Strong coupling results in $\mathcal{N} = 2$ gauge theories

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New perspectives on Quantum Field Theory with Boundaries Impurities, and Defects
Nordita, August 3, 2023

This talk is mainly based on:

- M. Billò, M. F., A. Lerda, A. Pini, P. Vallarino, *“Strong coupling expansions in N=2 quiver gauge theories”*, JHEP 01 (2023) 119, arXiv:2211.11795
- M. Billò, M. F., A. Lerda, A. Pini, P. Vallarino, *“Localization vs holography in 4d N=2 quiver theories”*, JHEP 10 (2022) 020, arXiv:2207.08846
- M. Billò, M. F., A. Lerda, A. Pini, P. Vallarino, *“Structure constants in N=2 superconformal quiver theories at strong coupling and holography”*, Phys. Rev. Lett. 129 (2022) 031602, arXiv:2206.13582
- M. Billò, M. F., F. Galvagno, A. Lerda, A. Pini, *“Strong coupling results for N=2 superconformal quivers and holography”*, JHEP 10 (2021) 161, arXiv:2109.0055

but it builds on a very vast literature …

Introduction

Introduction

- Perturbation theory is one of the few universal tools to study quantum field theories
- However, to fully understand an interacting quantum field theory, it is necessary to go beyond perturbation theory and eventually explore also the strong-coupling regime
- This is in general a very difficult problem but, when there is a high amount of symmetry, significant progress can be made, thanks to the combined use of
  - Integrability
  - Localization
  - Holography

Introduction

- This is in fact the case of $\mathcal{N} = 4$ SYM, where several exact results have been obtained over the years:
  - 2- and 3-point functions of protected scalar operators
  - v.e.v. of BPS Wilson-loop
  - cusp anomalous dimension
  - Brehmsstrahlung function
  - integrated 4-point functions of superconformal primaries
  - octagon form factors in 4-point functions of very heavy scalar operators
  - …

$\mathcal{N}=4$

- $\mathcal{N}=4$ SU(N) SYM is the "simplest" gauge theory
- It is a superconformal theory with $AdS_5 \times S_5$ as holographic dual
- Simplest operators involving the fields of the vector multiplet (in $\mathcal{N}=2$ language)
  - Chiral operators

    $\mathcal{O}_n(x) = \mathrm{tr}\,\phi^n(x)$ primary operators with dimension n

  - Wilson loop

$$W = \frac{1}{N}\,\mathrm{tr}\,\mathcal{P}\exp\Big[\oint_C d\tau\big(iA_\mu\,\dot{x}^\mu + \frac{1}{\sqrt{2}}(\phi+\bar{\phi})|\dot{x}|\big)\Big]$$

$\mathcal{N}=4$

- Since the theory is conformal the form of 2 and 3 point functions is fixed
- 2-point function $$\langle \mathcal{O}_n(x)\,\overline{\mathcal{O}}_n(y)\rangle = \frac{G_n}{|x-y|^{2n}}$$
- 3-point function $$\langle \mathcal{O}_{n_1}(x)\,\mathcal{O}_{n_2}(y)\,\overline{\mathcal{O}}_{n_3}(z)\rangle = \frac{G_{n_1,n_2,n_3}}{|x-z|^{2n_1}|y-z|^{2n_2}}$$
- In the t'Hooft planar limit ($N\to\infty$) the structure constants

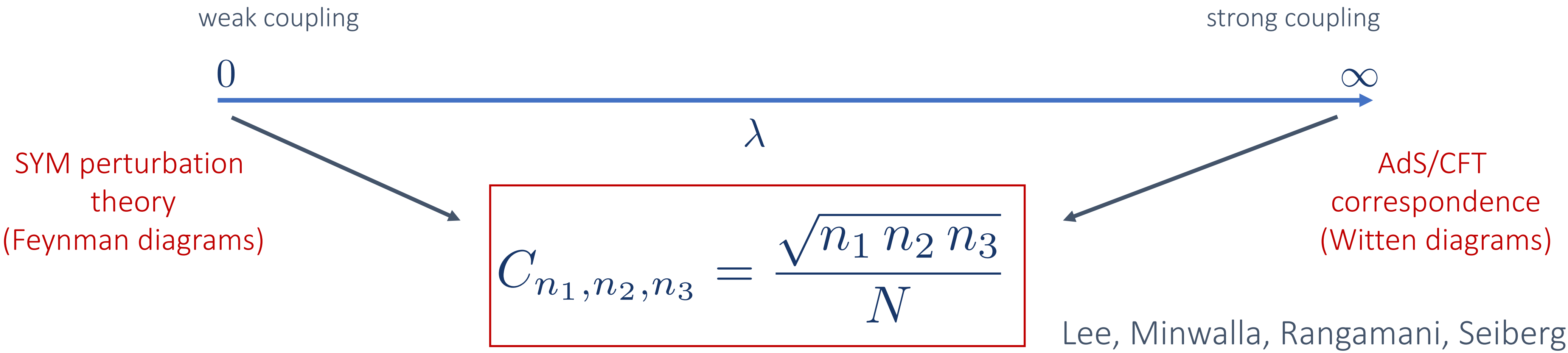
$$C_{n_1,n_2,n_3} = \frac{G_{n_1,n_2,n_3}}{\sqrt{G_{n_1}}\sqrt{G_{n_2}}\sqrt{G_{n_3}}} = \frac{\sqrt{n_1\,n_2\,n_3}}{N}$$

are independent on the coupling $\lambda = N g^2_{\mathrm{YM}}$!

Lee, Minwalla, Rangamani, Seiberg, 1998

$\mathcal{N}=4$ Structure Constant

- This result has been obtained in the weak and in the strong regime with very different techniques



weak coupling

strong coupling

0

∞

λ

SYM perturbation theory (Feynman diagrams)

AdS/CFT correspondence (Witten diagrams)

$$\boxed{C_{n_1,n_2,n_3}=\frac{\sqrt{n_1\,n_2\,n_3}}{N}}$$

Lee, Minwalla, Rangamani, Seiberg, 1998

- In this case we have a simple weak/strong extrapolation!

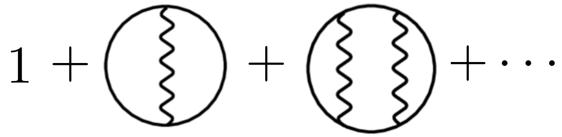
$\mathcal{N}=4$ Wilson loop

- The case is different for the circular Wilson

$$W = \frac{1}{N}\,\mathrm{tr}\,\mathcal{P}\exp\left[\oint_C d\tau\Big(iA_\mu\,\dot{x}^\mu + \frac{1}{\sqrt{2}}(\phi+\bar{\phi})|\dot{x}|\Big)\right]$$

$\mathcal{N}=4$ Wilson loop

- The case is different for the circular Wilson



$$\langle W\rangle = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \cdots$$

Erickson, Semenoff, Zarembo, 2000

$\mathcal{N}=4$ Wilson loop

- The case is different for the circular Wilson

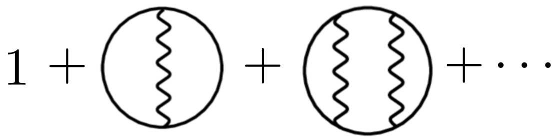
weak coupling

strong coupling

SYM perturbation theory

0 λ ∞

AdS/CFT correspondence



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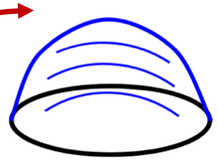
Erickson, Semenoff, Zarembo, 2000

minimal area law in AdS$_5$



$$\langle W\rangle = e^{\sqrt{\lambda}-\frac{3}{4}\log\lambda+\frac{1}{2}\log\frac{2}{\pi}+\cdots}$$

Maldacena, 1998

$\mathcal{N} = 4$ Wilson loop

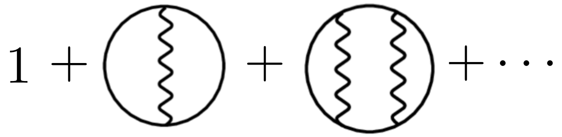
- The case is different for the circular Wilson

weak coupling

strong coupling

SYM perturbation theory

AdS/CFT correspondence



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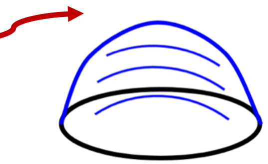


$$\langle W \rangle = e^{\sqrt{\lambda} - \frac{3}{4}\log\lambda + \frac{1}{2}\log\frac{2}{\pi} + \cdots}$$

Maldacena, 1998

$\lambda \to 0$

$$\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

$\lambda \to \infty$

Introduction

- As mentioned, there are many other examples of exact results in $\mathcal{N}=4$ SYM
- Finding exact results in non-maximally supersymmetric theories like $\mathcal{N}=2$ theories is more challenging!

- In this talk I discuss a class of $\mathcal{N}=2$ conformal theories in 4d where:
  - One can find exact results, valid for all values of the coupling constant
  - One can test the AdS/CFT holographic correspondence in a non-maximally supersymmetric context

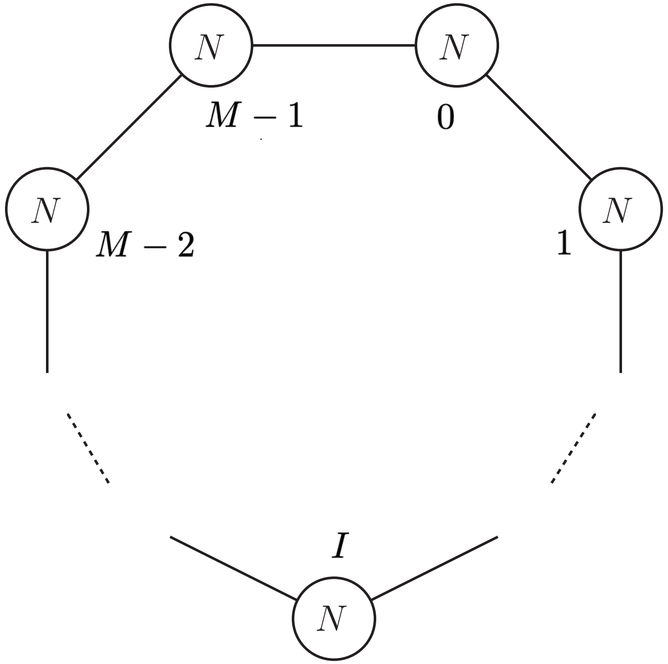
Introduction

- As mentioned, there are many other examples of exact results in $\mathcal{N}=4$ SYM
- Finding exact results in non-maximally supersymmetric theories like $\mathcal{N}=2$ theories is more challenging!

