non-trivial fixed point, we obtain a conformal interface that is a 1-dimensional object separating two conformal field theories and respecting the conformal symmetry.



In general it is hard to construct conformal interfaces between two different CFTs. For bulk RG flows between two non-trivial CFTs the RG interfaces were constructed by D. Gaiotto, 2012 for the RG flows between neighbouring Virasoro minimal models triggered by $\phi_{1,3}$ perturbation. The construction generalises to similar flows between other series of CFTs also considered in Ahn, Stanishkov, 2014; Stanishkov, 2016; A. Poghosyan, H. Poghosyan, G. Poghosyan (2013-2022) Gaiotto's construction is rather involved algebraically and does not allow to calculate many quantities related to the interface. Also some work was done on RG interfaces in higher dimensions.

RG boundaries

Most of perturbed CFTs are gapped so that the corresponding RG flows end up in a trivial CFT with a single vacuum state or a finite dimensional space of degenerate vacua. The corresponding RG interface is given by a conformal boundary condition in the UV theory which we call an RG boundary. For trivially gapped theories we get an irreducible conformal boundary while degenerate vacua are described by superpositions of conformal boundary conditions. For rational CFTs we know how to construct all symmetry preserving conformal boundary conditions.



Given a CFT one can study the space of its perturbations by relevant operators. Putting the theory on a circle of circumference R the perturbed Hamiltonian can be written as

$$H = H_0 + \sum_i \lambda_i \int_0^R \phi_i(0, y) dy$$

where

$$H_0 = \frac{2\pi}{R}(L_0 + \bar{L}_0 - \frac{c}{12})$$

is the CFT Hamiltonian and ϕ_i are relevant operators.

If the perturbed theory is trivially gapped then, for large R its vacuum state $|0\rangle_{\lambda}$ is approximated by the boundary state of the corresponding RG boundary:

$$|0\rangle_{\lambda} \sim |B\rangle\rangle$$

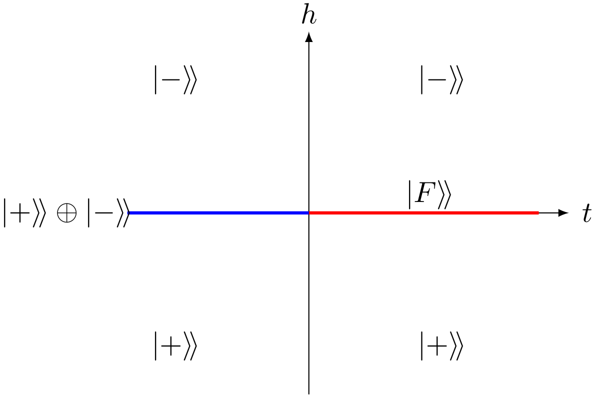
One can then consider the following problem: describe how the space of perturbed theories, parameterised by the relevant couplings: λ^i breaks into domains labelled by the RG boundaries. For example for the critical Ising model there are two relevant operators: the thermal ϕ_ϵ and magnetic ϕ_σ so that the general perturbation is

$$H = H_0 + t\int_0^R \phi_\epsilon(0,y)dy + h\int_0^R \phi_\sigma(0,y)dy\,.$$

The particle spectrum for mixed perturbations was studied extensively by P. Fonseca and A. Zamolodchikov.
The theory has three elementary conformal boundary conditions with boundary states

$$|+\rangle\rangle\,,\qquad |-\rangle\rangle\,,\qquad |F\rangle\rangle\,.$$

The mapping of the theory space in terms of RG boundaries was done by A.K., 2016 and J. Cardy, 2017.



General Questions

The following general questions arise

- What do the RG boundaries have to do with the usual phases, e.g. for the known models?
- In the description of the phase space that uses RG boundaries (and interfaces) certain surfaces separate domains labelled by different RG boundaries. The surfaces themselves are labelled by RG boundaries. What do such transition surfaces have to with the usual thermodynamic phase transitions?
- Can we build an analytic theory that would give us an assignment of RG boundaries (and interfaces) to perturbed CFTs? Such a theory potentially would be an improvement on Landau theory as it would target describing the phase structure of any perturbed CFT, including those for which we do not have conventional tools based on lattice or mean field theory description. Such a theory would take a lot of CFT data, including the exact scaling dimensions and OPE coefficients, as input.

Cardy's variational ansatz

To approximate the vacuum of a perturbed CFT when it is gapped J. Cardy, 2017 proposed to use a conformal boundary state $|\tilde{a}\rangle\rangle$ smoothed out by a translation:

$$|\tau, a\rangle = e^{-\tau H_0}|\tilde{a}\rangle\rangle \,.$$

The choice of conformal boundary condition a and the parameter τ are to be minimised over. Unlike the boundary state itself, the smoothed out state has a finite norm so that we can consider:

$$\frac{\langle \tau, a|H|\tau, a\rangle}{\langle \tau, a|\tau, a\rangle}$$

In the limit $\tau \ll R$ one obtains an explicit formula

$$\frac{\langle \tau, a|H|\tau, a\rangle}{\langle \tau, a|\tau, a\rangle} \to \boxed{E_a \equiv R\Big[\frac{\pi c}{24(2\tau)^2} + \sum_i \lambda^i \frac{A_i^a}{(2\tau)^{\Delta_i}}\Big]}$$

where A_i^a stands for a 1-point function of ϕ_i on a disc with boundary condition a.

