Cosmological Bootstrap in Slow Motion

Sadra Jazayeri

Institut d'Astrophysique de Paris

Based on works with

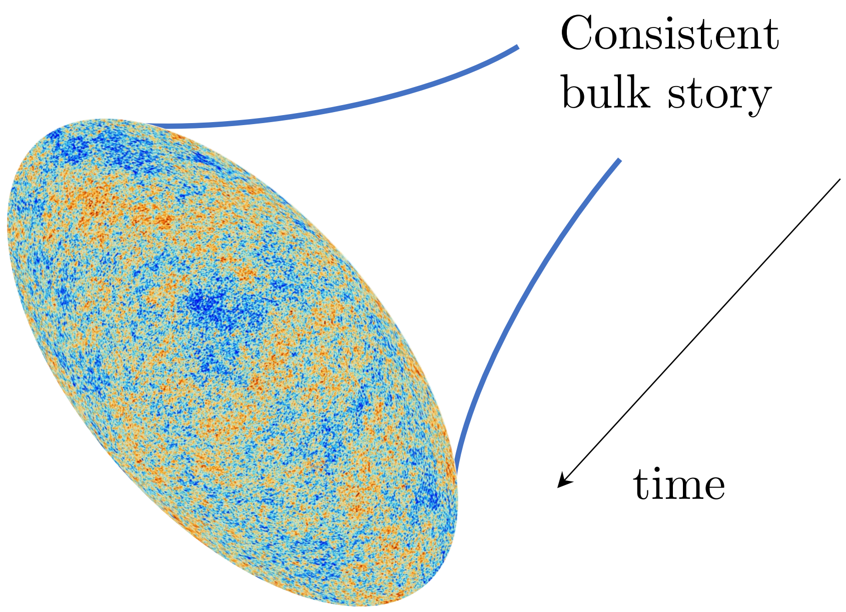
Sébastien Renaux-Petel, Enrico Pajer,
David Stefanyszyn, Harry Goodhew, Gordon Lee