- In this talk I discuss a class of $\mathcal{N}=2$ conformal theories in 4d:

$\mathcal{N}=2$ quiver gauge theories

$$\underbrace{\mathrm{SU}(N)\times\mathrm{SU}(N)\times\cdots\times\mathrm{SU}(N)}_{\text{M times}}$$



- For simplicity in this talk I will consider the case M=2

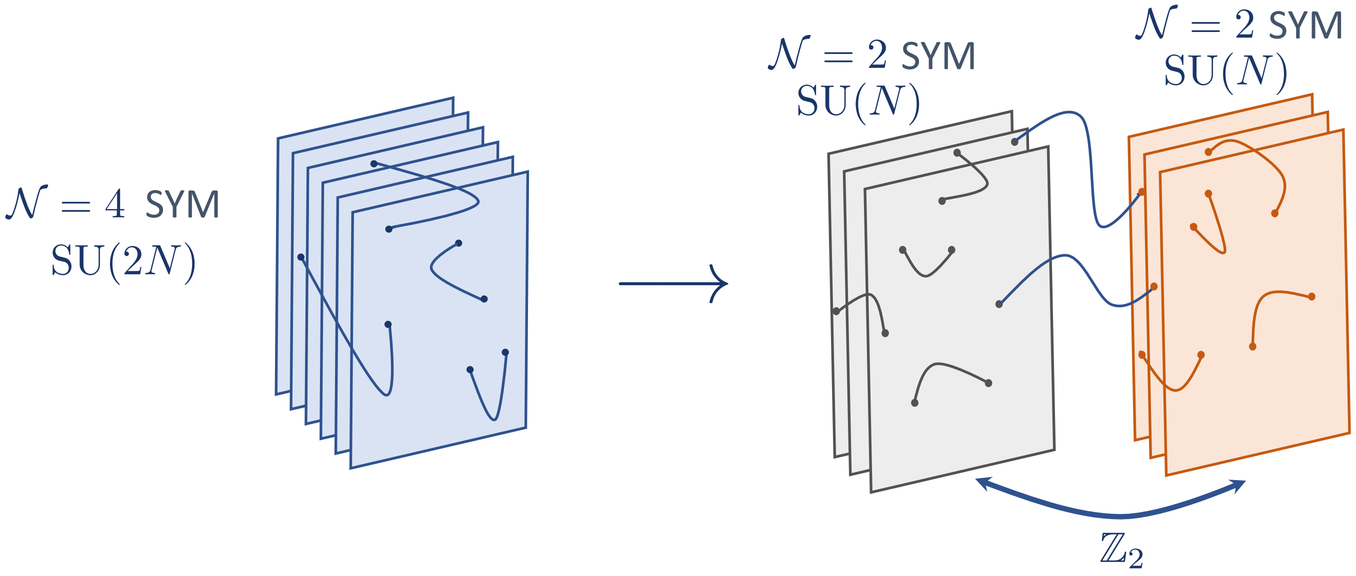
Plan of the talk

- $\mathcal{N}=2$ quiver gauge theories:
  - Exact results from localization
  - Strong coupling expansion
- Wilson loop
- Conclusions

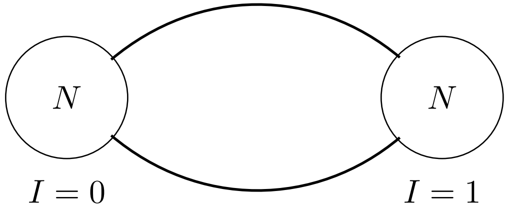
$\mathcal{N} = 2$ Quiver Theory

$\mathcal{N}=2$ Quiver Theory

- It is the "next-to-simplest" 4d gauge theory after $\mathcal{N}=4$ SYM
- It arises as a $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM
- It admits a simple string theory realization in terms of fractional D3-branes



$\mathcal{N}=2$ Quiver Theory



- SU(N) x SU(N) gauge group
- In each node:
  - 1 vector A_μ^I
  - 1 complex scalar ϕ^I in the adjoint
  - + fermions
- Between nodes: 2N bi-fundamental hypermultiplets

β – function = 0

- Local operators: $\mathcal{O}_k^{\pm}(x) = \frac{1}{\sqrt{2}}\left(\mathrm{tr}\,\phi_0(x)^k \pm \mathrm{tr}\,\phi_1(x)^k\right)$

$\mathcal{O}^+$untwisted
($\mathbb{Z}_2$ symmetric)

$\mathcal{O}^-$ twisted
($\mathbb{Z}_2$ anti-symmetric)

$\mathcal{N} = 2$ Quiver Theory

- We are interested in studying 2- and 3-point functions and the corresponding structure constants in the planar limit:

$$\langle \mathcal{O}_k^{\pm}(x)\, \overline{\mathcal{O}}_k^{\pm}(y) \rangle = \frac{G_k^{\pm}}{|x-y|^{2k}}$$

$$\langle \mathcal{O}_k^{+}(x)\, \mathcal{O}_\ell^{\pm}(y)\, \overline{\mathcal{O}}_p^{\pm}(z) \rangle = \frac{G_{k,\ell,p}^{\pm}}{|x-z|^{2k}|y-z|^{2\ell}} \quad \longrightarrow \quad C_{k,\ell,p}^{\pm} = \frac{G_{k,\ell,p}^{\pm}}{\sqrt{G_k^{+}}\sqrt{G_\ell^{\pm}}\sqrt{G_p^{\pm}}}$$

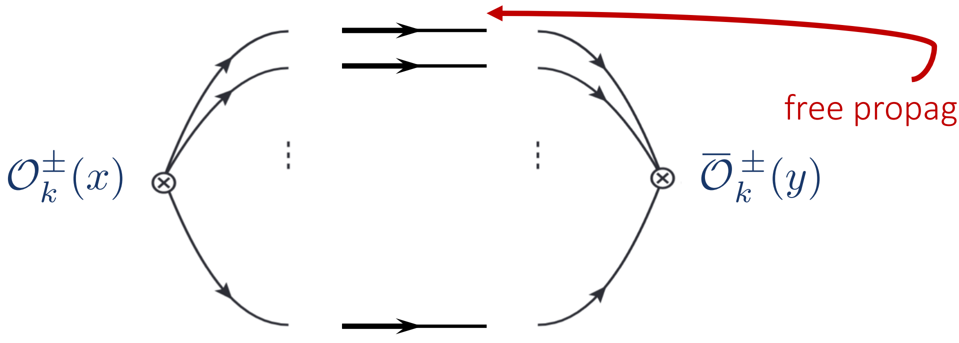
$(p = k + \ell)$

- The coefficients $G_k^{\pm}$, $G_{k,\ell,p}^{\pm}$, $C_{k,\ell,p}^{\pm}$ are non trivial functions of N and λ
- How can we compute them?

$\mathcal{N}=2$ Quiver Theory

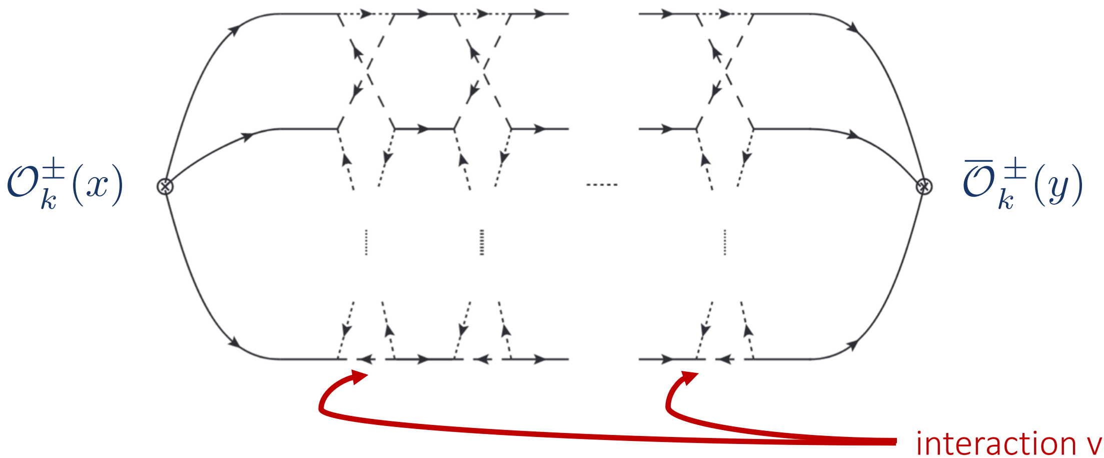
- At weak coupling one could use standard Feynman diagrams

- at tree level ($\lambda=0$)



free propagators

- at loop level



interaction vertices

done at the first orders….