By minimising $E_a(\tau)$ over τ we obtain a minimum at

$$\tau = \tau_a^*$$

Comparing the minimal values $E_a = E_a(\tau_a^*)$ and assuming that there is a single conformal boundary condition

$$a = \bar{a}$$

with the smallest $E_{\bar{a}}$ we obtain an approximation to the vacuum state of perturbed theory:

$$|\tau_{\bar{a}}^*, \bar{a}\rangle = e^{-\tau_{\bar{a}}^* H_0} |\tilde{\bar{a}}\rangle\rangle \, .$$

The state space for diagonal minimal models decomposes as

$$\mathcal{H} = \bigoplus_{i=1}^{P} \mathcal{H}_i^L \otimes \mathcal{H}_i^R$$

where each chiral irreducible representation space is graded by the descendant level

$$\mathcal{H}_i^L = \bigoplus_{N=0}^{\infty} \mathcal{H}_{i,N}^L \,, \qquad \mathcal{H}_i^R = \bigoplus_{N=0}^{\infty} \mathcal{H}_{i,N}^R \,.$$

Let $|k_{i,N}\rangle_L$, $k_{i,N} = 1, \ldots, \dim\mathcal{H}^L_{i,N}$ be the basis vectors of an orthonormal basis in $\mathcal{H}^L_{i,N}$ and let $|k_{i,N}\rangle_R$ denote the conjugate basis vectors. The Ishibashi states can then be written as

$$|i\rangle\rangle = \sum_N \sum_{k_{i,N}} |k_{i,N}\rangle_L \otimes |k_{i,N}\rangle_R \, .$$

Cardy's variational states then take the form

$$|\mathrm{var}\rangle = \sum_i \frac{a_i}{\sqrt{\mathcal{N}}} \sum_{N,k_{i,N}} e^{-\frac{2\pi\tau^*}{R}(2h_i+2N)} |k_{i,N}\rangle_L \otimes |k_{i,N}\rangle_R$$

For Virasoro minimal models

$$a_i = \frac{S_{\bar{a}i}}{\sqrt{S_{1i}}}$$

Cardy's ansatz in degenerate case

Sometimes the same minimum value is reached for several variational energies $E_{\bar{a}}$. In this case the subleading terms in the variational energy that go as positive powers of $e^{-R/\tau}$ cannot be neglected. The vacuum state is obtained by diagonalising the matrix

$$\mathcal{M}_{\bar{a}\bar{b}} = \frac{\langle \tau, \bar{a}|H|\tau, \bar{b}\rangle}{\sqrt{\langle \tau, \bar{a}|\tau, \bar{a}\rangle\langle \tau, \bar{b}|\tau, \bar{b}\rangle}} \,.$$

The off-diagonal terms in this matrix are exponentially suppressed. The eigenvalues are split by such exponential factors as expected for symmetry breaking systems at finite volume. The calculation of the matrix elements $\mathcal{M}_{\bar{a}\bar{b}}$ is more involved but can be done given enough CFT data. The vacuum $|\mathrm{var}\rangle$ at the leading order has the same form as above but with different coefficients a_i.

Difficulties with Cardy's ansatz

1. UV sensitivity

Explicit examples show that while the low weight components of the true vacuum vector are often well-approximated by Cardy's ansatz, the high weight tail looks different and sometimes is important. We will see explicit examples of that when discussing in detail the Ising model. Presumably the tail, not captured by Cardy's ansatz, is important when UV divergences are present. When they are absent, in many cases, the ansatz predictions fit very well with numerical TCSA results as shown by M. Lencsés, J. Viti and G. Takács, 2018 who studied single operator perturbations in IM and TIM.

2. Symmetry breaking in disordered region

When a unique energy minimum is achieved by a symmetric conformal boundary condition there are finite energy gaps with all symmetry-breaking boundary conditions. When we switch on a small symmetry-breaking perturbation by continuity the variational minimum is still given by the symmetric boundary condition but it cannot support non-vanishing VEV's of symmetry-breaking operators. Note that this is a problem arising for multiple perturbations (by more than one operator). It arises in both examples we will consider later:IM and TIM. We argue for a different resolution in each example. In the IM the non-trivial VEV is supported by the high weight tail of the vacuum while in TIM one needs to modify Cardy's ansatz so that even the low weight components look different.