Disclaimer: references incomplete

European Research Council
Established by the European Commission
GEODESI

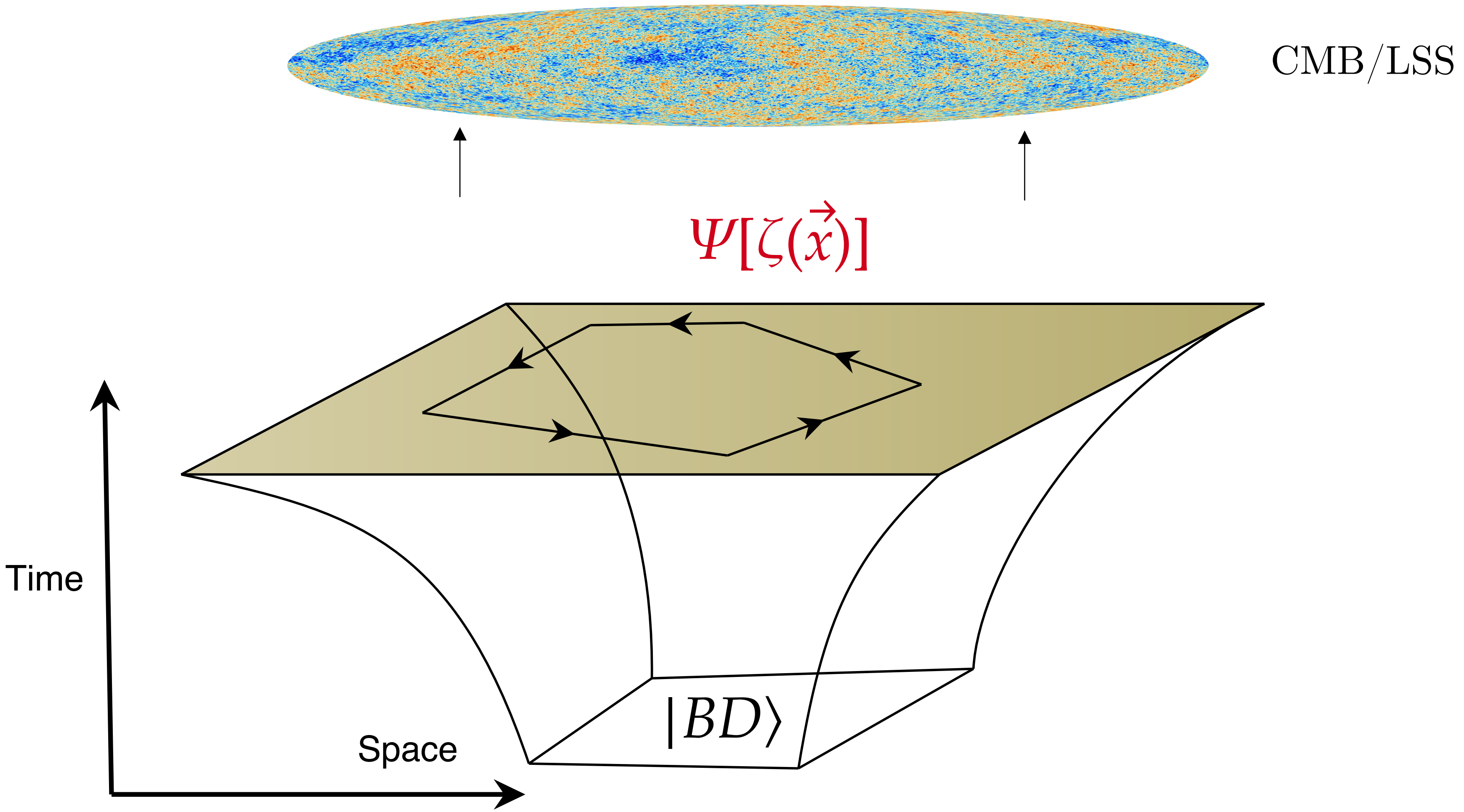
- Cosmological Correlators from a Boundary Perspective
- Bootstrap Elements
- Cosmological Phonon Collider
- Final Remarks