Billò, Fucito, Lerda, Morales, Stanev, Wen, 2017
Billò, Galvagno, Lerda, 2019

Localization

- A much more efficient way to compute these correlators is through

  localization

  which for a theory on a compact manifold (like a 4-sphere) reduces path integrals to finite dimensional integrals in a

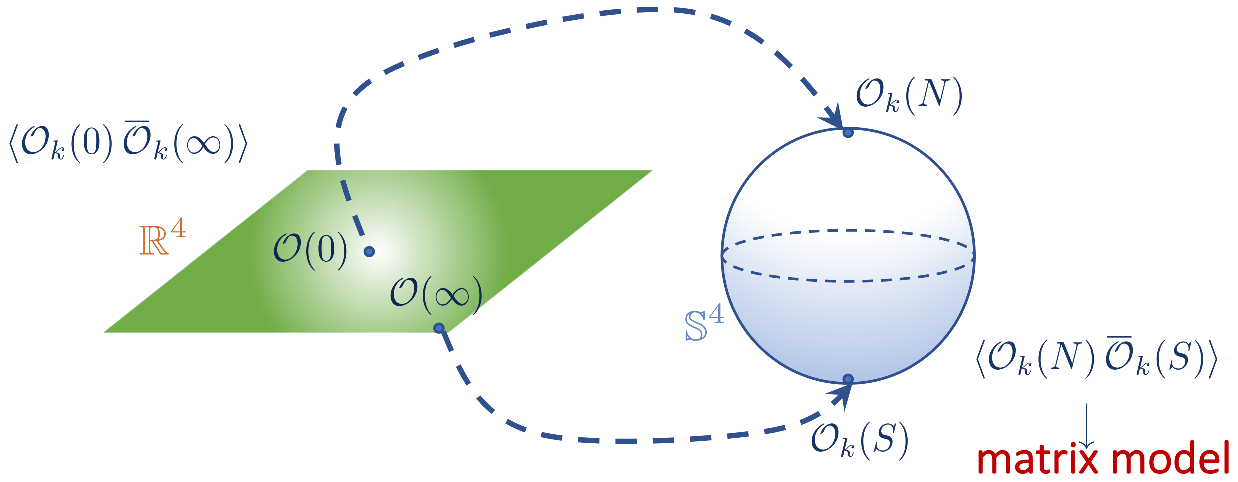
  matrix model Pestun, 2007

- This method applies to the partition function, the v.e.v. of circular Wilson loops, and the chiral/anti-chiral correlators

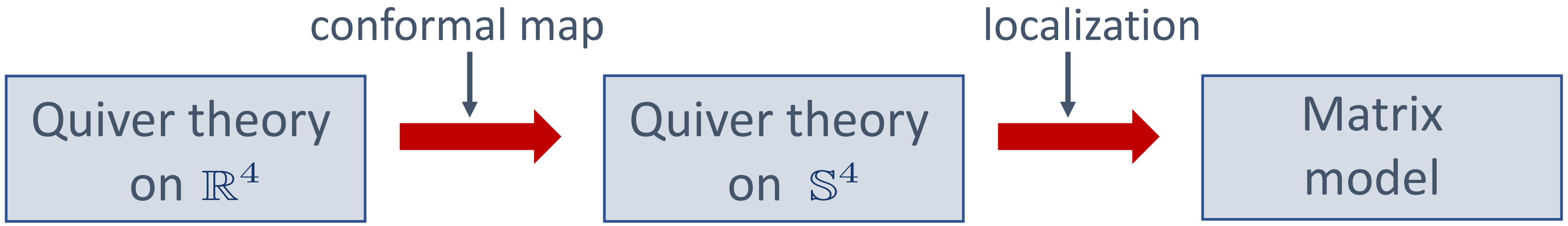
Baggio, Niarchos, Papadodimas, 2014, 2015
Gerchkovitz, Gomis, Ishtiaque et al, 2016
Rodriguez-Gomez, Russo, 2016
Billò, Fucito, Lerda, Morales, Stanev, Wen, 2017
…

Localization

- We are interested in computing correlators in $\mathbb{R}^4$ but, since our theory is conformal, we can easily map them to correlators in $\mathbb{S}^4$



- and then exploit the power of localization to reduce the computation to a matrix model correlator



Matrix model

- For our $\mathbb{Z}_2$ quiver theory the matrix model contains two N x N Hermitian matrices a_0 and a_1, corresponding to the v.e.v.'s of ϕ_0 and ϕ_1
- The partition function is

$$\mathcal{Z} = \int \Big(\prod_{I=0,1} da_I \; e^{-\mathrm{tr}\, a_I^2} \Big) \, |Z_{1-\mathrm{loop}} \, \cancel{Z_{\mathrm{inst}}}|^2$$

=1 when N $\longrightarrow \infty$

with $|Z_{1-\mathrm{loop}}|^2 = e^{-S_{\mathrm{int}}}$ and, in the large N limit

odd Riemann ζ-values

$$S_{\mathrm{int}} = 2 \sum_{m=2}^{\infty} \sum_{k=2}^{2m} (-1)^{m+k} \Big(\frac{\lambda}{8\pi^2 N} \Big)^m \binom{2m}{k} \frac{\zeta_{2m-1}}{2m} \big(\mathrm{tr}\, a_0^{2m-k} - \mathrm{tr}\, a_1^{2m-k} \big) \big(\mathrm{tr}\, a_0^k - \mathrm{tr}\, a_1^k \big)$$

Galvagno, Preti 2020

Matrix model

- Since $\phi_I(x) \longrightarrow a_I$ one may think that

$$\mathcal{O}_k^{\pm}(x) = \frac{1}{\sqrt{2}}\Big(\mathrm{tr}\,\phi_0(x)^k \pm \mathrm{tr}\,\phi_1(x)^k\Big) \longrightarrow \frac{1}{\sqrt{2}}\Big(\mathrm{tr}\,a_0^k \pm \mathrm{tr}\,a_1^k\Big) \equiv A_k^{\pm}$$

- However, $\mathcal{O}_k^{\pm}(x)$‘s do not have self-contractions, while $A_k^{\pm}$‘s do, so the correct map has to send $\mathcal{O}_k^{\pm}(x)$ into the normal-ordered operator

$$\mathcal{O}_k^{\pm}(x) \longrightarrow O_k^{\pm} = \,:A_k^{\pm}:\, = \sum_{\ell \leq k} \mathsf{M}_{k\ell}\, A_\ell^{\pm}$$

- and we have

$$\langle \mathcal{O}_k^{\pm}(x)\, \overline{\mathcal{O}}_k^{\pm}(y) \rangle = \frac{G_k^{\pm}}{|x-y|^2} \longleftrightarrow \langle O_k^{\pm}\, O_k^{\pm} \rangle = G_k^{\pm}$$

and similarly for the 3-point functions

Free matrix model $(S_{int} = 0)$

- In the free Gaussian model, in the planar limit, one finds

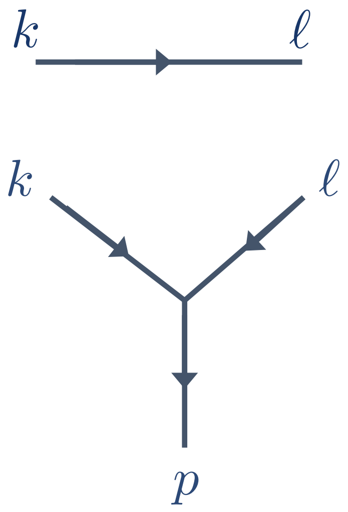
$$\left\langle O_k^\pm \, O_\ell^\pm \right\rangle_0 = k \left(\frac{N}{2}\right)^k \delta_{k,\ell} \qquad\qquad \left\langle O_k^+ \, O_\ell^\pm \, O_p^\pm \right\rangle_0 = \frac{k\,\ell\,p}{2\sqrt{2}} \left(\frac{N}{2}\right)^{\frac{k+\ell+p}{2}-1} \delta_{k+\ell,p}$$

- Defining the normalized operators $P_k^\pm = \frac{1}{\sqrt{\mathcal{G}_k}} \, O_k^\pm \Big|_0$, with $\mathcal{G}_k \equiv k\left(\frac{N}{2}\right)^k$ one has

$$\left\langle P_k^\pm \, P_\ell^\pm \right\rangle_0 = \delta_{k,\ell}$$

$$\left\langle P_k^+ \, P_\ell^\pm \, P_p^\pm \right\rangle_0 = \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N} \delta_{k+\ell,p}$$

like in $\mathcal{N} = 4$ SYM (up to the $\sqrt{2}$ due to the orbifold)



Interacting matrix model

- Let us notice that the interaction action

$$S_{\text{int}} = 2\sum_{m=2}^{\infty}\sum_{k=2}^{2m}(-1)^{m+k}\left(\frac{\lambda}{8\pi^2 N}\right)^m \binom{2m}{k}\frac{\zeta_{2m-1}}{2m}\left(\operatorname{tr} a_0^{2m-k} - \operatorname{tr} a_1^{2m-k}\right)\left(\operatorname{tr} a_0^{k} - \operatorname{tr} a_1^{k}\right)$$

depends only on the twisted operators:

$$S_{\text{int}} = -\frac{1}{2}\sum_{k,\ell} P_k^- \, \mathsf{X}_{k,\ell} \, P_\ell^-$$

with k, ℓ both even or both odd, otherwise zero

with

$$\mathsf{X}_{k,\ell} = -8\sqrt{k\,\ell}\sum_{p=0}^{\infty}(-1)^p \frac{(k+\ell+2p)!^{\,2}}{p!(k+p)!(\ell+p)!(k+\ell+p)!}\,\frac{\zeta_{k+\ell+2p-1}}{k+\ell+2p}\left(\frac{\lambda}{16\pi^2}\right)^{\frac{k+\ell+2p}{2}}$$

Interacting matrix model

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$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{e^t}{(e^t-1)^2}\,J_k\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_\ell\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

Beccaria, Billò, Galvagno, Hasan, Lerda, 2020
Beccaria, Dunne, Tseytlin, 2021;
Beccaria, Billo, M.F., Lerda, Pini, 2021 + …

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This convolution of Bessel functions contains the exact dependence of S_{int} on the coupling constant !

The X matrix

The structure of the X matrix is

$$X = \begin{pmatrix} X_{2,2} & 0 & X_{2,4} & 0 & X_{2,6} & 0 & \cdots \\ 0 & X_{3,3} & 0 & X_{3,5} & 0 & X_{3,7} & \cdots \\ X_{4,2} & 0 & X_{4,4} & 0 & X_{4,6} & 0 & \cdots \\ 0 & X_{5,3} & 0 & X_{5,5} & 0 & X_{5,7} & \cdots \\ X_{6,2} & 0 & X_{6,4} & 0 & X_{6,6} & 0 & \cdots \\ 0 & X_{7,3} & 0 & X_{7,5} & 0 & X_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus it is convenient to define

$$X^{\text{even}} = \begin{pmatrix} X_{2,2} & X_{2,4} & X_{2,6} & \cdots \\ X_{4,2} & X_{4,4} & X_{4,6} & \cdots \\ X_{6,2} & X_{6,4} & X_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad X^{\text{odd}} = \begin{pmatrix} X_{3,3} & X_{3,5} & X_{3,7} & \cdots \\ X_{5,3} & X_{5,5} & X_{5,7} & \cdots \\ X_{7,3} & X_{7,5} & X_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Interacting matrix model

- Given this expression for the interaction action

$$S_{\text{int}} = -\frac{1}{2}\sum_{k,\ell} P_k^- \,\mathsf{X}_{k,\ell}\, P_\ell^-$$