Stability with respect to boundary relevant perturbations

We can test whether the Cardy's ansatz states provide a stable minimum by looking at small perturbations by boundary relevant operators. Consider trial states of the form

$$e^{-\tau H_0}|a,\alpha\rangle$$

where $|a,\alpha\rangle$is the perturbed (non-conformal) boundary state. It is obtained by inserting

$$e^{-\alpha \int\limits_0^R \psi(y)dy}$$

on the a-boundary.

The leading order change in the variational energy is linear in α and comes from the average of the perturbing operators:

$$\frac{\langle\langle a,\alpha|e^{-\tau H_0}\phi_i(0,0)e^{-\tau H_0}|a,\alpha\rangle\rangle}{\langle\langle a,\alpha|e^{-2\tau H_0}|a,\alpha\rangle\rangle} = A_i^a\left(\frac{\pi}{4\tau}\right)^{\Delta_i}$$
$$-{}^{(a)}B_i^{\psi}\alpha\left(\frac{\pi}{4\tau}\right)^{\Delta_i+\Delta_\psi-1} f(\Delta_\psi) + \mathcal{O}(\alpha^2)$$

where ${}^{(a)}B_i^{\psi}$ is the bulk-boundary OPE coefficient and

$$f(\Delta_\psi) = \sqrt{\pi}\frac{\Gamma\left(\frac{\Delta_\psi}{2}\right)}{\Gamma\left(\frac{1+\Delta_\psi}{2}\right)}\,.$$

Since the sign of α is not fixed we see that the variational energy will always go down at the linear order in α unless a special condition takes place:

$$\boxed{\sum_i {}^{(a)}B_i^{\psi}\lambda_i\left(\frac{\pi}{4\tau_a^*}\right)^{\Delta_i} = 0}$$

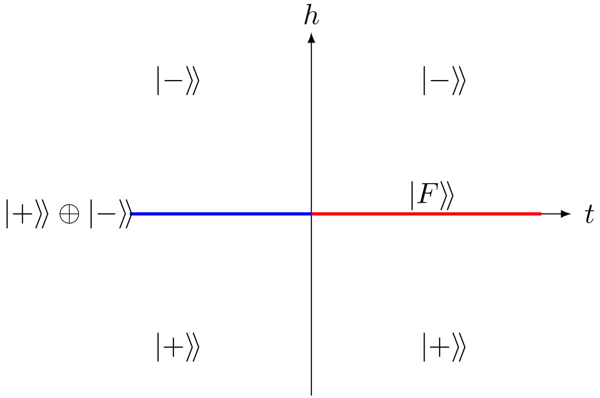
This condition can be satisfied trivially when either the boundary condition does not have any relevant operators or, less trivially, when all relevant bulk-boundary coefficients B_i^{ψ} vanish. We will see when looking at specific (multiple) perturbations of TIM that the condition can be satisfied by fine tuning the RG trajectory so that the sum vanishes via mutual cancellations. Away from this special line the given boundary condition is unstable. We conjecture that the corresponding RG boundaries are given as endpoints of the boundary RG flows triggered by ψ. We will argue that such lines (or more generally surfaces) are similar in some ways to the disorder line in the Ising model and that they are helpful in determining the detailed phase diagram of the theory.

Example 1: the Ising model

$$H = H_0 + t \int_0^R \phi_\epsilon(0,y)dy + h \int_0^R \phi_\sigma(0,y)dy\,.$$

Cardy's variational energies are

$$\begin{aligned}
E_+ &= R\Big[\frac{\pi}{48(2\tau)^2} + \frac{t\pi}{4\tau} + h2^{1/4}\left(\frac{\pi}{4\tau}\right)^{1/8}\Big],\\
E_- &= R\Big[\frac{\pi}{48(2\tau)^2} + \frac{t\pi}{4\tau} - h2^{1/4}\left(\frac{\pi}{4\tau}\right)^{1/8}\Big],\\
E_F &= \Big[\frac{\pi}{48(2\tau)^2} - \frac{t\pi}{4\tau}\Big].
\end{aligned}$$


h
$|-\rangle\rangle$
$|-\rangle\rangle$
$|F\rangle\rangle$
$|+\rangle\rangle \oplus |-\rangle\rangle$
t
$|+\rangle\rangle$
$|+\rangle\rangle$

Adding a small magnetic field in the disordered region

This is the region where we have a second type of problem with the ansatz: it predicts a symmetric vacuum. To fix the problem Cardy suggested to modify the ansatz by switching on an additional boundary magnetic field perturbation that breaks the symmetry:

$$e^{-\tau_{\rm uv}H_0}|h_b\rangle\rangle = g\exp\Big(-i\sum_{n=0}^{\infty} f_{n+1/2}\, a^{\dagger}_{n+1/2}\bar{a}^{\dagger}_{n+1/2}\Big)|0\rangle$$

$$+\tilde{g}\exp\Big(-i\sum_{n=1}^{\infty} f_n a^{\dagger}_n \bar{a}^{\dagger}_n\Big)|\sigma\rangle$$

where

$$f_k = f_k(\alpha,\tau_{\rm uv}/R) = \frac{k-\alpha}{k+\alpha}e^{-\frac{4\pi\tau_{\rm uv}}{R}k}\,,$$

and k is an integer or a half-integer, $\alpha = 2h_b^2 R$ is the dimensionless boundary magnetic field coupling.

