Entanglement entropies for Lifshitz fermionic fields at finite density

Erik Tonni

SISSA

- **Mihail Mintchev, Diego Pontello, Alberto Sartori and E.T.**
  [2201.04522] JHEP
- **Mihail Mintchev, Diego Pontello and E.T.**
  [2206.06187] JHEP

New Perspectives on Quantum Field Theory
with Boundaries, Impurities and Defects

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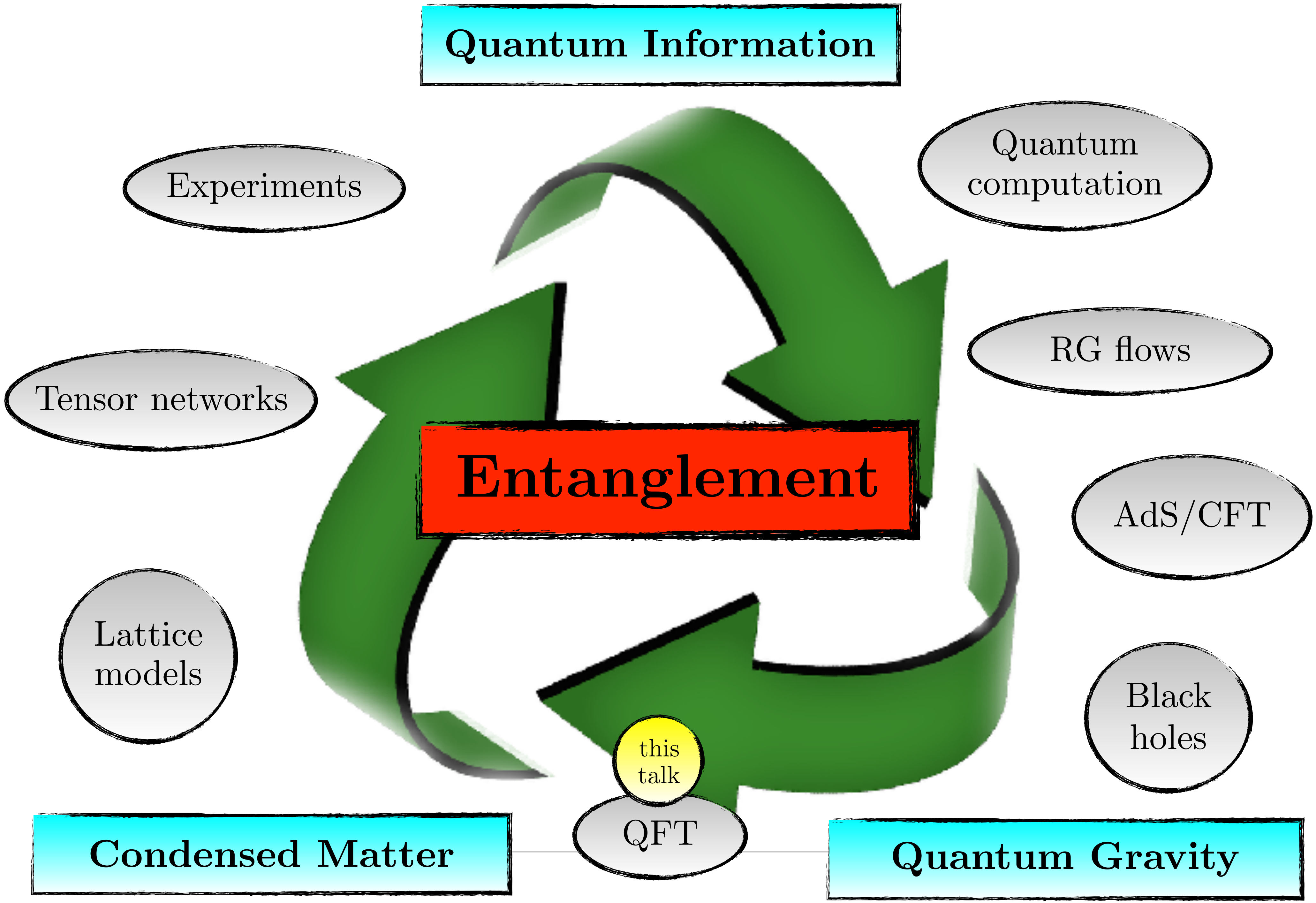
Entanglement: a crossroad of interests



Motivations

- Entanglement entropy in $1+1$ dimensional relativistic field theories:

  - CFT on the line and in the ground state [Holzey, Larsen, Wilczek, (1994)] [Calabrese, Cardy, (2004)]

$$S_A = \frac{c}{3} \log(\ell/\epsilon) + \dots \qquad \epsilon \to 0$$

  - Entropic C function [Casini, Huerta, (2004)]

$$C_A = \ell \, \partial_\ell S_A$$

- Entanglement entropy in $1+1$ non-relativistic field theories?

[Gioev, Klich, (2005)] [Wolf, (2005)] [Fradkin, Moore, (2006)] [Balasubramanian, McGreevy, (2008)]
[Kachru, Liu, Mulligan, (2008)] [Fradkin, (2009)] [Solodukhin, (2009)]
[Ogawa, Takayanagi, Ugajin, (2011)] [Leschke, Sobolev, Spitzer, (2013)]
[Cardy, (2016)] [Mozaffar, Mollabashi, (2017)] [Hartmann, Kavanagh, Vandoren, (2017)]
[Angel-Ramelli, Giangreco, Thorlacius, (2019)] [Daguerre, Medina, Solis, Torroba, (2020)]
. . . and many others

Outline

[Mintchev, Pontello, Sartori, E.T., (2022)]

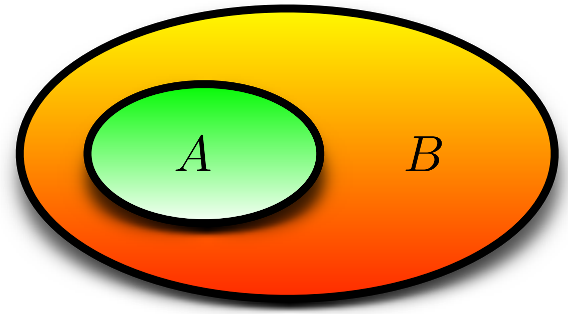
- Lifshitz fermions on the line at finite density
- Entanglement entropies of an interval
  - Expansions in the two limiting regimes

[Mintchev, Pontello, E.T., (2022)]

- Schrödinger field theory on the half line
  with either Neumann or Dirichlet boundary conditions
- Entanglement entropies of an interval
  at the beginning of the half line
  - Expansions in the two limiting regimes
- Cumulant expansion of the entanglement entropy

Entanglement: Hamiltonian, Spectrum and Entropies

- Quantum system in its ground state: $\rho = |\Psi\rangle\langle\Psi|$
  Factorised Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



e.g.: spatial bipartition

- A's **reduced density matrix** $\rho_A = \mathrm{Tr}_B \rho$

(normalisation: $\mathrm{Tr}\rho_A = 1$)

$$\rho_A \propto e^{-K_A}$$

Entanglement Hamiltonian (EH) K_A
(modular Hamiltonian)

- Entanglement spectrum $\{\lambda_j\}$: spectrum of K_A
- Entanglement entropies:
  - → Entanglement entropy
  - → Rényi entropies
  - → Single copy entanglement

$$S_A \equiv -\mathrm{Tr}\big(\rho_A \log \rho_A\big) = \lim_{\alpha \to 1} S_A^{(\alpha)}$$

$$S_A^{(\alpha)} \equiv \frac{1}{1-\alpha} \log\big[\mathrm{Tr}(\rho_A^\alpha)\big]$$

$$S_A^{(\infty)} = -\log\big(\lambda_{\max}\big)$$

Lifshitz fermions in one dimension on the line

- Family of Lifshitz fermion fields whose evolution is given by

$$\left[\mathrm{i}\,\partial_t - \frac{1}{(2m)^{z-1}}\,(-\mathrm{i}\,\partial_x)^z\right]\psi(t,x) = 0 \qquad z \in \mathbb{N}$$

  - $z = 1$ — Relativistic chiral massless fermion field
  - $z = 2$ — Schrödinger fermion field

- Dispersion relation

$$\omega_z(k) \equiv \frac{k^z}{(2m)^{z-1}}$$

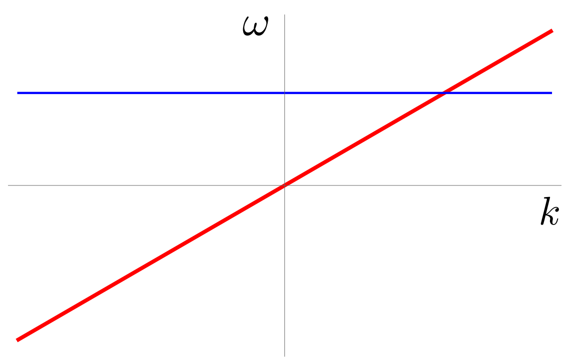
  - $z = 2n + 1$ — $\omega(k)$ unbounded from below
  - $z = 2n$ — $\omega(k)$ bounded from below

Lifshitz fermions at zero temperature and finite density

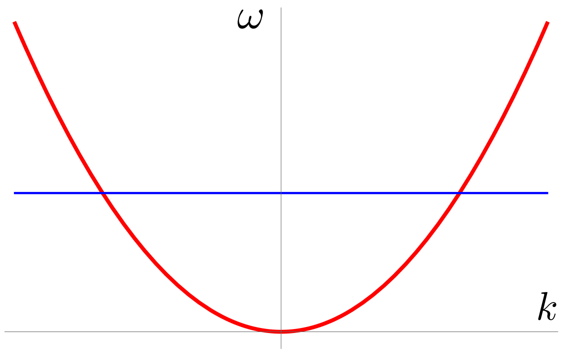
- Gibbs state at zero temperature and finite chemical potential μ

Fermi momentum $$k_{\mathrm{F},z} \equiv (2m)^{1-1/z}\mu^{1/z}$$

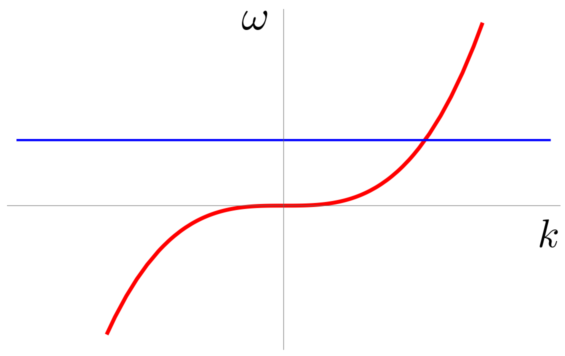
- $z = 1$



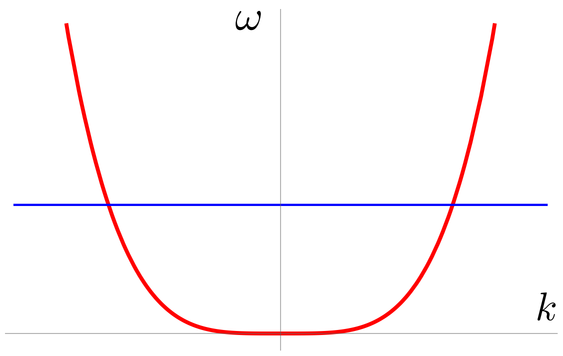
- $z = 2$



- $z = 3$



- $z = 4$



→ The Tomonaga-Luttinger approximation is not employed

Two point function at zero temperature and finite density

- The two point function depends on the distance $x_{12} \equiv x_1 - x_2$

$$\langle \psi^*(t,x_1)\,\psi(t,x_2)\rangle_{z,\infty,\mu} = \begin{cases} \dfrac{e^{-ik_{F,z}\,x_{12}}}{2\pi i\,(x_{12} - i\varepsilon)} & z = 2n+1 \\[2ex] \dfrac{\sin(k_{F,z}\,x_{12})}{\pi\,x_{12}} & z = 2n \end{cases}$$