Cosmological Correlators from a Boundary Perspective



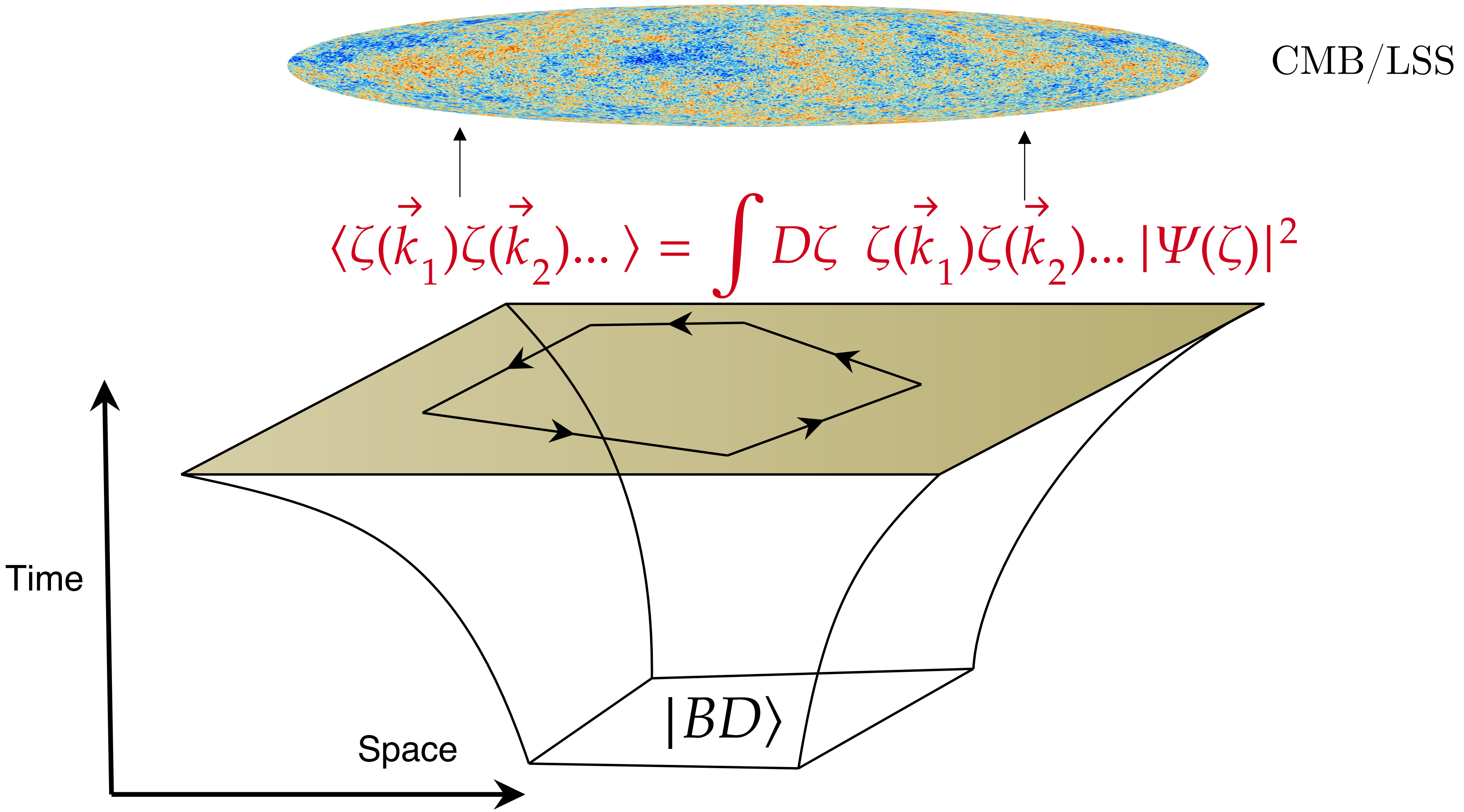
- Lat time wave function of the universe/cosmological correlators are the only fundamental observables in Cosmology

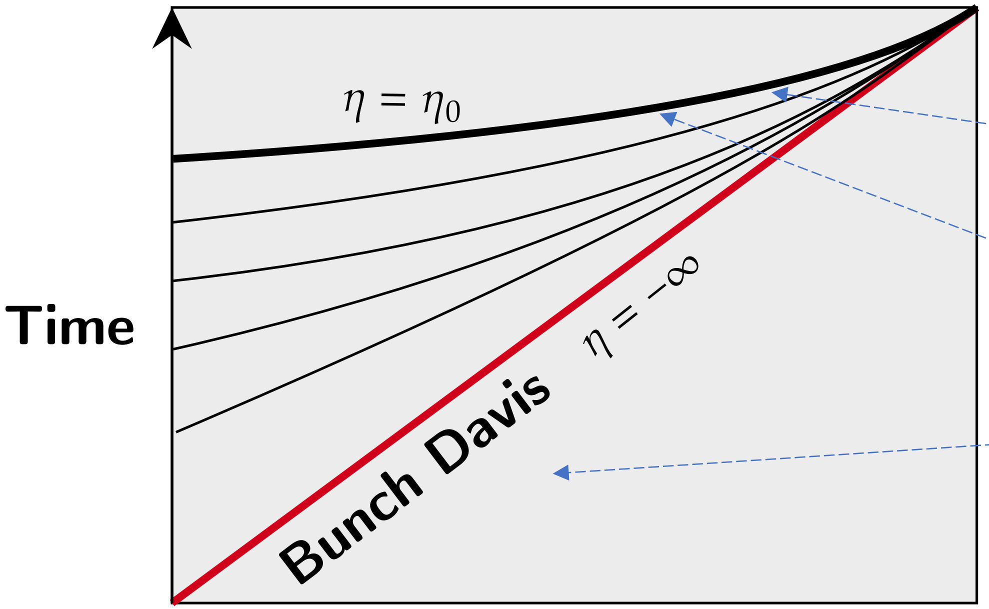
$$\langle \frac{\delta\rho}{\rho}(\mathbf{x}_1) \dots \frac{\delta\rho}{\rho}(\mathbf{x}_n) \rangle$$