This ansatz predicts a very particular behaviour of the vacuum vector components in the oscillator basis, as a function of R. We have compared this prediction with a numerical vacuum obtained using TCSA. For the latter we use the free massless fermion orthonormal basis that contains the states:

$$(\text{phase})a^{\dagger}_{k_1}\dots a^{\dagger}_{k_n}\bar{a}^{\dagger}_{\bar{k}_1}\dots \bar{a}^{\dagger}_{\bar{k}_m}|0\rangle \,, \quad n+m = \text{even}$$

$$k_i, \bar{k}_j \in 1/2 + \mathbb{Z} \text{ in NS sector}$$

$$(\text{phase})a^{\dagger}_{k_1}\dots a^{\dagger}_{k_n}\bar{a}^{\dagger}_{\bar{k}_1}\dots \bar{a}^{\dagger}_{\bar{k}_m}|\sigma\rangle \,, \quad n+m = \text{even}$$

$$k_i, \bar{k}_j \in \mathbb{Z} \text{ in Ramond sector}$$

In free fermion version of TCSA we restrict the Hamiltonian of perturbed theory onto a spin zero truncated subspace

$$\sum_i k_i = \sum_j \bar{k}_j \leq n_c$$

where n_c is the truncation parameter.

We then find numerically the truncated Hamiltonian eigenvalues and eigenvectors. The normalised vacuum vector can be written as

$$|0\rangle_{t,h} = \sum_i C_i |i\rangle$$

where $|i\rangle$ are the vectors of the orthonormal basis we described before. Each such vector has a conformal weight w_i assigned to it. We will present the vacuum state as a scattered plot of pairs

$$(w_i, C_i)$$

We label RG trajectories by a dimensionless parameter

$$\xi = \frac{h}{|t|^{15/8}}$$

We also use dimensionless radius

$$r = R|m| = R2\pi|t|$$

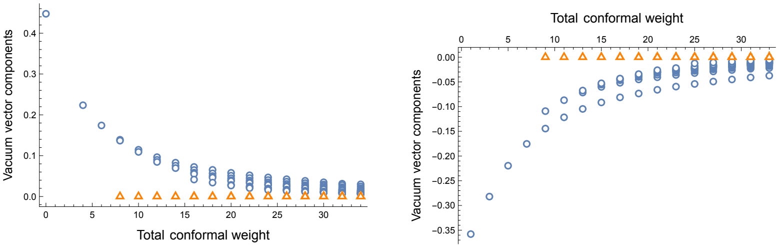


Figure: The TCSA vacuum vector components C_i for the mixed perturbation with $\xi = 1$, $t > 0$ and at $|m|R = 25$. The left plot represents the vacuum sector and the right plot – the ϵ-sector. The blue circles mark the diagonal components and the orange triangles – the non-diagonal ones.

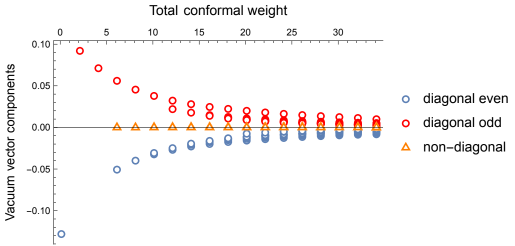


Figure: The TCSA vacuum vector components C_i in the σ-sector plotted against the total conformal weight of the basis vectors taken for the mixed perturbation with $\xi = 1$, $t > 0$ and at $|m|R = 25$.

When we change R we find the same behaviour of C_i as predicted by the boundary state of the boundary magnetic field model.

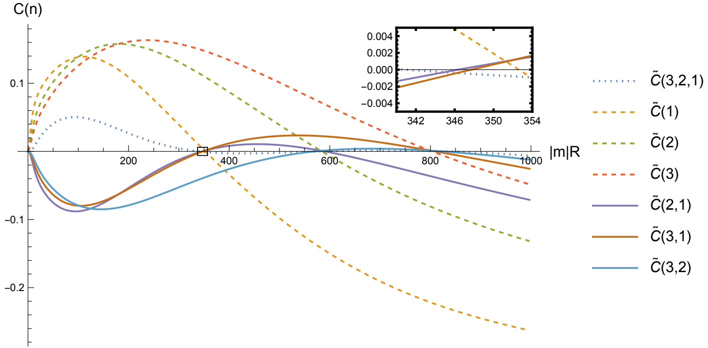


Figure: The first seven components $\tilde{C}(\mathbf{n})$ in the Ramond sector of the perturbed Ising model with $\xi = 1$, $t > 0$. These are obtained for TCSA using the oscillator basis and level truncation at $n_c = 17$. The insert plot shows a magnified region near the first crossings of the horizontal axis.