→ The behaviour at coincident points depends on the parity of z

$$\langle \psi^*(t,x_1)\,\psi(t,x_2)\rangle_{z,\infty,\mu} \;\to\; \begin{cases} z = 2n+1 & \text{singular at } x_1 = x_2 \\ z = 2n & \text{regular at } x_1 = x_2 \end{cases}$$

Spectral problem for the interval on the line

- $S_A^{(\alpha)}$ can be studied by considering the spectral problem associated to the two point function restricted to A [Peschel, (2002)]

$$A = (-R, R) \subset \mathbb{R}$$

- Even z: the kernel is finite in the coincident points limit [Eisler, Peschel, (2013)]

$$\int_{-1}^{1} K(\eta; x-y)\, f_n(\eta; y)\, \mathrm{d}y \;=\; \gamma_n\, f_n(\eta; x) \qquad n \in \mathbb{N}$$

$$K(\eta; x-y) \equiv \frac{\sin[\eta(x-y)]}{\pi(x-y)} \qquad\qquad \eta \equiv R\, k_{\mathrm{F},z}$$

- Odd z: the kernel is divergent in the coincident points limit

$$\int_{-R}^{R} \frac{\mathrm{e}^{-\mathrm{i}k_{\mathrm{F},z}(x-y)}}{2\pi\mathrm{i}\,(x-y-\mathrm{i}\varepsilon)}\, \phi_s(y)\, \mathrm{d}y \;=\; \gamma_s\, \phi_s(x) \qquad s \in \mathbb{R}$$

$\tilde{\phi}_s(x) \equiv \mathrm{e}^{\mathrm{i}k_{\mathrm{F},z}x}\, \phi_s(x) \quad \Longrightarrow \quad \gamma_s$ (hence also $S_A^{(\alpha)}$) independent of $k_{\mathrm{F},z}$

Even z : Spectrum of the sine kernel

- This spectral problem has been solved by Slepian, Pollak and Landau who studied the *time-frequency limiting process* for signals

[Slepian, Pollak, Landau; a series of papers in the early 1960's] [Osipov, Rokhlin, Xiao, (2013)]

$n \in \mathbb{N}_0$

$$\gamma_n(\eta) = \frac{2\eta}{\pi} \mathcal{R}_{0n}(\eta, 1)^2$$

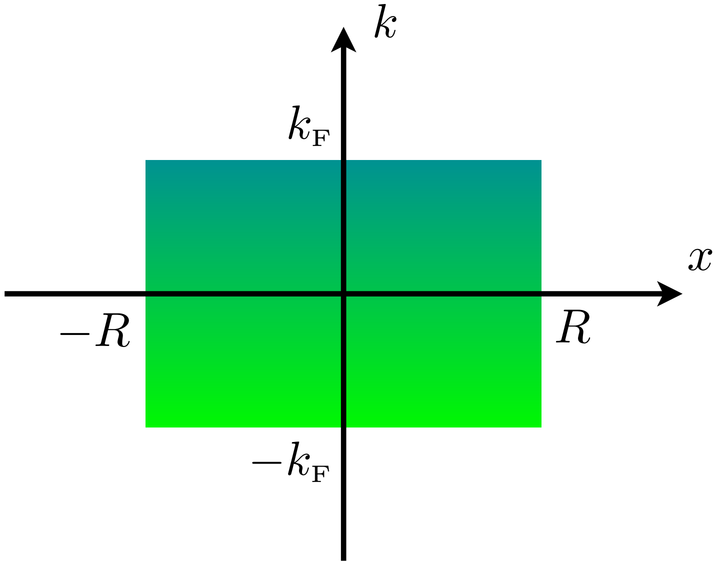
$0 < \gamma_n < 1$

radial Prolate Spheroidal Wave Functions (PSWF) of zero order $\mathcal{R}_{0n}$

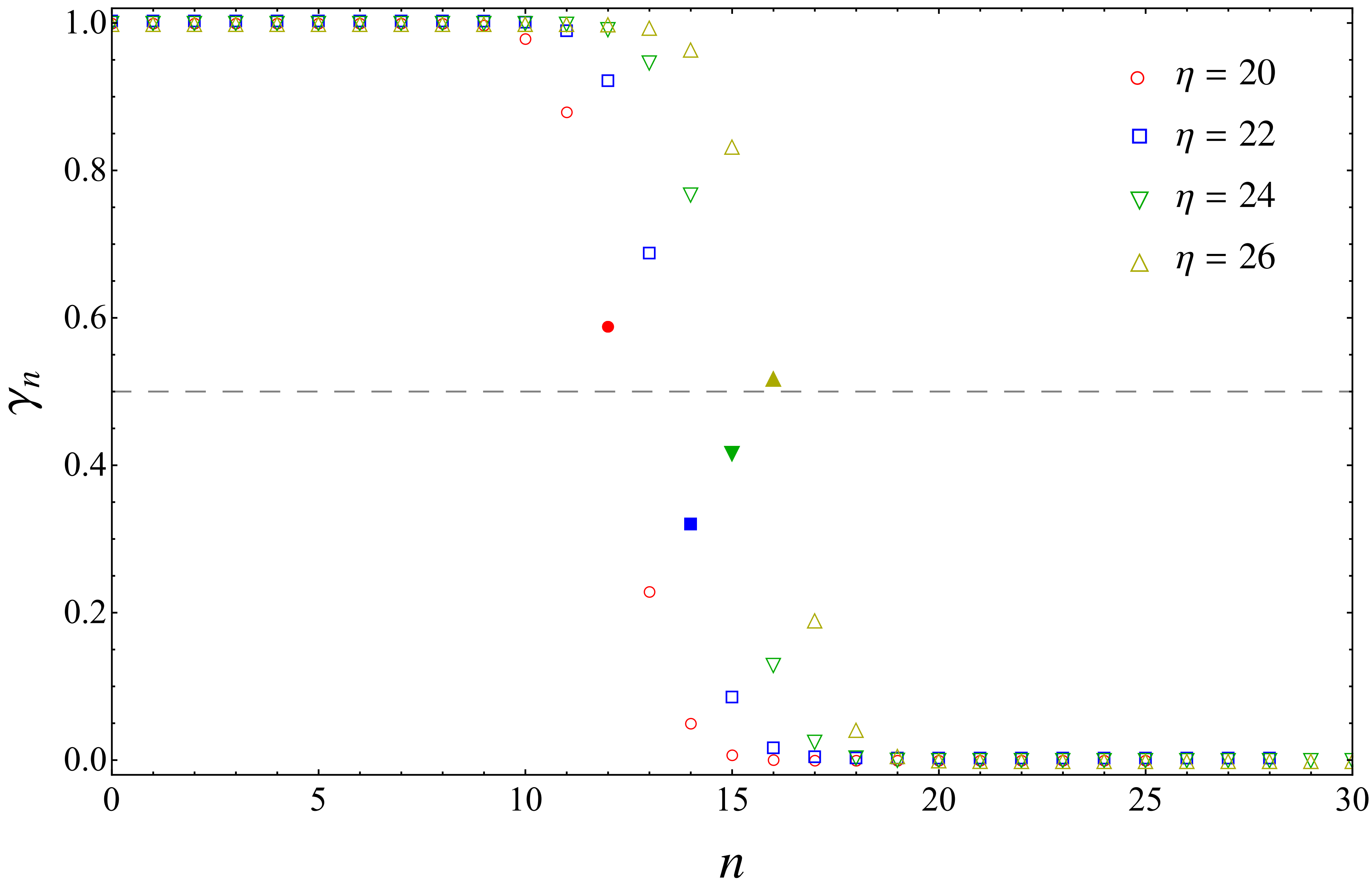
- *Space-momentum limiting process* :

$$\eta \equiv R\, k_{\mathrm{F}}$$

The parameter η is proportional to the area of a rectangular region in the phase space



Spectrum of the sine kernel



Entanglement entropies of the interval on the line

- $S_A \equiv -\,\mathrm{Tr}\big(\rho_A \log \rho_A\big) = \sum_{n=0}^{\infty} s(\gamma_n)$ $\qquad s(x) \equiv -\,x \log(x) - (1-x)\log(1-x)$

  $S_A^{(\alpha)} \equiv \dfrac{\log\big[\mathrm{Tr}(\rho_A^\alpha)\big]}{1-\alpha} = \sum_{n=0}^{\infty} s_\alpha(\gamma_n)$ $\qquad s_\alpha(x) \equiv \dfrac{1}{1-\alpha}\log\big[x^\alpha + (1-x)^\alpha\big]$

  The single copy entanglement $S_A^{(\infty)} = -\log(\lambda_{\mathrm{max}})$ has been also explored

- PSWF: $\gamma_n \to 0$ as $n \to \infty$ with a super-exponential decay rate [Widom, (1994)]

$$0 \leqslant \gamma_n \leqslant \tilde{\gamma}_n \qquad\qquad \tilde{\gamma}_n = \frac{\eta^{2n+1}\, e^{2n}}{(4n)^{2n+1}}\big[1 + O(1/n)\big]$$

→ Entanglement entropies $S_A^{(\alpha)}$ are finite functions of η

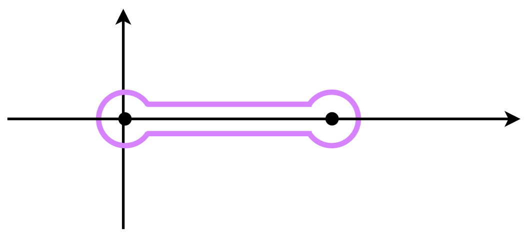
- $S_A^{(\alpha)}$ →
  - $z = 2n+1$ UV divergent as the UV cutoff vanishes
  - $z = 2n$ finite function of $\eta \equiv R\, k_{\mathrm{F},2n}$

Entanglement entropies: PSWF & tau function approaches

- We employ two ways to explore the entanglement entropies

$$S_A^{(\alpha)} = \sum_{n=0}^{\infty} s_\alpha(\gamma_n) \quad \longrightarrow \quad \text{PSWF approach}$$

$$= \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} s_\alpha(z)\, \partial_z \log(\tau)\, \mathrm{d}z \quad \longrightarrow \quad \tau \text{ function approach}$$