- the 2-point functions of the untwisted operators do not change, and in the planar limit one has

$$\left\langle P_k^+\, P_\ell^+ \right\rangle = \frac{\left\langle P_k^+\, P_\ell^+\, e^{-S_{\text{int}}} \right\rangle_0}{\left\langle e^{-S_{\text{int}}} \right\rangle_0} \underset{N\to\infty}{\sim} \frac{\left\langle P_k^+\, P_\ell^+ \right\rangle_0 \cancel{\left\langle e^{-S_{\text{int}}} \right\rangle_0}}{\cancel{\left\langle e^{-S_{\text{int}}} \right\rangle_0}} = \left\langle P_k^+\, P_\ell^+ \right\rangle_0$$

Interacting matrix model

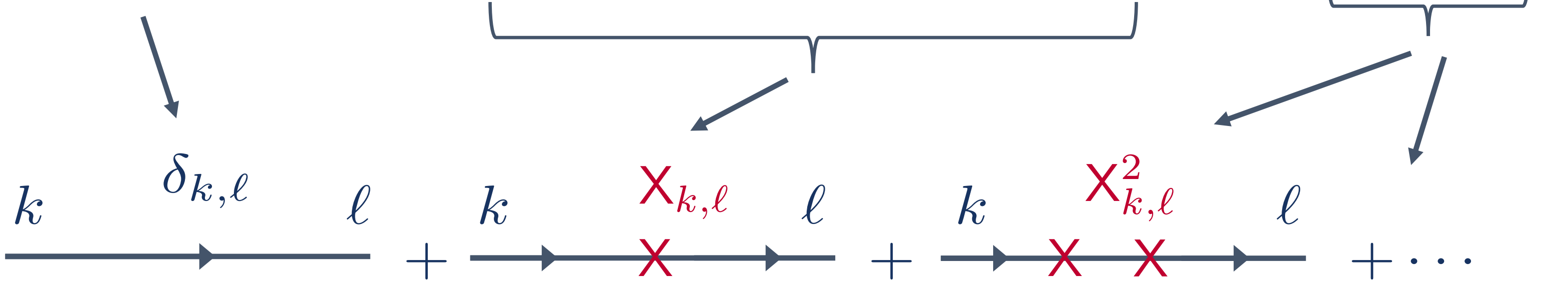
- Given this expression for the interaction action

$$S_{\text{int}} = -\frac{1}{2}\sum_{k,\ell} P_k^- \, \mathsf{X}_{k,\ell} \, P_\ell^-$$

- For the twisted operators instead

$$\langle P_k^- \, P_\ell^- \rangle \;=\; \frac{\langle P_k^- \, P_\ell^- \, e^{-S_{\text{int}}} \rangle_0}{\langle e^{-S_{\text{int}}} \rangle_0}$$

$$\underset{N\to\infty}{\sim} \; \langle P_k^- \, P_\ell^- \rangle_0 + \Big[- \underbrace{\langle P_k^- \, P_\ell^- \, S_{\text{int}} \rangle_0 + \langle P_k^- \, P_\ell^- \rangle_0 \langle S_{\text{int}} \rangle_0}_{} \Big] + \underbrace{\cdots\quad}_{}$$



Interacting matrix model

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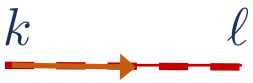
$$S_{\text{int}} = -\frac{1}{2} \sum_{k,\ell} P_k^- \, \mathsf{X}_{k,\ell} \, P_\ell^-$$

- For the twisted operators instead

$$\langle P_k^- \, P_\ell^- \rangle \; = \; \delta_{k,\ell} + \mathsf{X}_{k,\ell} + \mathsf{X}^2_{k,\ell} + \cdots \; = \; \left(\frac{1}{1-\mathsf{X}} \right)_{k,\ell} \; \equiv \; D_{k,\ell} \qquad \longleftrightarrow$$

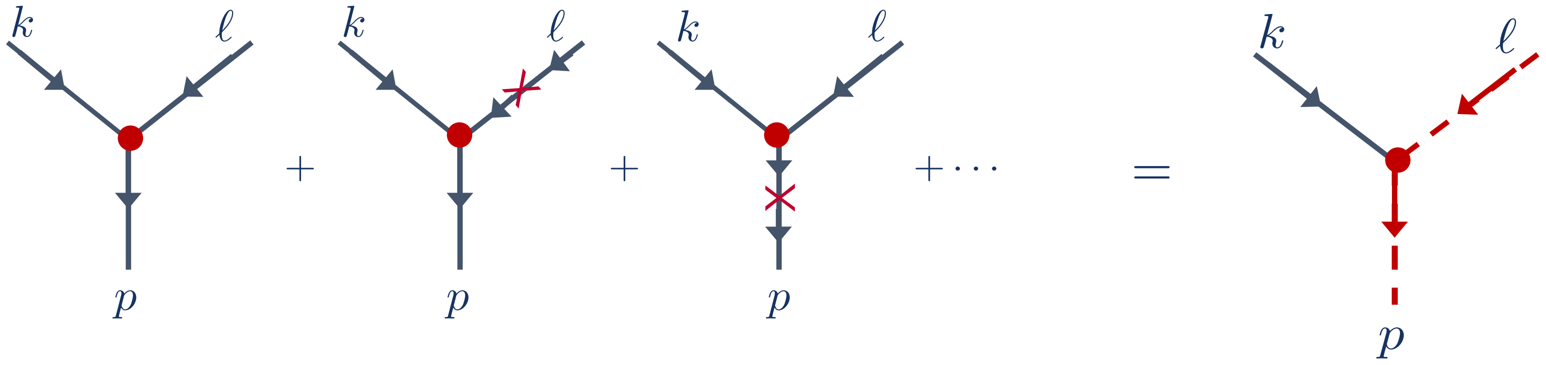


Exact expression in λ!

Interacting matrix model

- Also the 3-point functions can be computed in a similar way

$$\langle P_k^+ \, P_\ell^- \, P_p^- \rangle =$$



$$= \sum_{\ell', p'} \frac{\sqrt{k\,\ell'\,p'}}{\sqrt{2}\,N} \, \mathsf{D}_{\ell,\ell'} \, \mathsf{D}_{p,p'}$$

- Defining $\mathsf{d}_\ell = \sum_{\ell'} \sqrt{\ell'}\, \mathsf{D}_{\ell,\ell'} = \sum_{\ell'} \sqrt{\ell'} \left(\frac{1}{1 - \mathsf{X}} \right)_{\ell,\ell'}$ one gets

$$\langle P_k^+ \, P_\ell^- \, P_p^- \rangle = \frac{\sqrt{k}\,\mathsf{d}_\ell\,\mathsf{d}_p}{\sqrt{2}\,N}$$

Interacting matrix model ⟶ Recap

- The 2-point functions $$\langle P_k^- \, P_\ell^- \rangle = \left(\frac{1}{1-\mathsf{X}}\right)_{k,\ell} \equiv \mathsf{D}_{k,\ell}$$
- The 3-point functions $$\langle P_k^+ \, P_\ell^- \, P_p^- \rangle = \frac{\sqrt{k}\,\mathsf{d}_\ell\,\mathsf{d}_p}{\sqrt{2}\,N} \qquad \mathsf{d}_\ell = \sum_{\ell'} \sqrt{\ell'}\,\mathsf{D}_{\ell,\ell'}$$
- Everything is formally expressed in terms of the X matrix

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}}\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{e^t}{(e^t-1)^2}\,J_k\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_\ell\!\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

These formulas contains the exact dependence on the coupling constant !

Interacting matrix model ➡ Recap

- The 2-point functions $$\langle P_k^- \, P_\ell^- \rangle = \left(\frac{1}{1-\mathsf{X}}\right)_{k,\ell} \equiv \mathsf{D}_{k,\ell}$$
- The 3-point functions $$\langle P_k^+ \, P_\ell^- \, P_p^- \rangle = \frac{\sqrt{k}\,\mathsf{d}_\ell\,\mathsf{d}_p}{\sqrt{2}\,N} \qquad \mathsf{d}_\ell = \sum_{\ell'} \sqrt{\ell'}\,\mathsf{D}_{\ell,\ell'}$$
- However the P_k^- are NOT the operators O_k^- we are interested in, since $\sqrt{\mathcal{G}_k}\,P_k^- = O_k^-\big|_0$, but $O_k^- = \sqrt{\mathcal{G}_k}\,{:}P_k^-{:}$
- The correlators of the O_k^-'s are combinations of the one of the P_k^-'s

2-point functions

- After the normal ordering one finds

$$G^-_{2n} = \langle O^-_{2n} \, O^-_{2n} \rangle = \mathcal{G}_{2n} \, \frac{\det\left(1 - \mathsf{X}^{\text{even}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{even}}_{[n]}\right)}$$

$$G^-_{2n+1} = \langle O^-_{2n+1} \, O^-_{2n+1} \rangle = \mathcal{G}_{2n+1} \, \frac{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n]}\right)}$$

- where $\mathcal{G}_k \equiv k\left(\frac{N}{2}\right)^k$ and, for example,

$$\mathsf{X}^{\text{even}}_{[2]} = \left(\begin{array}{c|ccc} \mathsf{X}_{2,2} & \mathsf{X}_{2,4} & \mathsf{X}_{2,6} & \cdots \\ \hline \mathsf{X}_{4,2} & \mathsf{X}_{4,4} & \mathsf{X}_{4,6} & \cdots \\ \mathsf{X}_{6,2} & \mathsf{X}_{6,4} & \mathsf{X}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right) \qquad \mathsf{X}^{\text{odd}}_{[3]} = \left(\begin{array}{c|c|cc} \mathsf{X}_{3,3} & \mathsf{X}_{3,5} & \mathsf{X}_{3,7} & \cdots \\ \hline \mathsf{X}_{5,3} & \mathsf{X}_{5,5} & \mathsf{X}_{5,7} & \cdots \\ \cline{2-4} \mathsf{X}_{7,3} & \mathsf{X}_{7,5} & \mathsf{X}_{7,7} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

2-point functions

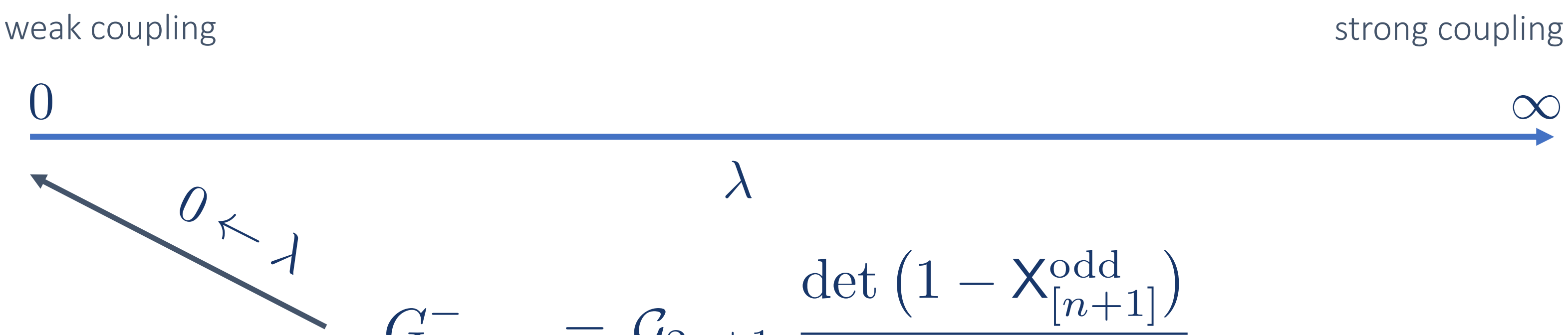
- After the normal ordering one finds

$$G^{-}_{2n} = \langle O^{-}_{2n}\, O^{-}_{2n} \rangle = \mathcal{G}_{2n} \frac{\det\left(1 - \mathsf{X}^{\text{even}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{even}}_{[n]}\right)}$$

$$G^{-}_{2n+1} = \langle O^{-}_{2n+1}\, O^{-}_{2n+1} \rangle = \mathcal{G}_{2n+1} \frac{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n]}\right)}$$

These formulas are valid for any value of λ !