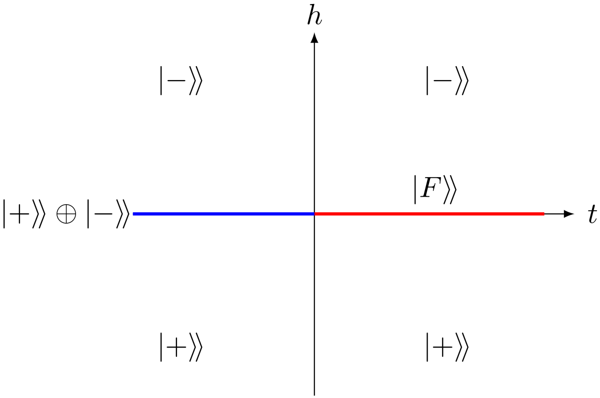
For sufficiently large R the low weight components of the vacuum take the form of a smeared boundary state

$$e^{-\tau_{\rm ir}H_0}|-\rangle\rangle$$

with $\tau_{\rm ir} \neq \tau_{\rm uv}$. The R-evolution describes a boundary RG flow

$$(F) \longrightarrow (-)$$

Thus for large R the low weight components are described by the ordinary Cardy ansatz state whose variational energies we analysed. The difference is in the high weight tail of the vacuum that differs from the standard Cardy ansatz and must be responsible for lowering the variational energy versus $e^{-\tau H_0}|F\rangle\rangle$ states.



$$(-) \longleftarrow (F) \longrightarrow (+)$$

$$(-) \longleftarrow (+) \oplus (-) \longrightarrow (+)$$

Tricritical Ising model

The TIM is the second model after Ising in the diagonal minimal model sequence. It has central charge $c = 7/10$, six primary states: $1, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'$ and six associated irreducible conformal boundary conditions. This model gives a universality class of critical phenomena in two dimensions in the presence of a tricritical point. The underlying lattice model which describes the same universality class as perturbed TIM is the spin-1 Ising model which was originally introduced by Blume and Capel and later generalised by Blume, Emery and Griffiths. For the latter (BEG) model the lattice Hamiltonian can be written as

$$\mathcal{H} = -H\sum_{j=1}^{N} s_j + \Delta\sum_{j=1}^{N} s_j^2 - J\sum_{\langle i,j\rangle} s_i s_j - H_3\sum_{\langle i,j\rangle} s_i s_j(s_i+s_j) - K\sum_{\langle i,j\rangle} s_i^2 s_j^2\,.$$

$$s_i \in \{0, 1, -1\}$$

The model has two magnetic couplings: H, H_3 and three energy couplings:Δ, J, K. The biquadratic coupling $K$is irrelevant with respect to the tricritical Ising fixed point. The remaining four couplings in the continuum limit give rise to the four relevant directions given by four CFT scaling operators. We will first focus on perturbations by two most relevant operators:

$$H = H_0 + \lambda_\epsilon \int_0^R \phi_\epsilon(0,y)dy + \lambda_\sigma \int_0^R \phi_\sigma(0,y)dy\,.$$

$$\Delta_\sigma = 3/40\,, \qquad \Delta_\epsilon = 1/5$$

the model has six elementary boundary conditions. In notation introduced by L. Chim

$$|0\rangle\rangle\,, \quad |+\rangle\rangle\,, \quad |-\rangle\rangle\,, \quad |+0\rangle\rangle\,, \quad |-0\rangle\rangle\,, \quad |d\rangle\rangle$$

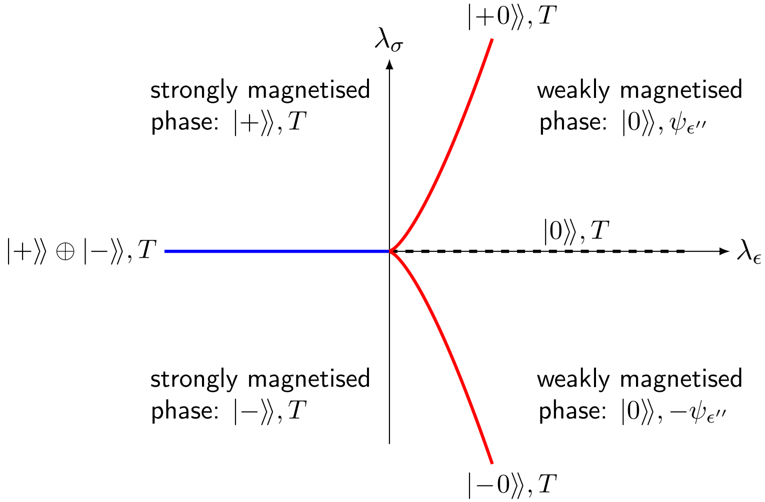


Figure: The phase diagram of TIM perturbed by ϕ_σ and ϕ_ϵ. The phases are labeled by conformal boundary states together with the leading irrelevant operator defining their perturbation at finite volume.