$\mathfrak{C}$ is a closed path encircling $(0,1)$

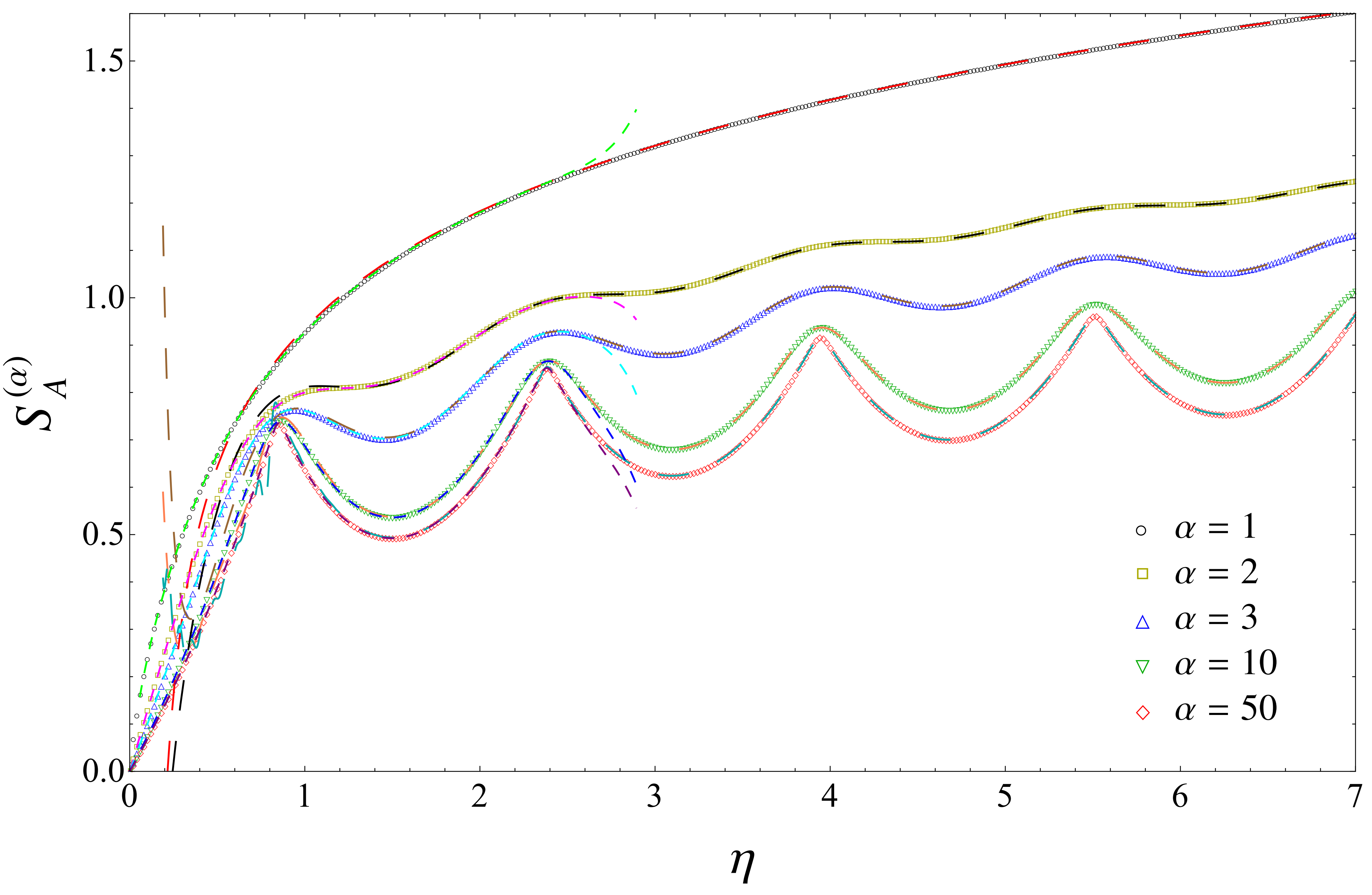
$$\tau \equiv \det\left(I - z^{-1} K\right)$$

Fredholm determinant of the sine kernel

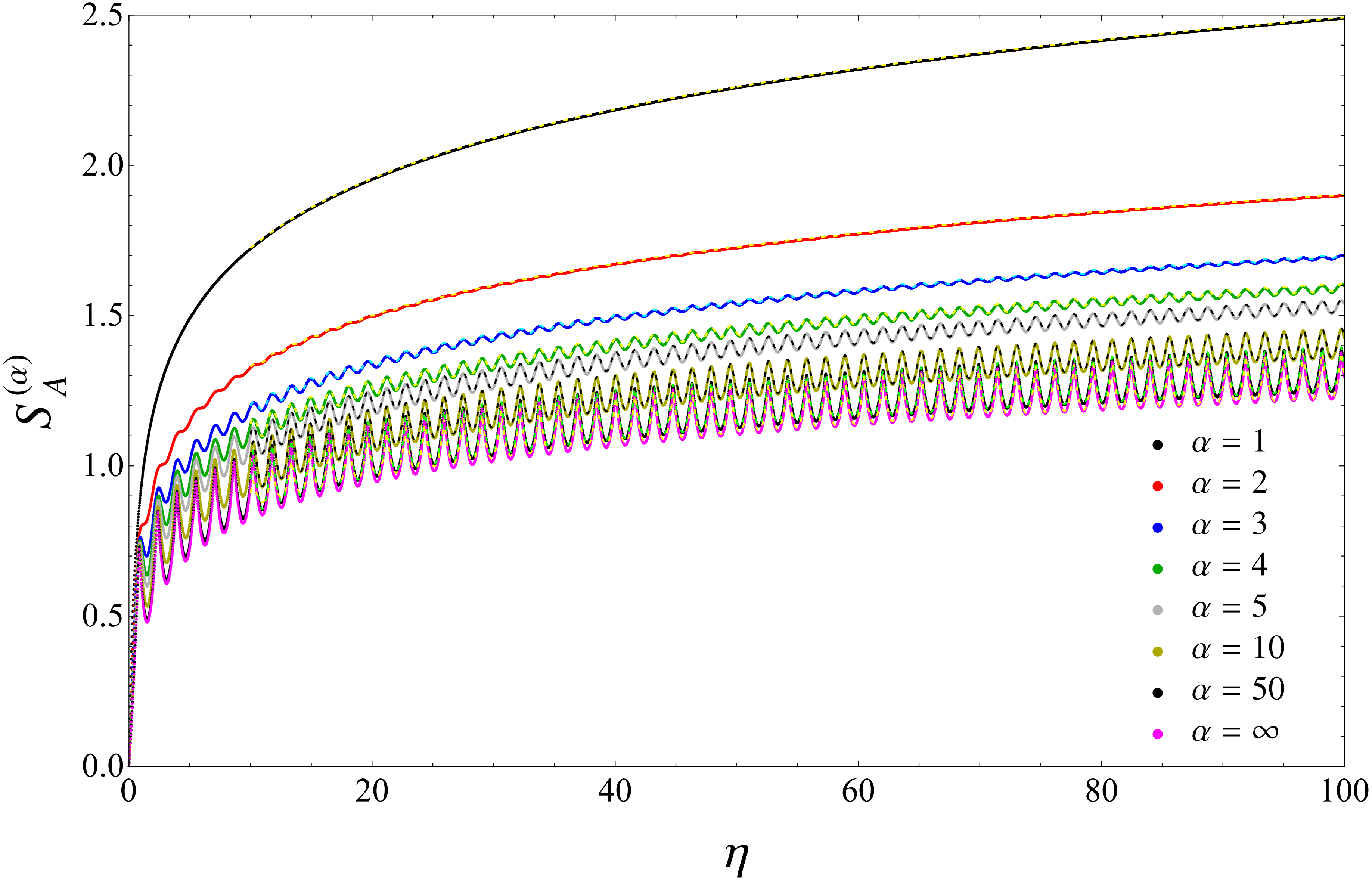
- Analogous methods employed in lattice calculations
  [Jin, Korepin, (2003)] [Keating, Mezzadri, (2004)]

- Tau function approach:
  - Sine kernel τ can be studied through the solution of a Painlevé V
    [Jimbo, Miwa, Mori, Sato, (1980)]
  - Important recent advances in the comprehension of the solutions of the Painlevé equations made through the Kyiv formula
    [Gamayun, Iorgov, Lisovyy, (2012), (2013)]

Even z : Entanglement entropies of an interval on the line



Even z : Entanglement entropies of an interval on the line



→ S_A is a strictly increasing function of η

Even z : Analogue of the relativistic entropic C function

- Consider the analogue of the entropic C function introduced for relativistic field theories in $1+1$ [Casini, Huerta, (2004)]

$$C \equiv \eta\, \partial_\eta S_A$$

- The PSWF approach or the τ function approach can be used

$$C = \sum_{n=0}^{\infty} \big[\log\big(1/\gamma_n - 1\big)\big]\, \eta\, \gamma_n' = \lim_{\epsilon,\delta \to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} s(z)\, \partial_z \sigma\, \mathrm{d}z$$

- The auxiliary function σ satisfies a Painlevé V [Jimbo, Miwa, Mori, Sato, (1980)] [McCoy, Tang, (1986)]

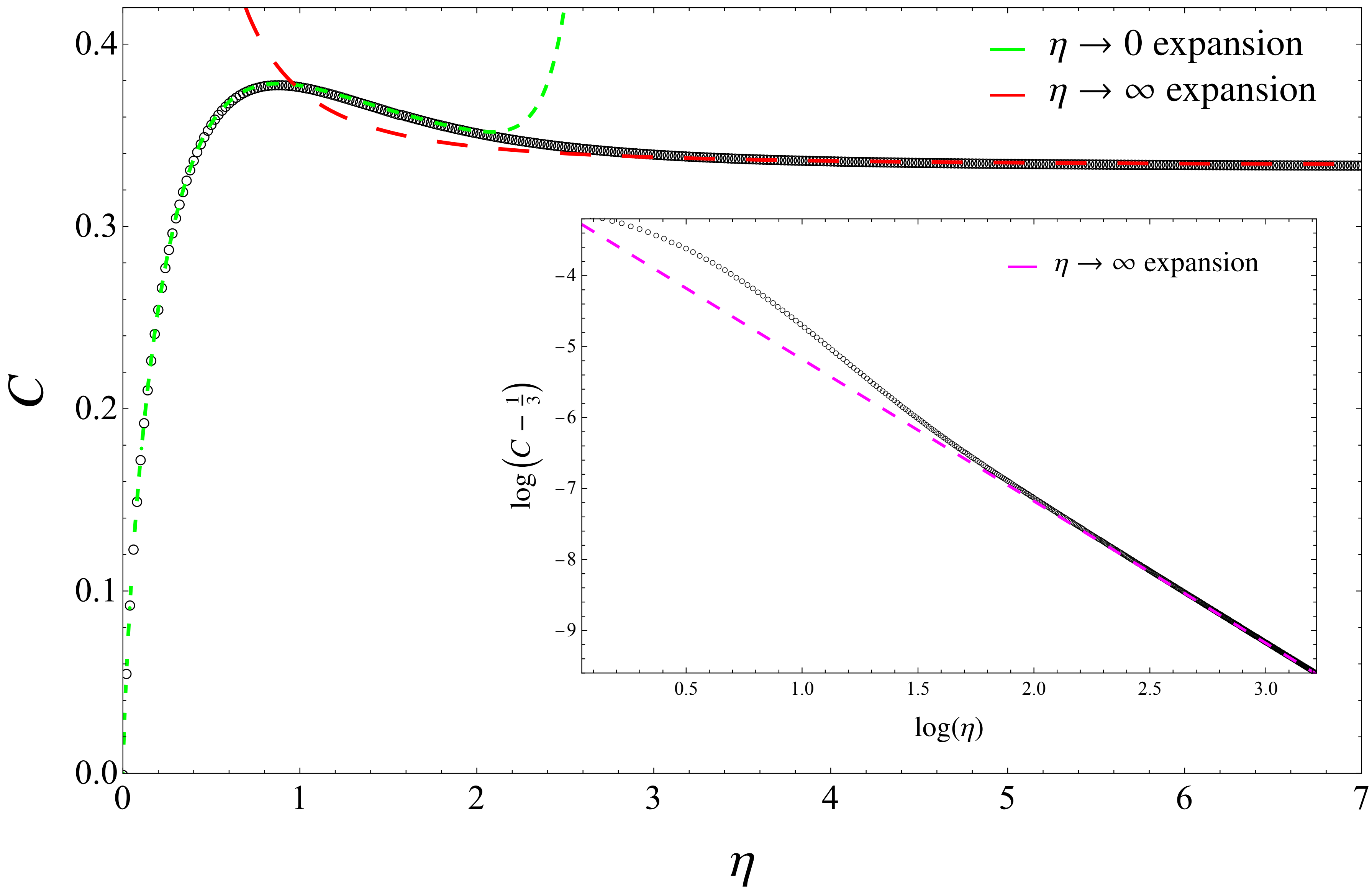
$$\sigma \equiv \eta\, \partial_\eta \log(\tau)$$

$$\begin{cases} \big(\eta\, \partial_\eta^2 \sigma\big)^2 + 16\, \big(\eta\, \partial_\eta \sigma - \sigma\big) \left[\eta\, \partial_\eta \sigma - \sigma + \frac{1}{4} (\partial_\eta \sigma)^2\right] = 0 \\ \sigma = -\frac{2\,\eta}{\pi\, z} - \left(\frac{2\,\eta}{\pi\, z}\right)^2 + O\big(\eta^3\big) \qquad \eta \to 0 \end{cases}$$