2-point functions


weak coupling

strong coupling

0

∞

λ

$\lambda \to 0$

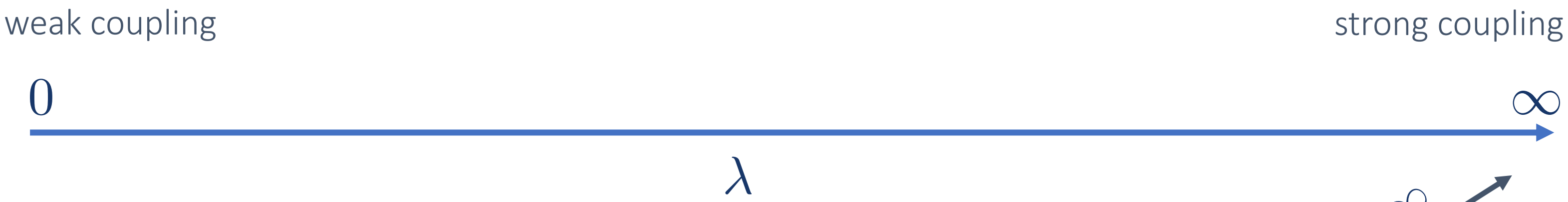
$$G^-_{2n+1} = \mathcal{G}_{2n+1} \, \frac{\det\left(1 - \mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\mathrm{odd}}_{[n]}\right)}$$

- Using the small λ expansion of the Bessel functions, one obtains the weak-coupling expansions. For example:

$$G^-_3 = \frac{3\,N^3}{8}\left[\,1 - \frac{5\,\zeta_5}{256\pi^6}\,\lambda^3 + \frac{105\,\zeta_7}{4096\pi^8}\,\lambda^4 - \frac{1701\,\zeta_9}{65536\pi^{10}}\,\lambda^5 + \cdots \; + \; O(\lambda^{160})\,\right]$$

- The perturbative expansions have a finite radius of convergence $\lambda \simeq \pi^2$, but they can be re-summed a la Padé and extended beyond that limit.

2-point functions



$$G^{-}_{2n+1} = \mathcal{G}_{2n+1}\,\frac{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n+1]}\right)}{\det\left(1-\mathsf{X}^{\mathrm{odd}}_{[n]}\right)}$$

$\lambda \to \infty$

- More interestingly, we can derive analytically the strong-coupling expansions of the 2-point functions. In fact

The X matrix at large λ

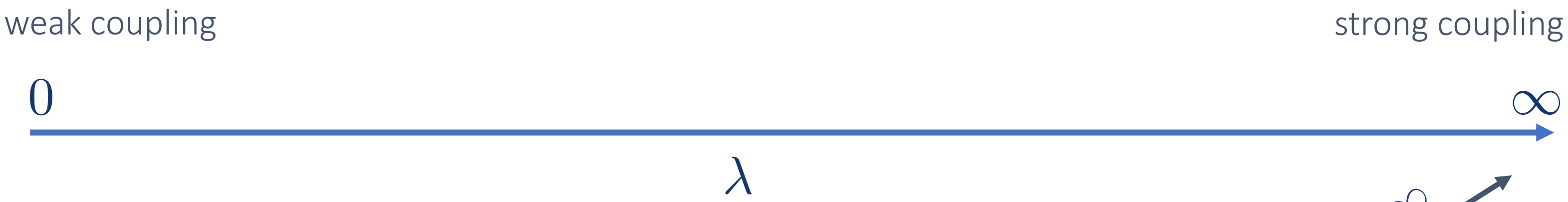
The large λ behaviour of the X matrix can be obtained by Mellin-Barnes techniques

$$\mathsf{X}^{\mathrm{odd}} \underset{\lambda\to\infty}{\sim} -\lambda\,\mathsf{S} + O(\lambda^0) \qquad \text{with}$$

Beccaria, Billò, M.F., Lerda , Pini, 2021
Beccaria, Dunne, Tseytlin, 2021 + …

$$\mathsf{S} = \frac{1}{16\pi^2}\begin{pmatrix} 1 & -\frac{1}{\sqrt{15}} & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{\sqrt{15}} & \frac{1}{3} & -\frac{2}{3\sqrt{35}} & 0 & 0 & 0 & \cdots \\ 0 & -\frac{2}{3\sqrt{35}} & \frac{1}{6} & -\frac{1}{6\sqrt{7}} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{1}{6\sqrt{7}} & \frac{1}{10} & -\frac{2}{15\sqrt{11}} & 0 & \cdots \\ 0 & 0 & 0 & -\frac{2}{15\sqrt{11}} & \frac{1}{15} & -\frac{1}{3\sqrt{143}} & \cdots \\ 0 & 0 & 0 & 0 & -\frac{1}{3\sqrt{143}} & \frac{1}{21} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2-point functions



$$G^{-}_{2n+1} = \mathcal{G}_{2n+1} \frac{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n]}\right)}$$

$\lambda \to \infty$

- More interestingly, we can derive analytically the strong-coupling expansions of the 2-point functions. In fact

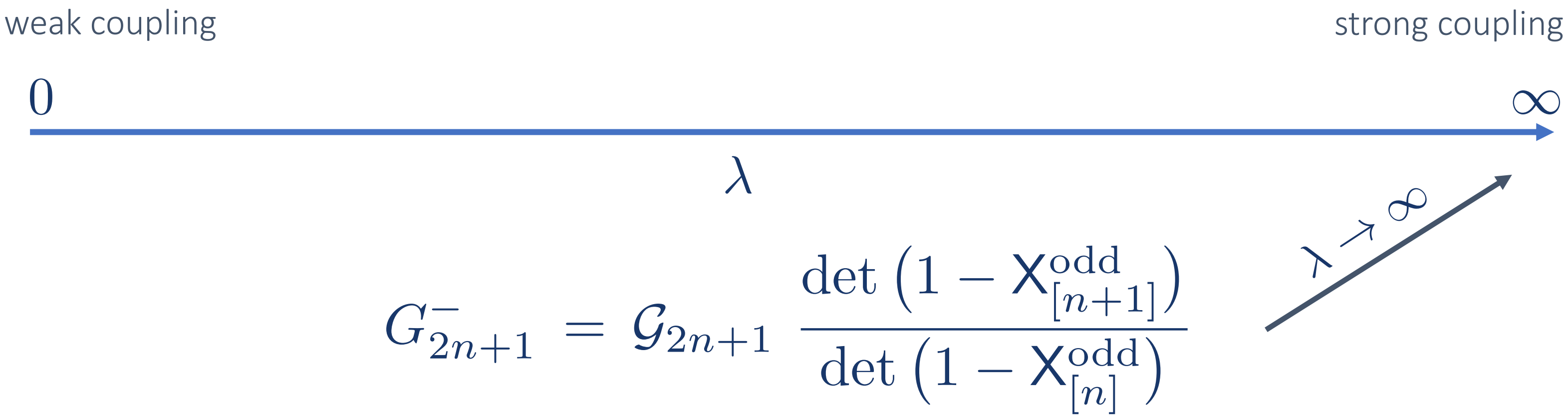
$$\mathsf{X}^{\text{odd}} \underset{\lambda\to\infty}{\sim} -\lambda\,\mathsf{S} + O(\lambda^0)$$

- Heuristically

$$\det\left(\cancel{1} - \mathsf{X}^{\text{odd}}\right) \underset{\lambda\to\infty}{\sim} \det(\lambda\,\mathsf{S})$$

$$\frac{\det\left(\cancel{1} - \mathsf{X}^{\text{odd}}_{[n+1]}\right)}{\det\left(\cancel{1} - \mathsf{X}^{\text{odd}}_{[n]}\right)} \underset{\lambda\to\infty}{\sim} \frac{1}{\lambda}$$

2-point functions



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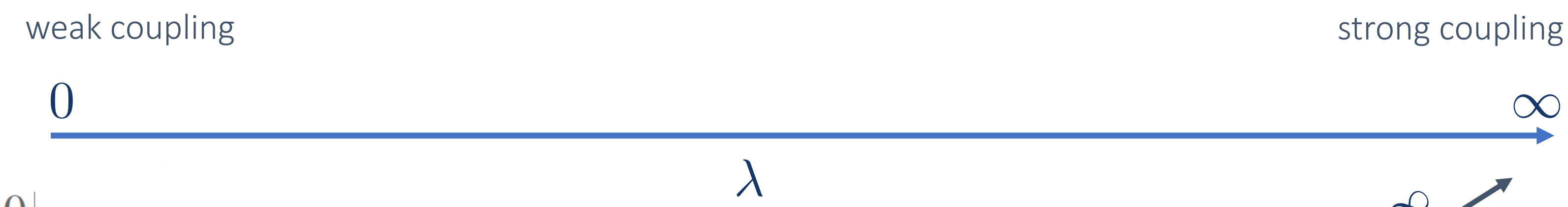
- More interestingly, we can derive analytically the strong-coupling expansions of the 2-point functions. In fact

$$G^{-}_{2n+1} \underset{\lambda\to\infty}{\sim} \mathcal{G}_{2n+1} \frac{8\pi^2\, n\,(2n+1)}{\lambda} + O(\lambda^{-\frac{3}{2}})$$