Weak magnetic field in the disordered region

The single coupling perturbations were analysed by Lencses, Viti and Takacs, 2018.
We label the RG trajectories by the parameter

$$\xi_\sigma = \frac{\lambda_\sigma}{|\lambda_\epsilon|^{77/72}}$$

Cardy's ansatz works really well quantitatively in the $\lambda_\epsilon < 0$ region, so we focus on the $\lambda_\epsilon > 0$ region which has the dashed line and the two red lines.
Near the dashed line, i.e. at weak magnetic fields, we have the same problem with Cardy's ansatz as in the Ising model. In TIM however TCSA shows that even the low weight components look different. We propose a modification of the ansatz which includes a boundary irrelevant $\psi_{3,1}$ field.

Kink transitions (the red lines)

The predictions for the red lines can be found from stability analysis of the Cardy's ansatz built on the $|+0\rangle\rangle$ and $|-0\rangle\rangle$ boundary conditions. The $(+0)$ boundary has a relevant boundary operator $\psi_{\epsilon'}^{(+0)}$ that is responsible for a pair of boundary flows:

$$(0) \longleftarrow (+0) \longrightarrow (+)$$

Both of our bulk perturbing fields: ϕ_ϵ and ϕ_σ couple to it via the bulk-boundary OPE coefficients: ${}^{(+0)}B_{\sigma}^{\epsilon'}$, ${}^{(+0)}B_{\epsilon}^{\epsilon'}$. These coefficients have the opposite signs so that the ϕ_ϵ perturbation with $\lambda_\epsilon > 0$ drives the system (by lowering the energy) towards the (0)-phase while ϕ_σ drives the system in the opposite direction towards the $(+)$-phase.

The two effects cancel each other at the leading order when

$$\lambda_{\epsilon}^{10/9}\Big(\xi_{\sigma}\frac{^{(+0)}B_{\sigma}^{\epsilon'}}{(\tilde{\tau}^{*})^{3/40}} + \frac{^{(+0)}B_{\epsilon}^{\epsilon'}}{(\tilde{\tau}^{*})^{1/5}}\Big) = 0$$

where

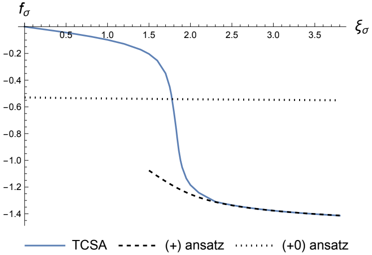
$$\tilde{\tau} = \frac{4}{\pi}\tau|\lambda_{\epsilon}|^{5/9}$$

and $\tilde{\tau}^{*} = \tilde{\tau}^{*}(\xi_{\sigma})$ is the value of $\tilde{\tau}$ that minimises the variational energy E_{+0}. We find numerically that the condition is satisfied at

$$\xi_{\sigma} = \xi_{\rm c}^{(3)} \equiv 1.78138...$$

Using TCSA we find numerical evidence that there is indeed a transition of the vacuum vector structure close to this value. To elucidate the physical meaning of the transition it is instructive to look at the VEVs of scaling operators as functions of ξ_{σ}.

$$\langle \phi_i \rangle_{\rm pl} = m^{\Delta_i} f_i(\xi_\sigma)$$



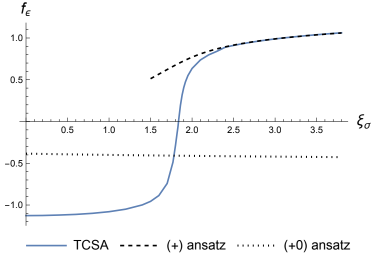


Figure: Scaled vacuum expectation values on the plane of ϕ_σ (left) and ϕ_ϵ (right) plotted against ξ_σ. The solid blue line is from TCSA with $n_c = 11$, the dashed line is from Cardy's ansatz for the $(+)$ boundary condition and the dotted line is from Cardy's ansatz for the $(+0)$ boundary condition.