- Asymptotic behaviour as $\eta \to 0$ and $\eta \to \infty$ is respectively

$$C = -\frac{2}{\pi}\, \eta \log(\eta) + \dots \qquad\qquad C = \frac{1}{3} + \frac{1}{24\, \eta^2} + \dots$$

Even z : Analogue of the relativistic entropic C function



→ The function C is not monotonous, but S_A is monotonous

Expansions of the entanglement entropies (I)

Regime $\eta \to 0$

- Tau function approach: Expansion of the τ function as $\eta \to 0$
  [Gamayun, Iorgov, Lisovyy, (2013)]

$$\tau = \sum_{n=0}^{\infty} (-1)^n \frac{G(1+n)^6}{G(1+2n)^2} \frac{(4\,\eta)^{n^2}}{(2\pi\, z)^n} \mathcal{B}_n(\eta)$$

→ Approximate expression for the entanglement entropies

$$\widetilde{S}^{(\alpha)}_{A;\mathcal{N}} \equiv \sum_j s_\alpha(\tilde{z}_j)$$

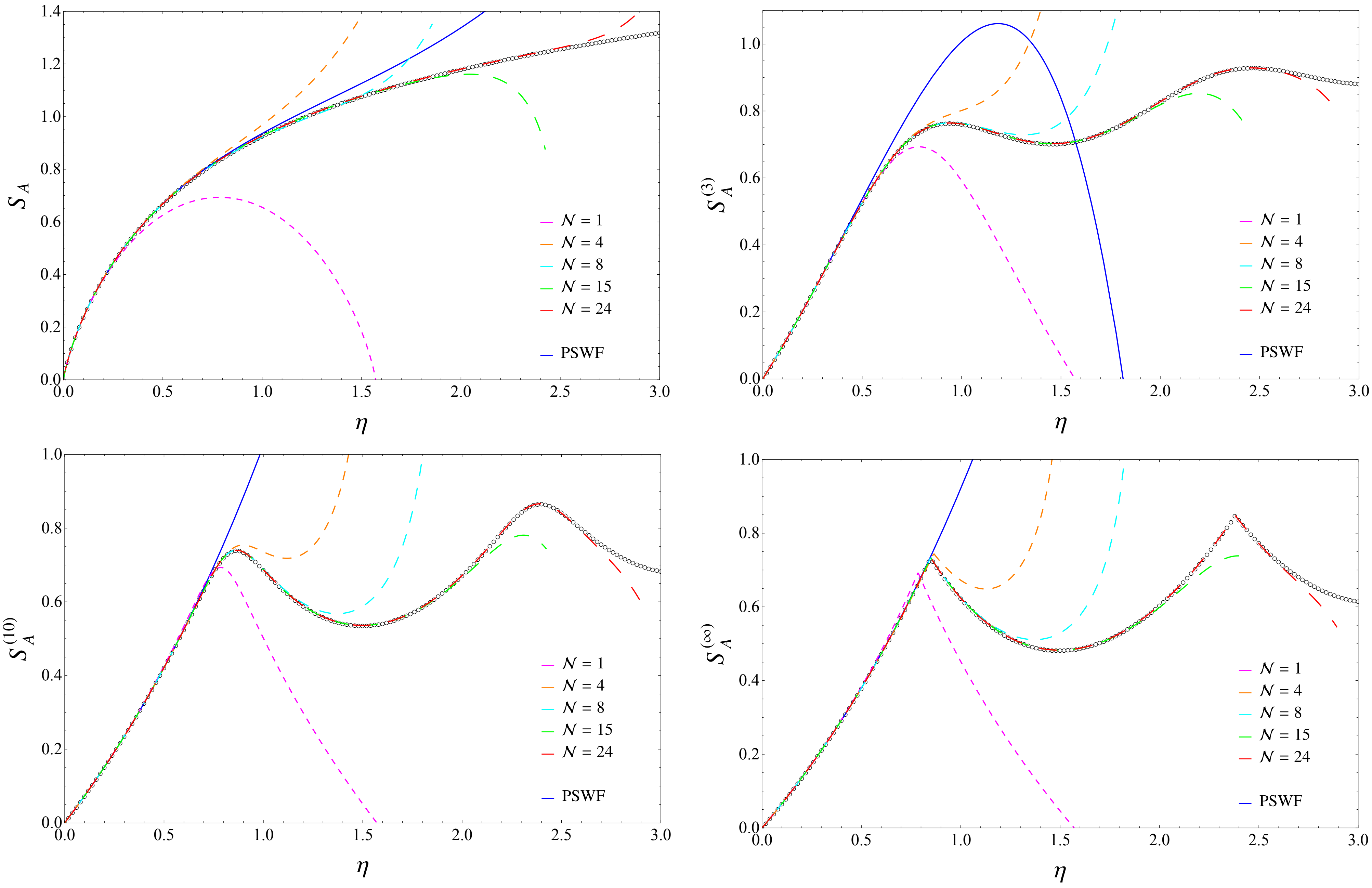
$\tilde{z}_j$ are zeros of the polynomials obtained by truncating the series for τ

- PSWF approach: Expansion of the eigenvalues γ_n as $\eta \to 0$ [Slepian, (1965)]

$$\gamma_n = \tilde{\gamma}_n \exp\left\{-\frac{(2n+1)\,\eta^2}{(2n-1)^2\,(2n+3)^2} + O(\eta^4)\right\}$$

$$\tilde{\gamma}_n \equiv \frac{2}{\pi}\left(\frac{2^{2n}\,(n!)^3}{(2n)!\,(2n+1)!}\right)^2 \eta^{2n+1}$$

Expansions of the entanglement entropies (I)



Expansions of the entanglement entropies (II)

- Expansion of the τ function as $\eta \to \infty$

[Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini, (2016)] [Lisovyy, Nagoya, Roussillon, (2018)]

$$\tau = \frac{\mathrm{e}^{4\mathrm{i}\nu_\star \eta}}{(4\eta)^{2\nu_\star^2}} \big[G(1-\nu_\star)\, G(1+\nu_\star) \big]^2$$

$$\times \sum_{n \in \mathbb{Z}} \frac{1}{(2\pi z)^{2n}} \left[\frac{G(1+\nu_\star+n)}{G(1+\nu_\star)} \right]^4 \frac{\mathrm{e}^{4\mathrm{i}n\eta}}{(4\mathrm{i}\eta)^{2n(n+2\nu_\star)}} \sum_{k=0}^{\infty} \frac{\mathcal{D}_k(\nu_\star+n)}{(4\mathrm{i}\eta)^k}$$

The polynomials $\mathcal{D}_k(\nu_\star)$ for $0 \leqslant k \leqslant 4$ are known and $\nu_\star = \dfrac{1}{2\pi\mathrm{i}} \log(1-1/z)$

- Large η expansion of the entanglement entropies

$$S_A^{(\alpha)} = \frac{1}{6}\left(1+\frac{1}{\alpha}\right)\log(4\eta) + E_\alpha + \widetilde{S}^{(\alpha)}_{A,\infty,0} + \frac{\widetilde{S}^{(\alpha)}_{A,\infty,1}}{\eta} + \frac{\widetilde{S}^{(\alpha)}_{A,\infty,2}}{\eta^2} + \frac{\widetilde{S}^{(\alpha)}_{A,\infty,3}}{\eta^3} + \dots$$

- The strong subadditivity and the logarithmic growth for large η allow to prove that S_A is a strictly increasing function of η

Expansions of the entanglement entropies (II)

The terms $\widetilde{S}^{-}_{A,\infty,k}$ with $\alpha \neq 1$ contain oscillating terms

$$\widetilde{S}^{(\alpha)}_{A,\infty,N} = \widetilde{S}^{(\alpha)}_{A,\infty,N,a} + \widetilde{S}^{(\alpha)}_{A,\infty,N,b} \qquad N \in \{0,1,2,3\}$$

$$\widetilde{S}^{(\alpha)}_{A,\infty,N,a} = 0 \qquad N \in \{0,1,3\} \qquad\qquad \widetilde{S}^{(\alpha)}_{A,\infty,2,a} = \frac{(1+\alpha)\,(3\,\alpha^2-7)}{384\,\alpha^3}$$