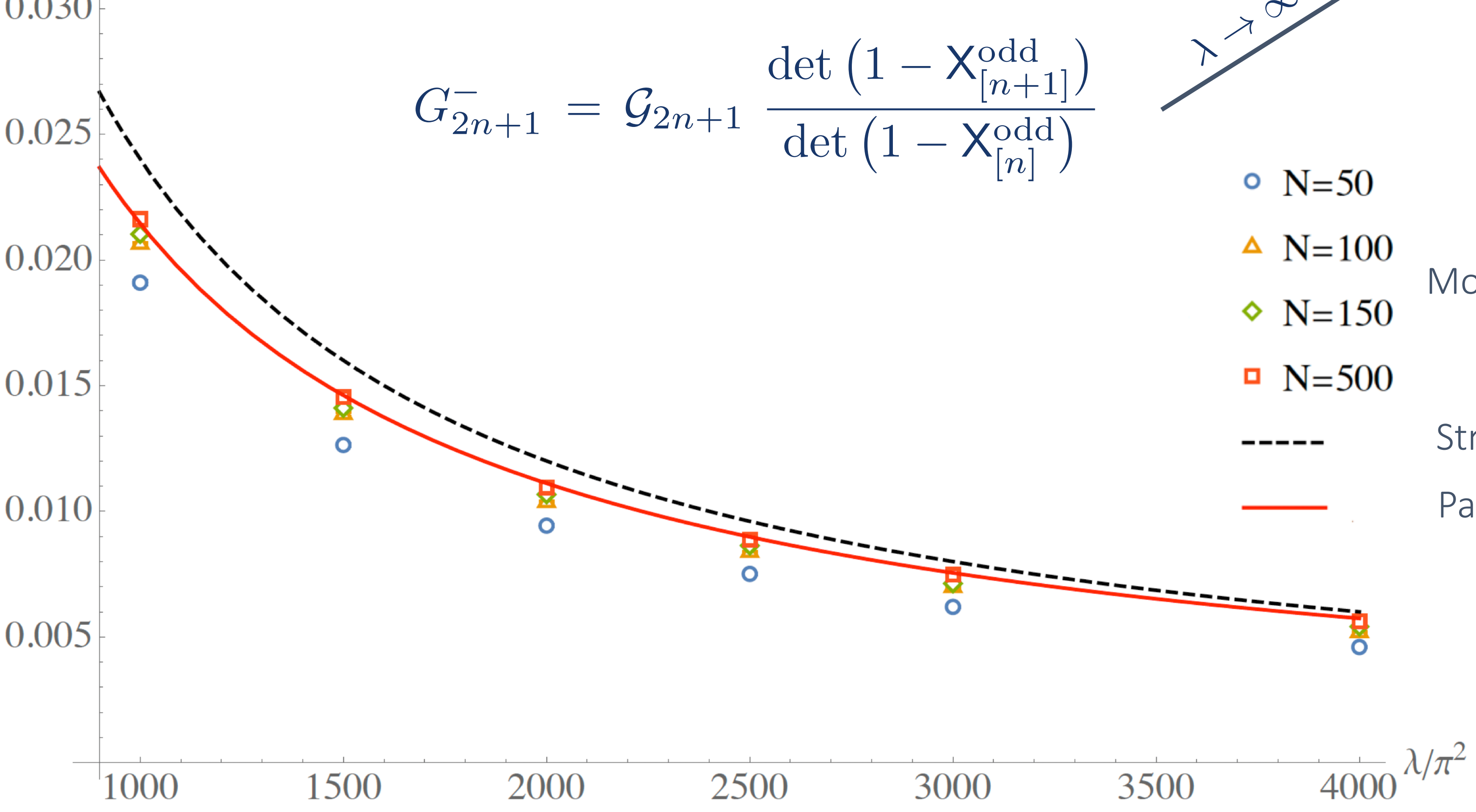
$$G^{-}_{2n} \underset{\lambda\to\infty}{\sim} \mathcal{G}_{2n} \frac{8\pi^2\, n\,(2n-1)}{\lambda} + O(\lambda^{-\frac{3}{2}})$$

2-point functions ➡ Numerical checks



$\lambda \to \infty$

$$G^{-}_{2n+1} = \mathcal{G}_{2n+1} \frac{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n+1]}\right)}{\det\left(1 - \mathsf{X}^{\text{odd}}_{[n]}\right)}$$



Monte Carlo simulations

Strong coupling

Padé

2-point functions

- Actually, one can do more and derive the full strong-coupling expansion

$$\log\left[\det\left(1-\mathsf{X}_{[n]}^{\text{even}}\right)\right]=\frac{\sqrt{\lambda}}{4}-\left(2n-\frac{3}{2}\right)\log\left(\frac{\sqrt{\lambda}}{4\pi}\right)+B_{2n-1}+f_{2n-1}$$

$$\log\left[\det\left(1-\mathsf{X}_{[n]}^{\text{odd}}\right)\right]=\frac{\sqrt{\lambda}}{4}-\left(2n-\frac{1}{2}\right)\log\left(\frac{\sqrt{\lambda}}{4\pi}\right)+B_{2n}+f_{2n}$$

Beccaria, Korchemsky, Tseytlin, 2022

where

$$B_k=-6\log\mathsf{A}+\frac{1}{2}+\frac{1}{6}\log 2-k\,\log 2+\log\Gamma(k)$$

A = Gleisher constant

$$f_k=\frac{1}{16}(2k-3)(2k-1)\log\left(\frac{\lambda'}{\lambda}\right)+(2k-5)(2k-3)(4k^2-1)\frac{\zeta_3}{32\lambda'^{3/2}}$$
$$-(2k-7)(2k-5)(4k^2-9)(4k^2-1)\frac{3\,\zeta_5}{256\lambda'^{5/2}}+O\left(\frac{1}{\lambda'^3}\right)$$ + non-perturbative terms

$$\sqrt{\lambda'}=\sqrt{\lambda}-4\log 2$$

2-point functions

$$G_k^- = \mathcal{G}_k \, \frac{4\pi^2 \, k\,(k-1)}{\lambda} \left(\frac{\lambda'}{\lambda}\right)^{k-1} \Bigg[1 + (k-1)(2k-1)(2k-3)\,\frac{\zeta_3}{\lambda'^{\,3/2}} - (k-1)(2k-3)(2k-5)(4k^2-1)\,\frac{9\,\zeta_5}{16\lambda'^{\,5/2}} + O\!\left(\frac{1}{\lambda'^{\,3}}\right)\Bigg]$$

Sub-leading corrections

Leading Order term

+ non-perturbative terms

$$\begin{tikzcd}
 & G_k^- \arrow[dl, "\lambda\to 0"'] \arrow[dr, "\lambda\to\infty"] & \\
G_k^- \underset{\lambda\to 0}{\sim} \mathcal{G}_k + O(\lambda) & & G_k^- \underset{\lambda\to\infty}{\sim} \mathcal{G}_k \, \dfrac{4\pi^2 \, k\,(k-1)}{\lambda} + O(\lambda^{-\frac{3}{2}})
\end{tikzcd}$$

3-point functions

- A similar analysis can be done for 3-point functions
- The calculations are simplified by observing that 3-point functions are related to the 2-point functions by an exact Ward-like identity

$$G^{-}_{k,\ell,p} = \langle O^{+}_{k}\, O^{-}_{\ell}\, O^{-}_{p} \rangle = \frac{1}{\sqrt{2}\,N} \sqrt{\big(k+\lambda\partial_\lambda\big)G^{+}_{k}}\, \sqrt{\big(\ell+\lambda\partial_\lambda\big)G^{-}_{\ell}}\, \sqrt{\big(p+\lambda\partial_\lambda\big)G^{-}_{p}}$$

Billò, M.F.,Lerda, Pini, Vallarino, 2022

- From these results one sees that at the leading order in λ

$$G^{-}_{k,\ell,p} = \langle O^{+}_{k}\, O^{-}_{\ell}\, O^{-}_{p} \rangle = \frac{1}{\sqrt{2}\,N}\,\frac{1}{\lambda} \sqrt{k(\ell-1)(p-1)}\sqrt{G^{+}_{k}}\, \sqrt{G^{-}_{\ell}}\, \sqrt{G^{-}_{p}}$$

3-point functions

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- The calculations are simplified by observing that 3-point functions are related to the 2-point functions by an exact Ward-like identity

$$G^{-}_{k,\ell,p} = \langle O^{+}_{k}\, O^{-}_{\ell}\, O^{-}_{p} \rangle = \frac{1}{\sqrt{2}\,N} \sqrt{(k+\lambda\partial_\lambda)G^{+}_{k}}\, \sqrt{(\ell+\lambda\partial_\lambda)G^{-}_{\ell}}\, \sqrt{(p+\lambda\partial_\lambda)G^{-}_{p}}$$

Billò, M.F.,Lerda, Pini, Vallarino, 2022

- From these results one can derive the structure constants

$$C^{-}_{k,\ell,p} = \frac{G^{-}_{k,\ell,p}}{\sqrt{G^{+}_{k}\,G^{-}_{\ell}\,G^{-}_{p}}} = \frac{1}{\sqrt{2}\,N}\sqrt{k+\lambda\partial_\lambda\big(\log G^{+}_{k}\big)}\, \sqrt{\ell+\lambda\partial_\lambda\big(\log G^{-}_{\ell}\big)}\, \sqrt{p+\lambda\partial_\lambda\big(\log G^{-}_{p}\big)}$$

Structure constants

weak coupling — strong coupling

$$0 \xrightarrow[\lambda]{} \infty$$

$$C^{-}_{k,\ell,p} = \frac{1}{\sqrt{2}\,N}\sqrt{k+\lambda\partial_\lambda\big(\log G^{+}_{k}\big)}\;\sqrt{\ell+\lambda\partial_\lambda\big(\log G^{-}_{\ell}\big)}\;\sqrt{p+\lambda\partial_\lambda\big(\log G^{-}_{p}\big)}$$

$\lambda \to 0$ — $\lambda \to \infty$

$$C^{-}_{k,\ell,p} \underset{\lambda\to 0}{\sim} \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N}$$

$$C^{-}_{k,\ell,p} \underset{\lambda\to \infty}{\sim} \frac{\sqrt{k\,(\ell-1)\,(p-1)}}{\sqrt{2}\,N}$$

⬇

It follows from the AdS/CFT correspondence !