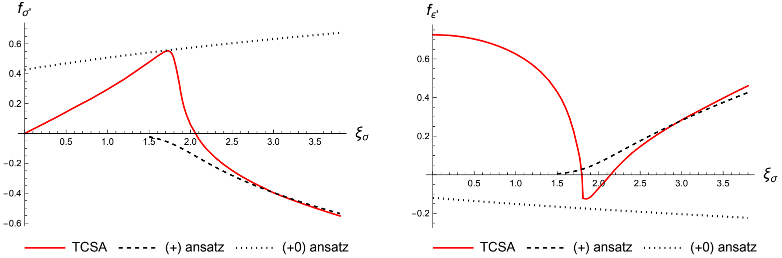


Figure: Scaled vacuum expectation values on the plane of $\phi_{\sigma'}$ (left) and $\phi_{\epsilon'}$ (right) plotted against ξ_σ. The solid line is from TCSA at $n_c = 11$, the dashed line is from Cardy's ansatz for the $(+)$ boundary condition and the dotted line is from Cardy's ansatz for the $(+0)$ boundary condition.

The mass gap

For a finite size system one can define an effective scaling exponent for the energy gap as

$$b = R\frac{d}{dR}\ln(E_1 - E_0)$$

It can be measured in TCSA with relatively little computing power.

$$b \sim 0 \qquad \text{massive theory with unique vacuum}\,,$$

$$b \sim -1 \qquad \text{criticality (second order phase transition)}$$

$$b \sim A - BR \qquad \text{degenerate vacuum (first order phase transition)}$$

For the kink transition we have a massive system with unique vacuum.

We can associate the following pairs of boundary RG flows with the symmetry breaking (blue) and the kink lines (red)

$$(0) \longleftarrow (+0) \longrightarrow (+)$$

$$(0) \longleftarrow (-0) \longrightarrow (-)$$

$$(-) \longleftarrow (+) \oplus (-) \longrightarrow (+)$$

Switching on the third coupling: $\lambda_{\sigma'}$

We add a third perturbation - by the subleading magnetisation $\phi_{\sigma'}$ of dimension $\Delta_{\sigma'} = 7/8$.

$$H = H_0 + \lambda_{\epsilon} \int_0^R \phi_{\epsilon}(0,y)dy + \lambda_{\sigma} \int_0^R \phi_{\sigma}(0,y)dy + \lambda_{\sigma'} \int_0^R \phi_{\sigma}(0,y)dy \,.$$

The RG trajectories are parameterised by two parameters: ξ_{σ} and

$$\xi_{\sigma'} = \frac{\lambda_{\sigma'}}{|\lambda_{\epsilon}|^{5/8}} \,.$$

The new field does not couple to the relevant boundary field on $(+0)$ but it shifts the position of the minimum of E_{+0}. The first order stability condition gives us a 2D surface – a "kink surface", which can be described by a function $\xi_{\sigma} = \xi_{\sigma,c}(\xi_{\sigma'})$.

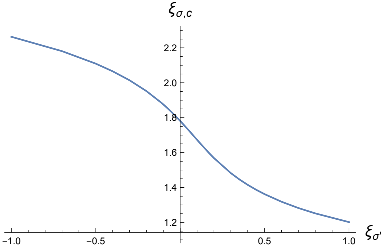


Figure: The curve $\xi_\sigma = \xi_{\sigma,c}(\xi_{\sigma'})$ that specifies a surface of kink phase transitions in the λ_σ, λ_ϵ, $\lambda_{\sigma'}$ space.

As we move along this surface from $\xi_{\sigma'} = 0$ towards positive $\xi_{\sigma'}$ we find using TCSA that the mass gap closes. Optimising the region $b \approx -1$ we find a symmetry breaking critical line at

$$\xi_\sigma^* = 1.682... \qquad \xi_{\sigma'}^* = 0.096... \qquad n_c = 11$$

This is quite close to the kink surface obtained via the Cardy's ansaz: c.f. a point on the surface $(\xi_\sigma, \xi_{\sigma'}) = (1.677..., 0.096)$

The symmetry breaking critical lines

This critical line leads to the critical Ising model universality class

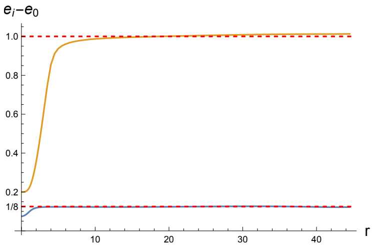


Figure: On the left plot, the lowest two scaled gaps $e_i - e_0 = (E_i - E_0)R/2\pi$ are shown together with the expected values: $1/8$ and 1. On the right plot, scaled gaps for the first 11 eigenstates in the σ-sector of the Ising model.

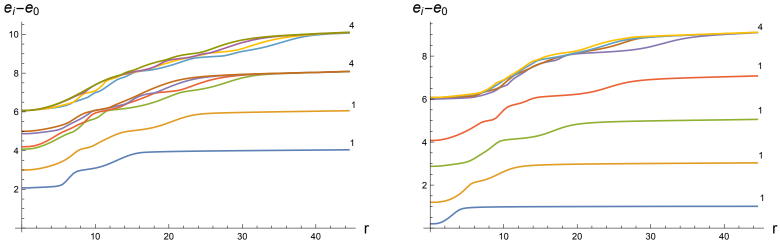


Figure: Scaled gaps $e_i - e_0 = (E_i - E_0)R/2\pi$ for the first 10 states in the vacuum sector (left plot) and for the first 8 states in the ϵ-sector of the Ising model (right plot).