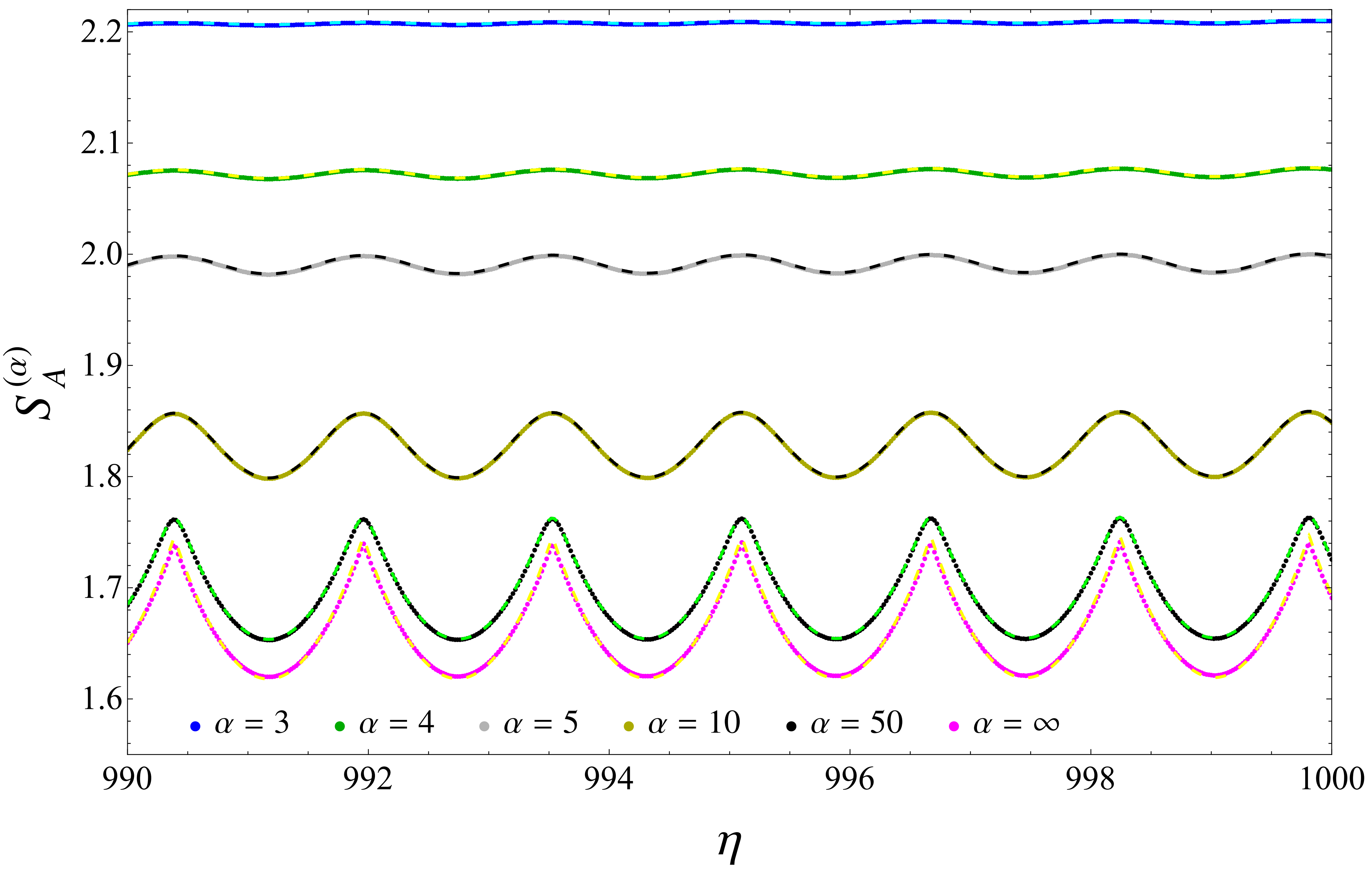
$$\widetilde{S}^{(\alpha)}_{A,\infty,N,b} = \frac{\kappa_N}{(\alpha-1)\,2^{2N-1}} \sum_{\substack{j=1\\k=0}}^{\infty} (-1)^j \, \frac{\left[\Omega(-\mathrm{i}y-1/2)^j\, \widetilde{\mathcal{D}}^{-}_{N,j-1}\right]\Big|_{\tilde{y}_k}}{(4\eta)^{2j(2k+1)/\alpha}} \cdot \begin{cases} \cos(4\eta j) & N \text{ even} \\ \sin(4\eta j) & N \text{ odd} \end{cases}$$

where $\Omega(z) \equiv \frac{\Gamma(1+z)^2}{\Gamma(-z)^2}$, $\kappa_N \equiv \sin(\pi N/2) + \cos(\pi N/2)$ is constant

$\widetilde{D}^{-}_{N,j-1}$ for $N \in \{0,1,2,3\}$ are known polynomials and $\tilde{y}_k \equiv \dfrac{\mathrm{i}}{\alpha}\left(k+\dfrac{1}{2}\right)$

- Agreement with previous lattice calculations [Jin, Korepin, (2003)] [Calabrese, Essler, (2010)]

Expansions of the entanglement entropies (II)



Modular Hamiltonian for odd Lifshitz exponents

- Odd Lifshitz exponent: Spectral problem

$$\int_{-R}^{R} \frac{\mathrm{e}^{-\mathrm{i}k_{\mathrm{F},z}(x-y)}}{2\pi\mathrm{i}\,(x-y-\mathrm{i}\varepsilon)}\,\phi_s(y)\,\mathrm{d}y \;=\; \gamma_s\,\phi_s(x) \qquad\qquad s\in\mathbb{R}$$

$$\tilde{\phi}_s(x)\equiv \mathrm{e}^{\mathrm{i}k_{\mathrm{F},z}x}\,\phi_s(x) \qquad \Longrightarrow \qquad \gamma_s\ \text{(hence also } S_A^{(\alpha)}\text{) independent of } k_{\mathrm{F},z}$$

⟶
$$S_A^{(\alpha)} = \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\log(2R/\epsilon) + O(1)$$

- The modular Hamiltonian and its modular flow depend on $k_{\mathrm{F},z}$

$$K_A \;=\; -\,2\pi\int_{-R}^{R}\frac{R^2-x^2}{2R}\,\big[\,T_{tt}(0,x)-k_{\mathrm{F},z}\,\varrho(0,x)\,\big]\,\mathrm{d}x$$

⟶ Also the modular flow of the field depends on $k_{\mathrm{F},z}$

Entanglement entropies of an interval

in the free Schrödinger field theory on the half line

Schroedinger field theory on the half line at finite density

- $$\left(\mathrm{i}\,\partial_t + \frac{1}{2m}\,\partial_x^2\right)\psi(t,x) = 0$$

  0 ———————— - - - $x \geqslant 0$

  Either Neumann $(+)$ or Dirichlet $(-)$ b.c. are imposed at the origin

  $$\lim_{x\to 0^+} \partial_x \psi_+(t,x) = 0 \qquad\qquad \lim_{x\to 0^+} \psi_-(t,x) = 0$$

- Equal time $t_1 = t_2 \equiv t$, zero temperature $\beta \to \infty$ two point function

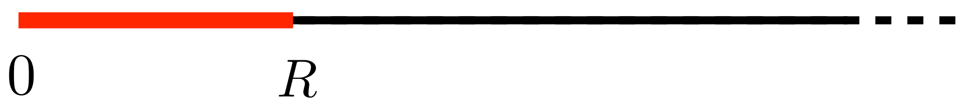
  $$\langle \psi_\pm^*(t,x_1)\,\psi_\pm(t,x_2)\rangle_{\infty,\mu} = \frac{\sin(k_{\mathrm{F}} x_{12})}{\pi\, x_{12}} \pm \frac{\sin(k_{\mathrm{F}} \widetilde{x}_{12})}{\pi\, \widetilde{x}_{12}} \qquad k_{\mathrm{F}} \equiv \sqrt{2m\mu}$$

  It depends both on x_{12} and on $\widetilde{x}_{12} \equiv x_1 + x_2$

- The particle density mean value exhibits *Friedel oscillations*

  $$\langle \varrho_\pm(t,x)\rangle_{\infty,\mu} = \frac{k_{\mathrm{F}}}{\pi} \pm \frac{\sin(2k_{\mathrm{F}} x)}{2\pi\, x} = \langle \varrho(t,x)\rangle_{\infty,\mu} \pm \frac{\sin(2k_{\mathrm{F}} x)}{2\pi\, x}$$

Entanglement entropies of an interval



- The entanglement entropies of $A = (0, R)$ read $\quad S^{(\alpha)}_{A,\pm} = \sum_{n=0}^{\infty} s_\alpha(\gamma_n^\pm)$

  where $\gamma_n^\pm$ are the eigenvalues of the spectral problems

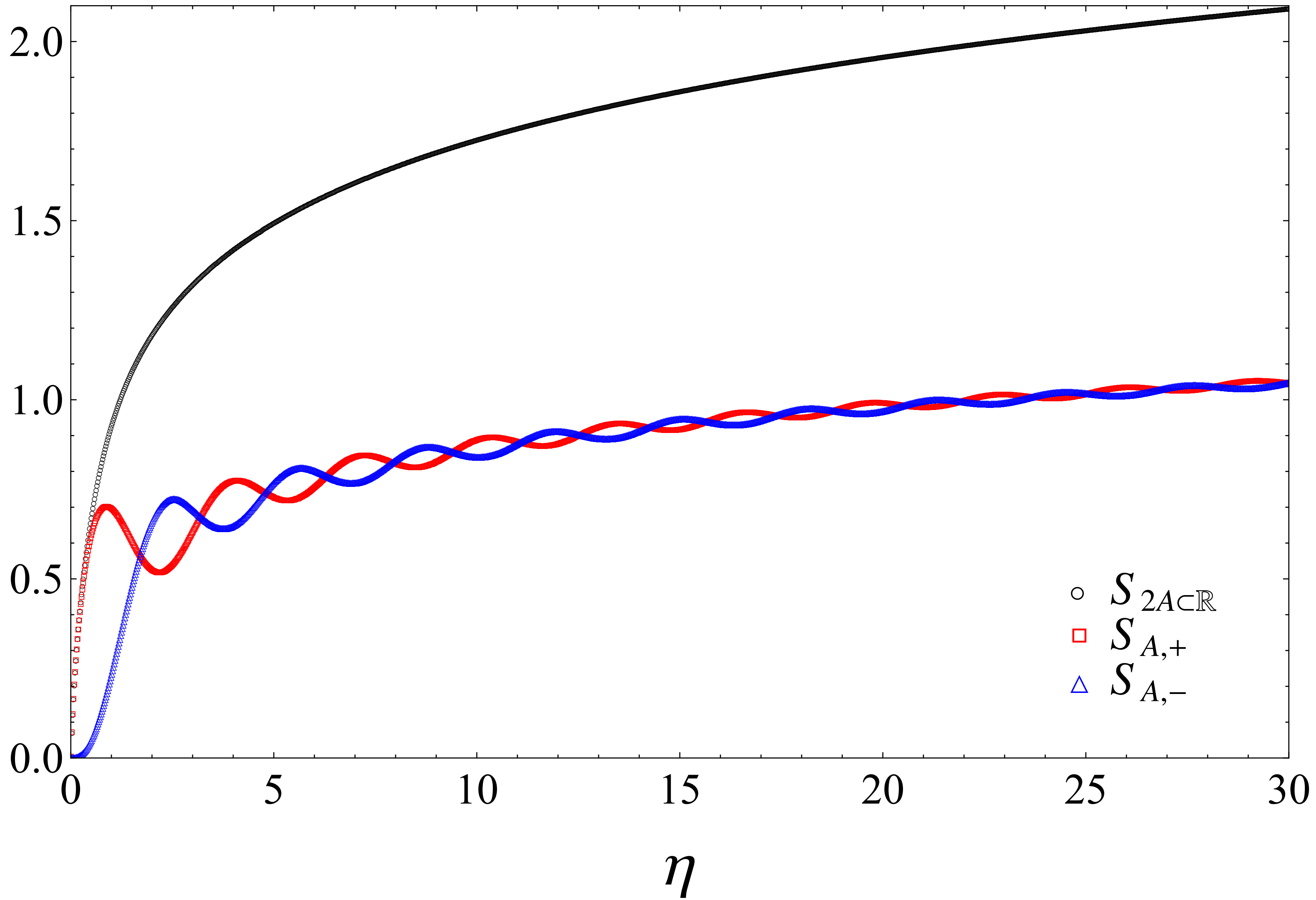
$$\int_0^1 K_\pm(\eta; x, y)\, f_n^\pm(\eta; y)\, \mathrm{d}y \;=\; \gamma_n^\pm\, f_n^\pm(\eta; x) \qquad x \in [0,1]$$

$$\boxed{K_\pm(k_{\text{F}}; x, y) \equiv \frac{\sin[k_{\text{F}}(x-y)]}{\pi(x-y)} \pm \frac{\sin[k_{\text{F}}(x+y)]}{\pi(x+y)}}$$