Billò, M.F.,Lerda, Pini, Vallarino, 2022

Holographic description

- The holographic dual of the quiver theory is the orbifold $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_2$ whose fixed locus is $\mathrm{AdS}_5 \times S^1$
- As in any orbifold, we have the untwisted and the twisted sectors

Holographic description

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- As in any orbifold, we have the untwisted and the twisted sectors

- The untwisted operators $\mathcal{O}_k^+$ are dual to K.K. modes of the metric and the R-R 4-form fluctuations (as in $\mathcal{N}=4$) s_k , whose effective action (derived from Type II B sugra in d=10) is

$$S = \frac{1}{2\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\left[\sum_{k\geq 2} A_k\Big(\nabla_\mu s_k^*\,\nabla^\mu s_k + k(k-4)s_k^*\,s_k\Big) + \sum_{k,\ell,p}\Big(V_{k,\ell,p}\,s_k^*\,s_\ell^*\,s_p + \mathrm{c.c.}\Big)\right]\frac{\pi^3}{2}$$

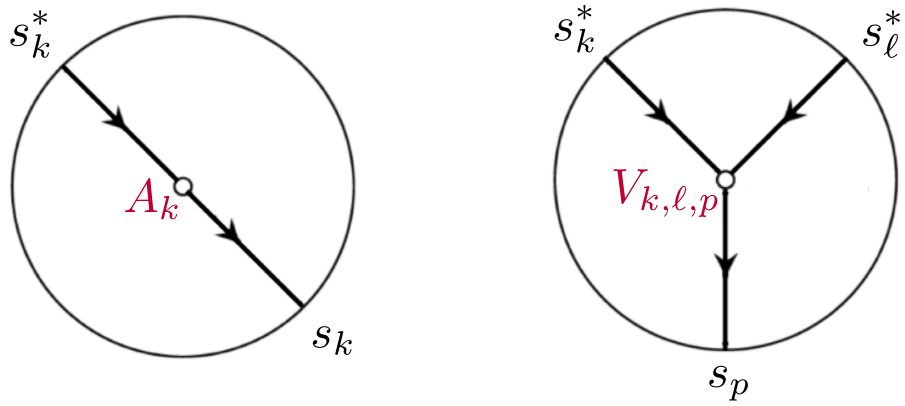
Maldacena, 1997;
Lee, Minwalla, Rangamani, Seiberg, 1998; …

Holographic description

- From this action, using the holographic dictionary

$$8\pi N g_s = \lambda\,, \quad \alpha'\sqrt{\lambda} = R^2 \quad \longrightarrow \quad \frac{1}{2\kappa_{10}^2} = \frac{1}{(2\pi)^7 g_s^2\,\alpha'^{\,4}} = \cdots = \frac{4(2N)^2}{(2\pi)^5}\,\xcancel{\frac{1}{R^8}}$$

- one computes the 2- and 3- point functions from Witten diagrams



- and gets

$$C^+_{k,\ell,p} \underset{\lambda\to 0}{\sim} \frac{\sqrt{k\,\ell\,p}}{\sqrt{2}\,N} = C^+_{k,\ell,p}\Big|_{\lambda=0}$$

like in $\mathcal{N}=4$

Holographic description

- The twisted operators $\mathcal{O}_k^-$ are dual to K.K. modes of the twisted scalars η_k obtained by wrapping the NS-NS and R-R 2-forms on the exceptional 2-cycle

$$\frac{1}{2\pi\alpha'}\int_e B_{(2)} \qquad \frac{1}{2\pi\alpha'}\int_e C_{(2)}$$

Gukov, 1998;
Billò, M.F., Galvagno, Lerda,Pini, 2021

- Their effective action (derived from localizing Type II B sugra at the orbifold fixed point) is

$$S = \frac{(2\pi\alpha')^2}{4\kappa_{10}^2}\int_{\mathrm{AdS}_5} d^5z\,\sqrt{g}\left[\sum_{k\geq 2}\frac{1}{2}\Big(\underbrace{\nabla_\mu\eta_k^*\,\nabla^\mu\eta_k + k(k-4)}_{\text{Gukov, 1998}}\eta_k^*\,\eta_k\Big) + \sum_{k,\ell,p}\Big(W_{k,\ell,p}\,s_k^*\,\eta_\ell^*\,\eta_p + \mathrm{c.c.}\Big)\right]2\pi$$

$$W_{k,\ell,p} = -\frac{(k+\ell-p)(k+p-\ell)(k+l+p-2)(k+\ell+p-4)}{2^{\frac{k}{2}}(k+1)}$$

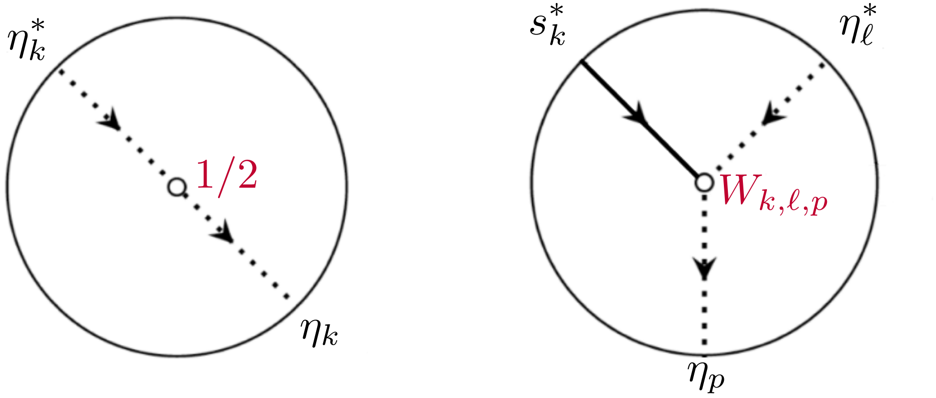
Billò, M.F., Lerda, Pini, Vallarino, 2022

Holographic description

- From this action, using the holographic dictionary

$$8\pi N g_s = \lambda\,, \quad \alpha'\sqrt{\lambda} = R^2 \quad \longrightarrow \quad \frac{(2\pi\alpha')^2}{4\kappa_{10}^2} = \frac{1}{2(2\pi)^5 g_s^2 \alpha'^2} = \cdots = \frac{2(2N)^2}{(2\pi)^3}\,\frac{1}{\lambda}\,\frac{1}{R^4}$$

- one computes the 2- and 3- point functions from Witten diagrams



- and gets
- $C^{-}_{k,\ell,p} \underset{\lambda\to\infty}{\sim} \dfrac{\sqrt{k\,(\ell-1)\,(p-1)}}{\sqrt{2}\,N}$ in agreement with the localization result!

First explicit check of the AdS/CFT correspondence in a non-maximally susy set-up.

Holographic description ➡ open questions

- The systematic strong-coupling expansion found for the structure constant

$$C^{-}_{k,\ell,p} = C^{-(\mathrm{LO})}_{k,\ell,p} \left(\frac{\lambda}{\lambda'}\right)^{\frac{1}{2}} \left[1 + c^{(1)}_{k,\ell,p}\frac{\zeta_3}{\lambda'^{3/2}} + c^{(2)}_{k,\ell,p}\frac{\zeta_5}{\lambda'^{5/2}} + c^{(3)}_{k,\ell,p}\frac{\zeta_3^2}{\lambda'^{3}} + \cdots\right]$$

  once the gauge/gravity dictionary $\alpha'\sqrt{\lambda} = R^2$ is used, has the same form expected for closed string amplitudes and, since at the moment explicit calculations beyond the supergravity limit do not seem to be possible, this analysis provides a very strong prediction for the string corrections.

- There is however a case in which a check of the AdS/CFT correspondence at the string level seems possible…

Ashok, Billò, M.F., Lerda, work in progress

Holographic description ➡ open questions

- When we consider a $\mathbb{Z}_M$ quiver theory with $M \geq 3$, also 3-point functions of only twisted operator are non trivial

$$G^{T}_{k,\ell,p} = \langle O^{\alpha}_{k}\, O^{\beta}_{\ell}\, O^{\gamma}_{p} \rangle\,, \qquad \alpha + \beta + \gamma = 0 \bmod M$$

Billò, M.F., Lerda, Pini, Vallarino, 2022

- From the field (and matrix) theory they are on the same footing of the others and, at strong coupling

$$C^{T}_{k,\ell,p} = \frac{G^{T}_{k,\ell,p}}{\sqrt{G^{\alpha}_{k}\, G^{\beta}_{\ell}\, G^{\gamma}_{p}}} \underset{\lambda\to\infty}{\sim} \frac{\sqrt{(k-1)\,(\ell-1)\,(p-1)}}{\sqrt{M}\,N} \quad \longrightarrow \quad G^{T}_{k,\ell,p} \underset{\lambda\to\infty}{\propto} \frac{1}{\lambda^{3/2}}$$

Holographic description ⟶ open questions

- When we consider a $\mathbb{Z}_M$ quiver theory with $M \geq 3$, also 3-point functions of only twisted operator are non trivial

$$G^T_{k,\ell,p} = \langle O^\alpha_k \, O^\beta_\ell \, O^\gamma_p \rangle \,, \qquad \alpha + \beta + \gamma = 0 \mod M$$

Billò, M.F., Lerda, Pini, Vallarino, 2022

- From the holographic point of view, since $\lambda^{-1/2} \propto \alpha'$, this means that the TTT-couplings must have an extra power of α' with respect to the UTT ones

$$G^T_{k,\ell,p} \underset{\lambda \to \infty}{\propto} \frac{1}{\lambda^{3/2}} \quad \longrightarrow \quad G^T_{k,\ell,p} \propto \alpha'^3$$

stringy coupling!