Going beyond the critical line along the kink surface we find the 1st order transition surface (two phase coexistence).

Interpretation using Landau theory

Consider Landau free energy in a three coupling space:

$$\mathcal{F}(\mu) = -h\mu + \frac{t_2}{2}\mu^2 - \frac{t_3}{3}\mu^3 + \frac{\mu^6}{6}$$

We can consider a 2D surface in the coupling constant space specified by two equations

$$\frac{d\mathcal{F}}{d\mu} = 0\,, \qquad \frac{d^3\mathcal{F}}{d\mu^3} = 0$$

In a single phase region we have for the second derivative of magnetisation

$$\frac{\partial^2\mu}{\partial h^2} = -\frac{\frac{\partial^2 h}{\partial \mu^2}}{\left(\frac{\partial h}{\partial \mu}\right)^3} = -\frac{\frac{\partial^3 \mathcal{F}}{\partial \mu^3}}{\left(\frac{\partial^2 \mathcal{F}}{\partial \mu^2}\right)^3}$$

This means that away from criticality, on the surface at hand the magnetisation has an inflection point. Expanding near it in the order parameter as

$$\mu = \left(\frac{t_3}{10}\right)^{1/3} + \xi$$

we obtain

$$h - h^* = |t_3|^{2/3} 10^{1/3} \xi^3 + T\xi + \frac{5}{10^{1/3}} |t_3|^{1/3} \xi^4 + \xi^5 \,,$$

$$T = t_2 - \frac{3}{2 \cdot 10^{1/3}} |t_3|^{4/3} \,.$$

When

$$|\xi| \ll \frac{10^{2/3}}{5} |t_3|^{1/3}$$

we can neglect the quartic and quintic terms. We get then the same magnetisation behaviour as on the $h = 0$ line of the Ising model.

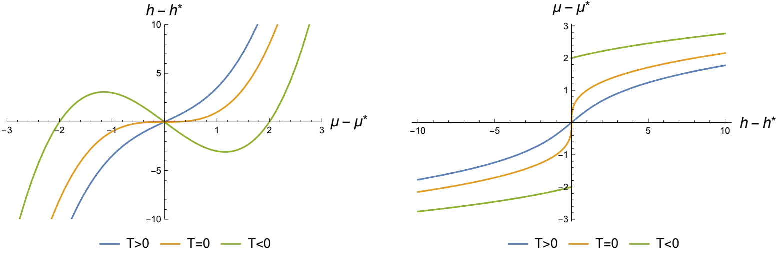


Figure: The external field as a function of magnetisation (left) and the magnetisation as a function of external field (right) at different values of parameter T.

The surface at hand (of kinks or inflection points) that starts from the single phase region ends on the critical line which is defined by the conditions

$$\frac{d\mathcal{F}}{d\mu} = 0 \,, \qquad \frac{d^2\mathcal{F}}{d\mu^2} = 0 \,, \qquad \frac{d^3\mathcal{F}}{d\mu^3} = 0$$

which in our parameterisation is given by $T = 0$.

Attempt at a summary

1. What have we learned about Cardy's ansatz?
   - Sometimes the ansatz works really well quantitatively. Typically this happens for strongly relevant operators in a certain region of phase diagram.
   - Sometimes the ansatz describes qualitatively the low weight components of the vacuum but the variational energy has significant contributions from the high weight tail which is not correctly described by the ansatz. The numerical value of the τ parameter may be incorrectly predicted by the ansatz. This behaviour most likely has to do with UV divergences.
   - Sometimes the ansatz does not describe even qualitatively the low weight components and needs to be modified. This may have to do with the presence of boundary irrelevant operators of dimension $1 < \Delta < 2$ as in TIM.

2. What have we learned about RG boundaries?

- Given a CFT with all conformal boundary conditions known, to construct its phase diagram one can start with stable boundary conditions, with no relevant fields on them, to represent the phases in the bulk of the theory space.
- The unstable conformal boundary conditions then label the transition surfaces. In particular the direct sums of stable boundary conditions are associated with first order phase transition surfaces (phase coexistence).
- One can also use the stability condition (and information from boundary RG flows) to find the surfaces in the theory space on which other unstable boundary conditions (in particular the irreducible ones) stabilise (at least at the leading order). These would be analogues of the disorder line in IM and the kink surface in TIM
- Such surfaces give some indication as to where second order transitions (critical lines) can be found. With an additional input (e.g. from TCSA) they may be located precisely.

3. What is missing?

Need information on the mass gap. Perhaps we do not need to know the whole RG interfaces to chart the phase diagram. An ansatz for the first excited state? Some other indicator that we cannot have a conformal boundary for a given RG trajectory and must have an interface?

END OF TALK