- The eigenvalues $\gamma_n^\pm$ can be read from the spectrum of the sine kernel spectral problem for the interval $(-R, R)$ on the line

$$\boxed{\begin{aligned} \gamma_n^+ &= \gamma_{2n} \\ \gamma_n^- &= \gamma_{2n+1} \end{aligned}} \qquad n \in \mathbb{N}_0 \qquad \Longrightarrow \qquad S^{(\alpha)}_{2A\,\subset\,\mathbb{R}} = S^{(\alpha)}_{A,+} + S^{(\alpha)}_{A,-}$$

Entanglement entropies of an interval



Entanglement entropies: tau function approach

- The τ function approach gives

$$S_{A,\pm}^{(\alpha)} = \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} s_\alpha(z)\, \partial_z \log(\tau_\pm)\, \mathrm{d}z \qquad\qquad \tau_\pm \equiv \det\big(I - z^{-1} K_\pm\big)$$

The product of $\tau_\pm$ gives the sine kernel tau function $\tau_{\text{sine}} = \tau_+ \, \tau_-$

- The kernels $K_\pm$ are special cases of a Bessel kernel whose auxiliary σ function satisfies a Painlevé III'_1 [Tracy, Widom, (1994)]

$$\tau_\pm(\eta) = \frac{2^{1/8}}{\pi^{\pm 1/2}\, \mathrm{e}^{\eta^2/8}\, \eta^{1/8}}\; \tau_{\text{III}'}(\eta^2/4)\Big|_{\theta_* = \theta_\star = \pm 1/4}$$

The auxiliary function $\sigma_{\text{III}'}(t) \equiv t\,\partial_t \log\big[\tau_{\text{III}'}(t)\big]$ satisfies the Painlevé III'_1

$$\big(t\,\sigma''_{\text{III}'}\big)^2 - \Big[4\big(\sigma'_{\text{III}'}\big)^2 - 1\Big]\big(\sigma_{\text{III}'} - t\,\sigma'_{\text{III}'}\big) + 4\,\theta_*\theta_\star\,\sigma'_{\text{III}'} - \big(\theta_*^2 + \theta_\star^2\big) = 0$$

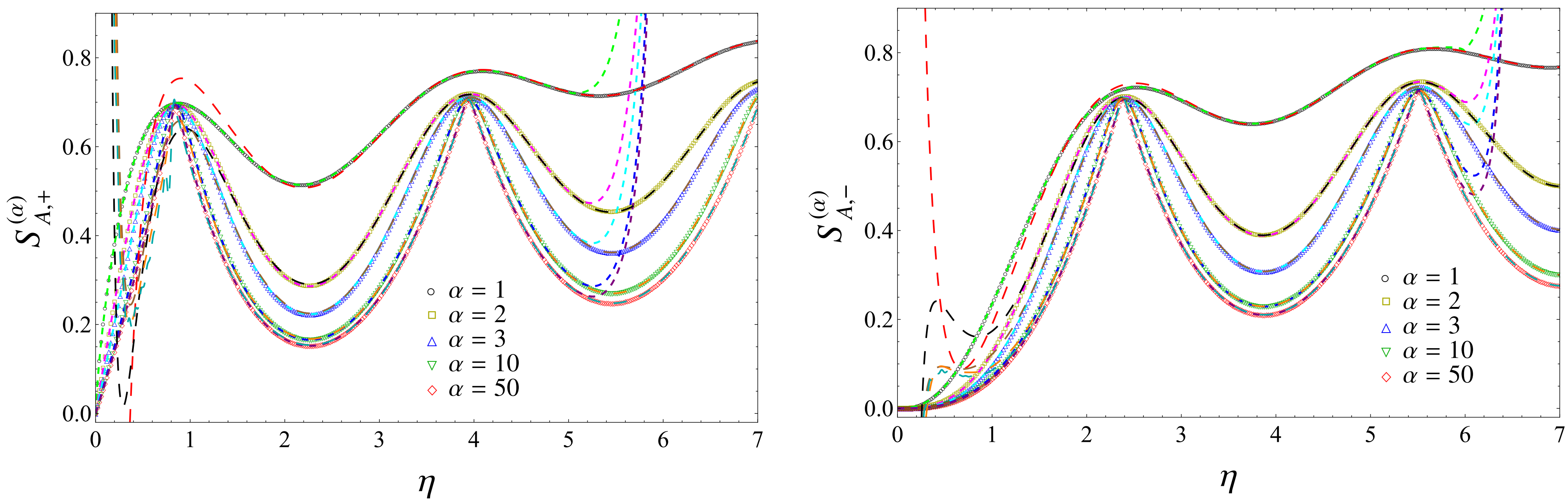
Expansions of the entanglement entropies

- Expansion of the τ function as $\eta \to 0$ [Gamayun, Iorgov, Lisovyy, (2013)]

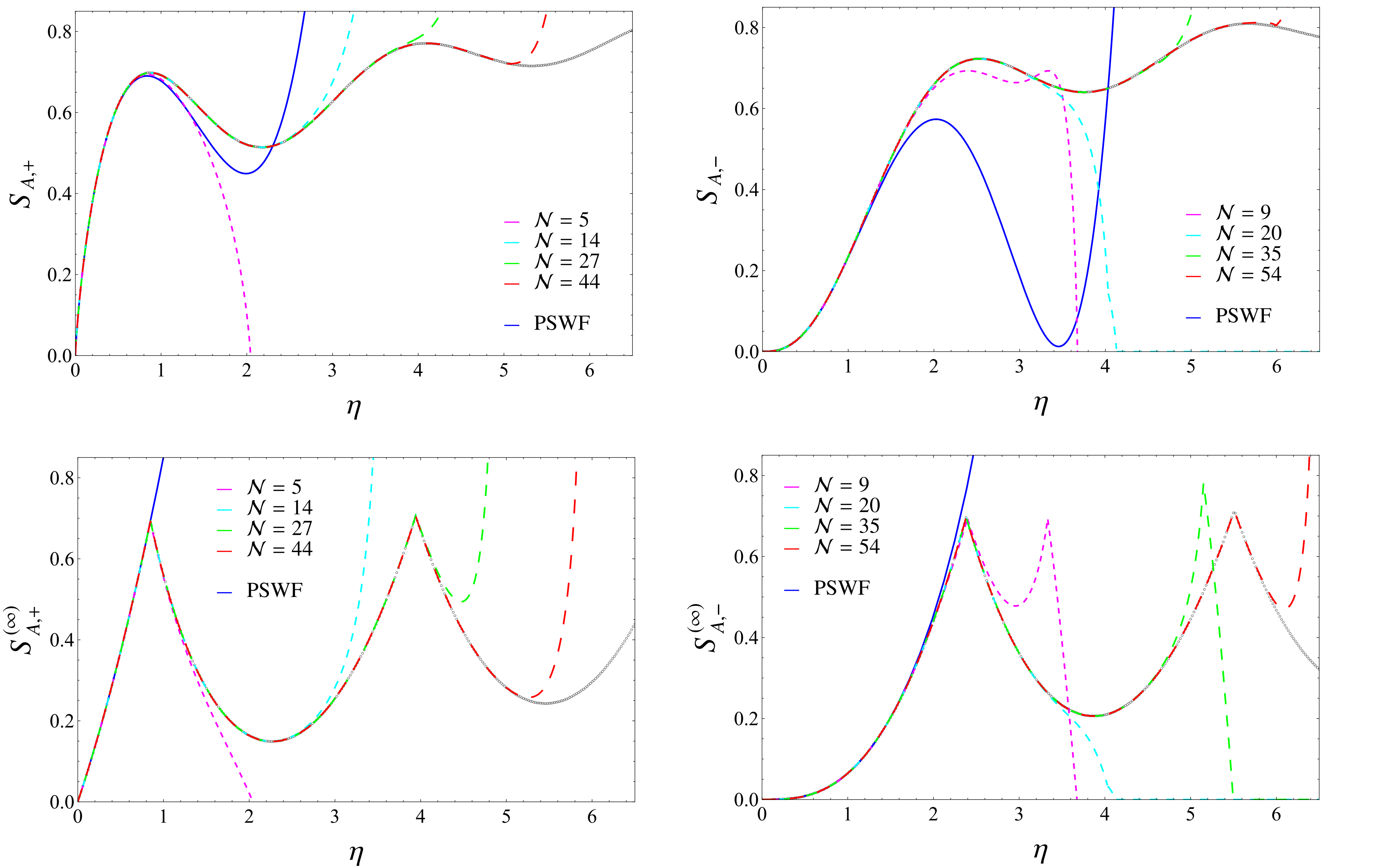
$$\tau_{\pm} = \frac{1}{\pi^{\pm 1/2}\, e^{\eta^2/4}} \sum_{n=0}^{\infty} C_{\mathrm{III}'}\left(\pm\tfrac{1}{4}, \pm\tfrac{1}{4}, \tfrac{1}{4} \mp n\right) \frac{(\eta/2)^{n(2n\mp 1)}}{z^n}\, \mathcal{B}_{\pm}(n; \eta^2/4)$$

$$C_{\mathrm{III}'}(\theta, \theta, \hat{\sigma}) \equiv \frac{\left[G(1+\theta+\hat{\sigma})\, G(1+\theta-\hat{\sigma})\right]^2}{G(1+2\hat{\sigma})\, G(1-2\hat{\sigma})}$$

This gives e.g. $S_{A,+} = -\frac{2}{\pi}\,\eta \log(\eta) + O(\eta)$ and $S_{A,-} = -\frac{2}{3\pi}\,\eta^3 \log(\eta) + O(\eta^3)$



Expansions of the entanglement entropies



Expansions of the entanglement entropies

- Expansion of the τ function as $\eta \to \infty$

[Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini, (2016)]

$$\tau_{\pm} = \sum_{n \in \mathbb{Z}} e^{\pm i \frac{\pi}{2}(\nu+n)} \frac{e^{i2\eta(n+\nu)}}{(4\eta)^{(\nu+n)^2}} G(1+\nu+n)\, G(1-\nu-n) \sum_{p=0}^{\infty} \frac{\mathcal{D}_p(\nu+n)}{(2\eta)^p}$$