Ashok, Billò, M.F., Lerda, work in progress

Wilson loop

Wilson loop

- Localization can also be exploited to compute correlators of local operators in a defect CFT with Wilson loop.
- In case of circular Wilson loop in the fundamental representation, this amounts to insert in the matrix model correlators the operator

$$W = \frac{1}{N}\,\mathrm{tr}\exp\left(\frac{g}{\sqrt{2}}\,a\right)$$

- In $\mathcal{N}=4$ many exacts results are known and in particular

$$\langle W\rangle = \frac{2}{\sqrt{\lambda}}\,I_1(\sqrt{\lambda})$$

Erickson, Semenoff, Zarembo, 2000

$$\langle W\,\mathcal{O}_n(x)\rangle = \frac{\sqrt{n\,\mathcal{G}_n}}{N}\,\frac{I_n(\sqrt{\lambda})}{\left(2\pi\,|x|\right)^n}$$

Semenoff, Zarembo, 2001

Wilson loop

- In $\mathcal{N}=2$ some exacts results are known for the large N limit in the quiver theory (or its orientifold → E-theory)
Pini, Vallarino 2023

- 1-point function of twisted chiral operator in presence of Wilson loop:

$$\langle W\mathcal{O}_n(x)\rangle = \frac{w_n\left(\lambda,N\right)}{\left(2\pi\left|x\right|\right)^n}$$

where $w_n\left(\lambda,N\right)$ can be obtained in the same way discussed for local operators, for instance

$$w_3\left(\lambda,N\right) = \frac{1}{N}\sum_{m=1}^{\infty}\sqrt{m}\,\mathcal{D}_{3,m}\,I_m\left(\sqrt{\lambda}\right) \qquad \mathcal{D}_{3,m} = \sqrt{\mathcal{G}_3}\,\mathsf{D}_{3,m}$$

Wilson loop

- In $\mathcal{N}=2$ some exacts results are known for the large N limit in the quiver theory (or its orientifold ⟶ E-theory)

Pini, Vallarino 2023

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where $w_n\left(\lambda, N\right)$ ca be obtained in the same way discussed for local operators, for odd n

$$w_n\left(\lambda, N\right) = \frac{1}{N}\sum_{m=1}^{\infty}\sqrt{m}\,\mathcal{D}_{n,m}\,I_m\big(\sqrt{\lambda}\big)$$

$$\mathcal{D}_{n,m} = \sum_{\ell=3}^{n}\sum_{k=3}^{\ell}\alpha_{nl}^{\ell}\left(\lambda\right)\sqrt{\mathcal{G}_l}\,\mathsf{D}_{m,k}$$

These formulas are valid for any value of λ !

Wilson loop

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Pini, Vallarino 2023

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$$\langle W\mathcal{O}_n(x)\rangle = \frac{w_n(\lambda, N)}{\left(2\pi\,|x|\right)^n} \qquad\qquad w_n(\lambda, N) = \frac{1}{N}\sum_{m=1}^{\infty}\sqrt{m}\,\mathcal{D}_{n,m}\,I_m(\sqrt{\lambda})$$

- The strong coupling regime is obtained by carefully studying the ratio $\dfrac{\langle W\mathcal{O}_n\rangle}{\langle W\mathcal{O}_n\rangle|_0}$ with a mix of analytical and numerical techniques

$$\frac{\langle W\mathcal{O}_n\rangle}{\langle W\mathcal{O}_n\rangle|_0} = \frac{n-1}{\sqrt{\lambda}}(4\pi - 2.69)$$

Pini, Vallarino 2023
Korchemsky, Pini, Vallarino, work in progress

Mass deformation of $\mathcal{N} = 4$

$\mathcal{N}=2^*$ Gauge theories

- $\mathcal{N}=2^*$ gauge theories are mass deformation of $\mathcal{N}=4$ in which we give mass to the scalars of the adjoint hypers
- The sphere partition function of $\mathcal{N}=2^*$ has a non-trivial mass dependence coming from $Z_{1-\text{loop}}$ and, at leading order in m^2,

$$\mathcal{Z}_{\mathcal{N}=2^*} = \mathcal{Z}_{\mathcal{N}=4}\left(1+m^2\,M(\lambda)+O(m^4)\right) \qquad M(\lambda)=\langle\,\mathbf{M}(\lambda,a)\,\rangle_0$$

$\mathcal{N}=2^*$ Gauge theories

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- The sphere partition function of $\mathcal{N}=2^*$ has a non-trivial mass dependence coming from $Z_{1-\text{loop}}$ and, at leading order in m^2,

$$\mathcal{Z}_{\mathcal{N}=2^*}=\mathcal{Z}_{\mathcal{N}=4}\left(1+m^2\,M(\lambda)+O(m^4)\right) \qquad M(\lambda)=\langle\,\mathbf{M}(\lambda,a)\,\rangle_0$$

- This leading term is interesting because

$$\partial_\tau\,\partial_{\bar\tau}\,\partial_{m^2}\mathcal{Z}_{\mathcal{N}=2^*}\big|_{m=0}\propto\int d\mu\,\langle\mathcal{O}_v\mathcal{O}_v\mathcal{O}_h\mathcal{O}_h\rangle \qquad \mathcal{O}_v=\operatorname{tr}\phi_v^2 \qquad \mathcal{O}_h=\operatorname{tr}\phi_h^2$$

Binder, Chester, Pufu, Wang 2019
Dorigoni, Green, Wen 2021 + …

$\mathcal{N}=2^*$ Gauge theories

- $\mathcal{N}=2^*$ gauge theories are mass deformation of $\mathcal{N}=4$ in which we give mass to the scalars of the adjoint hypers
- The sphere partition function of $\mathcal{N}=2^*$ has a non-trivial mass dependence coming from $Z_{1-\text{loop}}$ and, at leading order in m^2,

$$\mathcal{Z}_{\mathcal{N}=2^*}=\mathcal{Z}_{\mathcal{N}=4}\left(1+m^2\,M(\lambda)+O(m^4)\right) \qquad M(\lambda)=\langle\,\mathbf{M}(\lambda,a)\,\rangle_0$$

- The local operator $\mathbf{M}(\lambda,a)$ can be expressed in a form very similar to S_{int} of the quiver theory in term of an infinite matrix $\mathsf{K}_{k,\ell}$

$$\mathbf{M}=\sum_{k,\ell=0}^{\infty}\mathsf{K}_{k,\ell}\,\mathcal{P}_k\mathcal{P}_\ell \qquad \underset{k,\ell\geq 1}{\mathsf{K}_{k,\ell}}=-(-1)^{\frac{k+\ell}{2}}\sqrt{k\,\ell}\int_0^\infty\frac{dt}{t}\,\frac{t^2\,e^t}{(e^t-1)^2}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

slightly different kernel from **X** !

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$$\mathcal{Z}_{\mathcal{N}=2^*}=\mathcal{Z}_{\mathcal{N}=4}\big(1+m^2\,M(\lambda)+O(m^4)\big) \qquad\qquad M(\lambda)=\langle\,\mathbf{M}(\lambda,a)\,\rangle_0$$

- In terms of $\mathsf{K}_{k,\ell}$ one easily finds

$$M(\lambda)=N^2\,\mathsf{K}_{00}=N^2\int_0^\infty\frac{dt}{t}\,\frac{t^2\,e^t}{(e^t-1)^2}\left[1-\frac{16\pi^2}{t^2\lambda}J_1^2\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\right]$$

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- In terms of $\mathsf{K}_{k,\ell}$ one easily finds

$$\partial_\tau\,\partial_{\bar\tau}\,\partial_{m^2}\mathcal{Z}_{\mathcal{N}=2^*}\big|_{m=0} = \frac{N^2}{4}\,\left(2\,\mathsf{K}_{11}-\mathsf{K}_{22}\right)$$

- These are exact expressions in λ and their strong coupling limit can be studied with the techniques discussed for the scalar correlators!

Binder, Chester, Pufu, Wang 2019
Dorigoni, Green, Wen 2021 +…

$\mathcal{N} = 2^*$ Gauge theories

- The same reasoning can be applied to the study of the vev of a circular Wilson loop in $\mathcal{N} = 2^*$ gauge theories, where we may expect that

$$\partial_{m^2} \langle W \rangle_{\mathcal{N}=2^*} \big|_{m=0} \propto \int d\mu \, \langle W \mathcal{O}_h \mathcal{O}_h \rangle$$

Pufu, Rodriguez, Wang 2023
Billò, M.F., Galvagno, Lerda, work in progress

- Using the method discussed before one sees that, in the large N limit

$$\frac{\partial_{m^2} \langle W \rangle_{\mathcal{N}=2^*} \big|_{m=0}}{\langle W \rangle_{\mathcal{N}=4}} = \frac{\sqrt{\lambda}}{I_1(\sqrt{\lambda})} \sum_{r=1}^{\infty} \sqrt{2r} \, I_{2r}(\sqrt{\lambda}) \, \mathsf{K}_{0,2r}$$

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Russo, Zarembo 2013

$$\frac{2\pi\sqrt{\lambda}}{I_1(\sqrt{\lambda})} \int_0^\infty d\omega \, \frac{\omega}{(\omega^2 + \pi^2) \sinh^2 \omega} \, J_1(\frac{\omega}{\pi}\sqrt{\lambda}) \left[\pi I_0(\sqrt{\lambda}) \, J_1(\frac{\omega}{\pi}\sqrt{\lambda}) - \omega I_1(\sqrt{\lambda}) \, J_0(\frac{\omega}{\pi}\sqrt{\lambda})\right]$$

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Pufu, Rodriguez, Wang 2023
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- This method reproduce the known results in the simple case of $\mathcal{N} = 2^*$, where the leading m^2 contributions can be computed in the free matrix model

- But it can be used also to compute the effects of mass deformations in theories (like the quiver gauge theory we have discussed) that are associated to interacting matrix models!

Conclusions and Perspectives

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- The X matrix we have used for the study of the $\mathcal{N}=2$ quiver theory is very similar to the K matrix that appear in the study of the mass deformed theories and also to the matrix that is used for the analysis of the cusp anomalous dimension in $\mathcal{N}=4$ SYM via the BES equation

Beisert, Eden, Staudacher, 2006

$$\mathsf{X}^{\mathrm{BES}}_{k,\ell} = -4(-1)^{\frac{k+\ell+2k\ell}{2}}\,\sqrt{k\,\ell}\int_0^\infty \frac{dt}{t}\,\frac{1}{e^t-1}\,J_k\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)\,J_\ell\Big(\frac{t\sqrt{\lambda}}{2\pi}\Big)$$

- In this case non-perturbative exponentially small corrections were systematically studied and a resurgent transseries structure for the cusp anomalous dimension was discovered.
- It would be nice to see if the same studies could be done for the observables we just discussed along the lines initiated in Beccaria, Korchemsky, Tseytlin, 2022

Conclusions and Perspectives

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- In any case the formalism of the “X matrix” is much more general than it seems at first sight and it would be nice to uncover a general pattern!

Thank you for your attention!