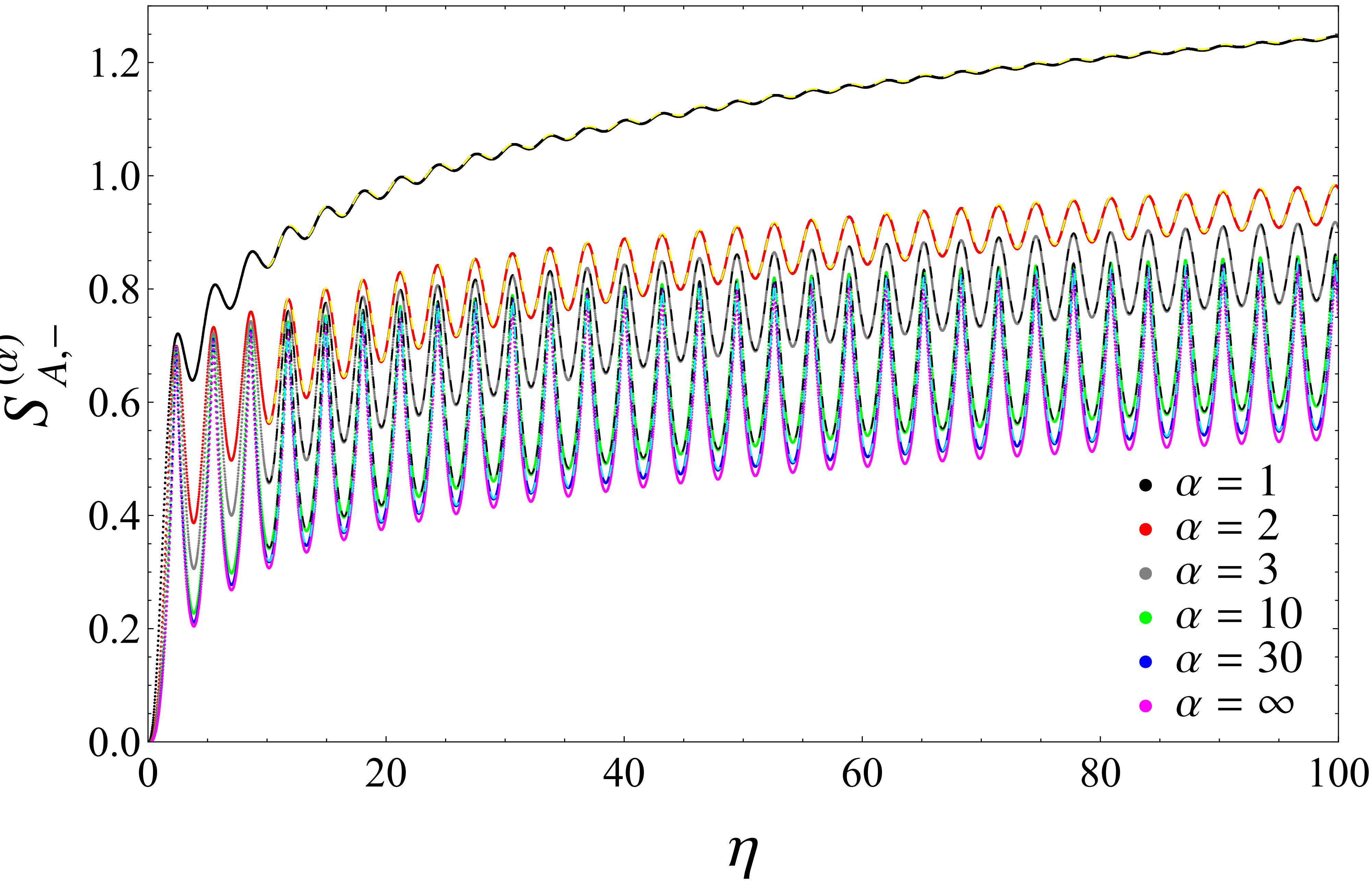
where $\nu = \frac{1}{2\pi i} \log(1-1/z)$ and the first polynomials $\mathcal{D}_p$ are known

- Large η expansion of the entanglement entropies

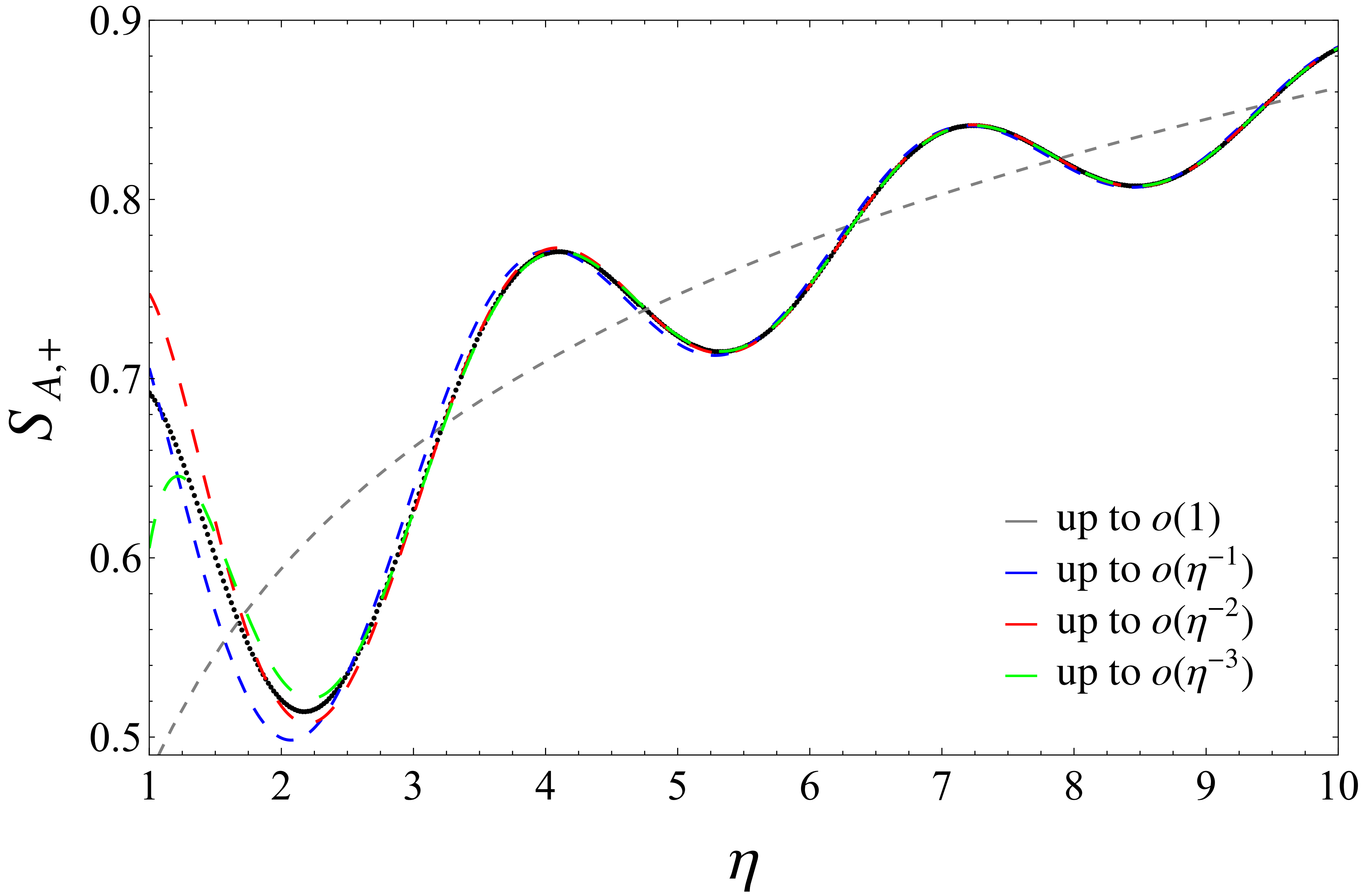
$$S^{(\alpha)}_{A,\pm} = \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\log(4\eta) + \frac{E_\alpha}{2} + \sum_{N=0}^{\infty} \frac{\widetilde{S}^{(\alpha)}_{A,\pm,\infty,N}}{(4\eta)^N}$$

  - In this cases also $S_{A,\pm}$ display oscillations
  - Agreement with previous lattice calculations [Fagotti, Calabrese, (2010)]

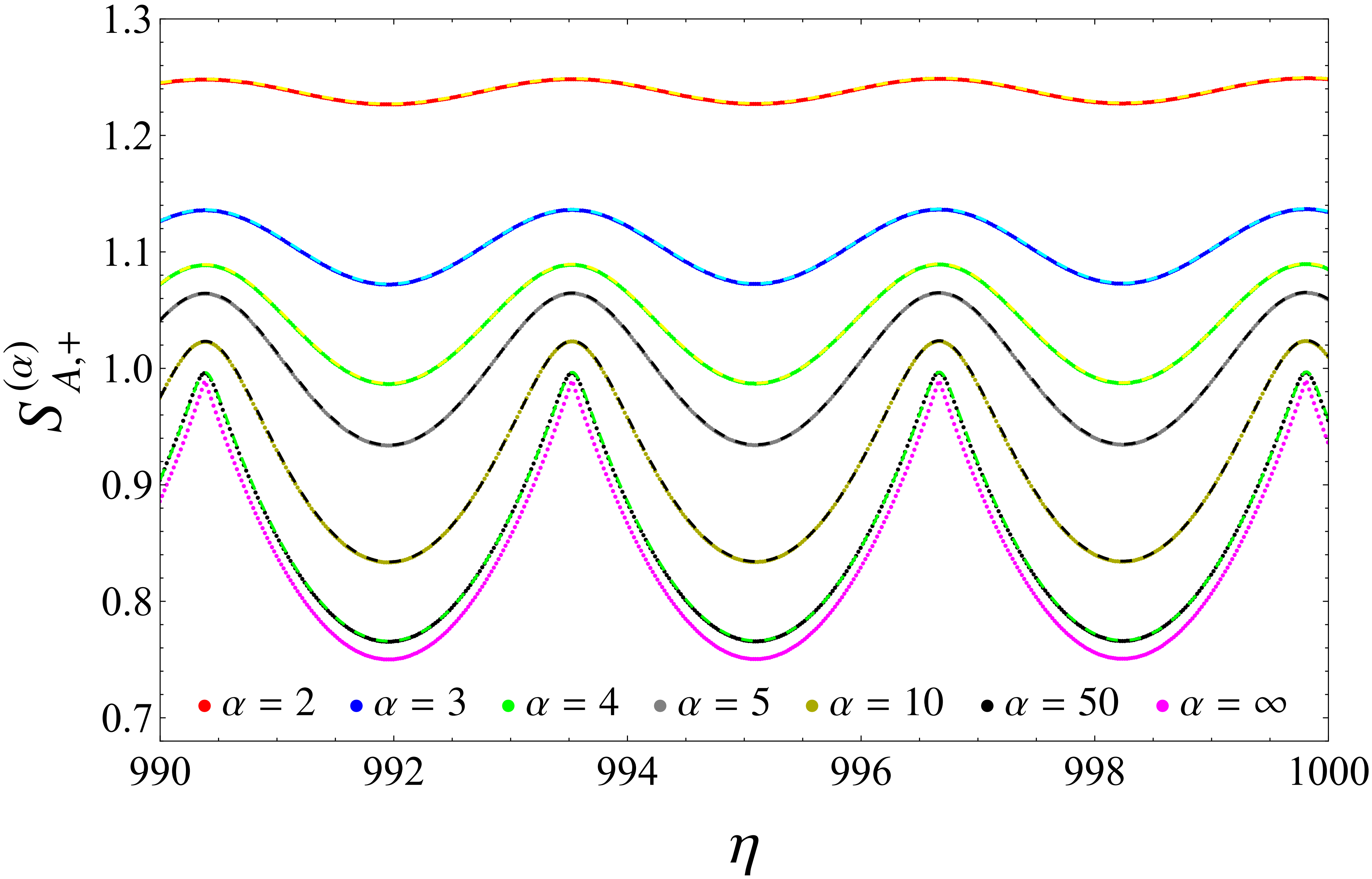
Expansions of the entanglement entropies



Expansions of the entanglement entropies



Expansions of the entanglement entropies



Entanglement entropies & Friedel oscillations

- The large η expansion of $S_{A,\pm}$ can be written as

$$S_{A,\pm} = \frac{1}{6}\log(4\eta) + \frac{E_1}{2} + \frac{\langle \varrho_\pm(t,R)\rangle_{\infty,\mu} - \langle \varrho(t,R)\rangle_{\infty,\mu}}{2k_{\mathrm{F}}/\pi} + O\big(1/\eta^2\big)$$

$$S^{(\alpha)}_{A,\pm} - \frac{1}{2}\, S^{(\alpha)}_{2A\,\subset\,\mathbb{R}} = \pm\, \frac{S^{(\alpha)}_{A,+} - S^{(\alpha)}_{A,-}}{2}$$

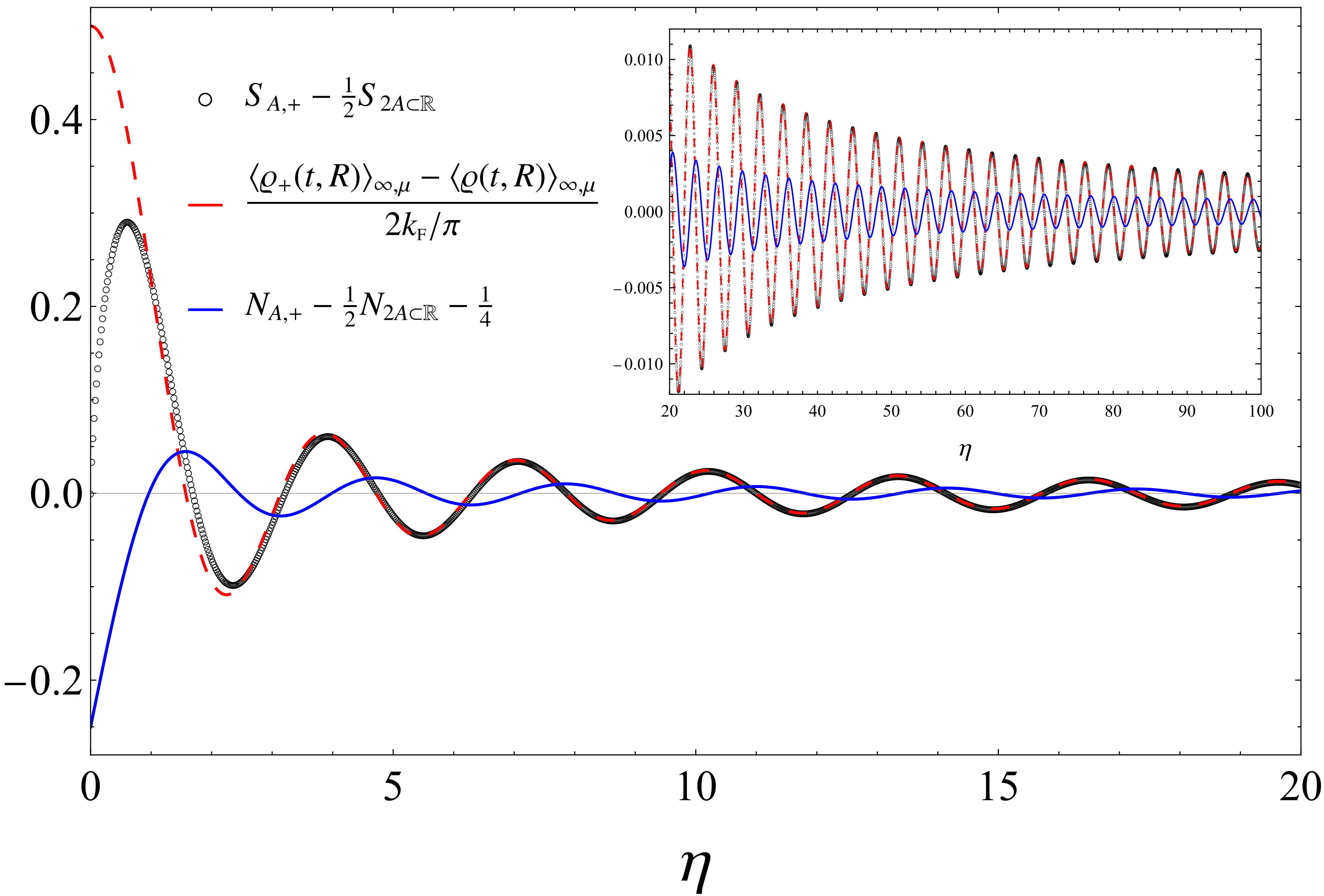
- Mean value of the particle densities at the entangling point

$$\frac{\langle \varrho_\pm(t,R)\rangle_{\infty,\mu}}{k_{\mathrm{F}}} = \frac{\langle \varrho(t,R)\rangle_{\infty,\mu}}{k_{\mathrm{F}}} \pm \frac{\sin(2\eta)}{2\pi\,\eta}$$

- Mean particle number in the interval

$$N_{A,\pm} \equiv \int_0^R \langle \varrho_\pm(t,x)\rangle_{\infty,\mu}\, \mathrm{d}x = \frac{\eta}{\pi} \pm \frac{\mathrm{Si}(2\eta)}{2\pi}$$

Entanglement entropies & Friedel oscillations



Schatten norms & Cumulants expansion

- Schatten p-norm of the kernels $K_\pm$

$$\mathrm{Tr}\big(K_\pm^p\big) = \sum_{n=0}^{\infty} (\gamma_n^\pm)^p = \lim_{\epsilon,\delta\to 0} \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{C}} z^p\, \partial_z \log(\tau_\pm)\, \mathrm{d}z$$

➔ $$\mathrm{Tr}(K_\pm^p) = \frac{\eta}{\pi} - \frac{\gamma_{\mathrm{E}} + \psi(p)}{2\pi^2} \log(4\eta) \pm \frac{1}{4} + C_0(p) + o(1) \qquad \eta \to \infty$$

➔ Cumulants of the local charge operator $Q_{A,\pm} = \int_A \varrho_\pm(t=0,x)\,\mathrm{d}x$

$$\mathcal{C}^{(1)}_{A,\pm} = \mathrm{Tr}\big(K_\pm\big) \qquad \mathcal{C}^{(2)}_{A,\pm} = \mathrm{Tr}\big(K_\pm - K_\pm^2\big) \qquad \mathcal{C}^{(3)}_{A,\pm} = \mathrm{Tr}\big(K_\pm - 3K_\pm^2 + 2K_\pm^3\big)$$

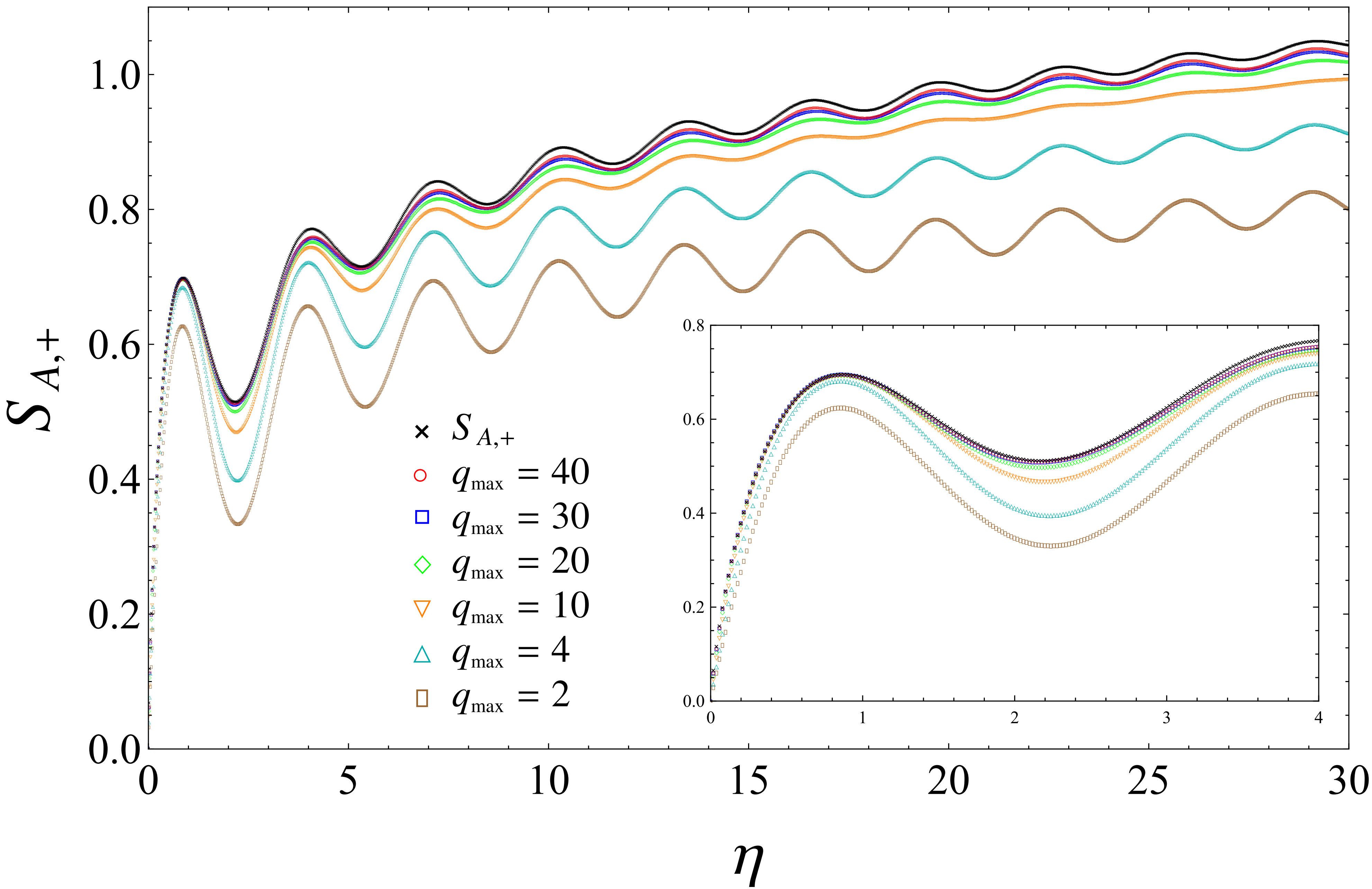
- Expansion of the entanglement entropy in terms of the charge cumulants

[Klich, Levitov, (2008)] [Song, Flindt, Rachel, Klich, Le Hur, (2010)]

$$S_{A,\pm} = \lim_{q\to\infty} \sum_{n=1}^{q+1} a_n(q)\, \mathcal{C}^{(n)}_{A,\pm}$$

$$a_n(q) \equiv \begin{cases} 0 & \text{odd } n \\ 2\displaystyle\sum_{k=n-1}^{q} \frac{S_1(k,n-1)}{k!\,k} & \text{even } n \end{cases}$$

Cumulants expansion



Conclusions

- Entanglement entropies of an interval for free fermionic spinless fields with integer Lifshitz exponent z, on the line and on the half line (either Neumann or Dirichlet b.c.)

Even z:

→ The entanglement entropies are finite functions of η

→ On the line, the entanglement entropy is strictly increasing

→ $\eta \to 0$ and $\eta \to \infty$ expansions: PSWF and τ function approaches

→ On the half line, the oscillations of the entanglement entropy are related to the Friedel oscillations at the entangling point

- Some future directions:
  - Finite temperature
  - Interactions [Benfatto, Gallavotti, (1995)]
  - Bosonic fields
  - Higher dimensions
  - Holography

Thank you!