- Lat time wave function of the universe/cosmological correlators are the only fundamental observables in Cosmology

$$\langle \frac{\delta\rho}{\rho}(\mathbf{x}_1)\dots\frac{\delta\rho}{\rho}(\mathbf{x}_n)\rangle$$



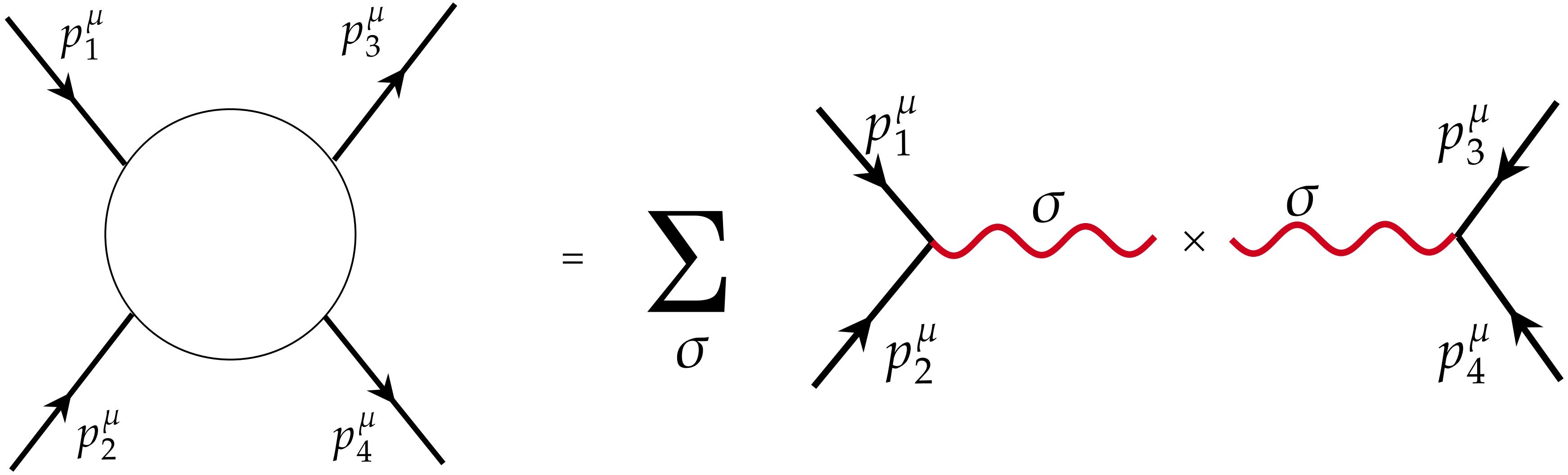


$$\Psi[\phi(\mathbf{k})] = \int_{\tilde\phi(-\infty(1-i\epsilon))=0}^{\tilde\phi(\eta_0)=\phi} \mathcal{D}\tilde\phi(\eta,\mathbf{x})\ e^{iS[\tilde\phi]}$$

quasi-dS background	$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2)$	$a(\eta) \sim -\frac{1}{\eta H}$
weakly coupled QFT	$S = S_2[\phi] + S_{\rm I}[\phi]$	

Pros	Cons
Explicitly unitary (Hermitian Hamiltonian)	**Complex nested time integrals** (lack of time translation in cosmology)
Explicitly local (local interactions at vertices)	**Complex massive field mode functions**
Explicit invariance under putative symmetries (e.g. de Sitter isometries)	**Redundancies** field redefinitions Gauge/Diff transformations

- The way out in flat space: on-shell methods



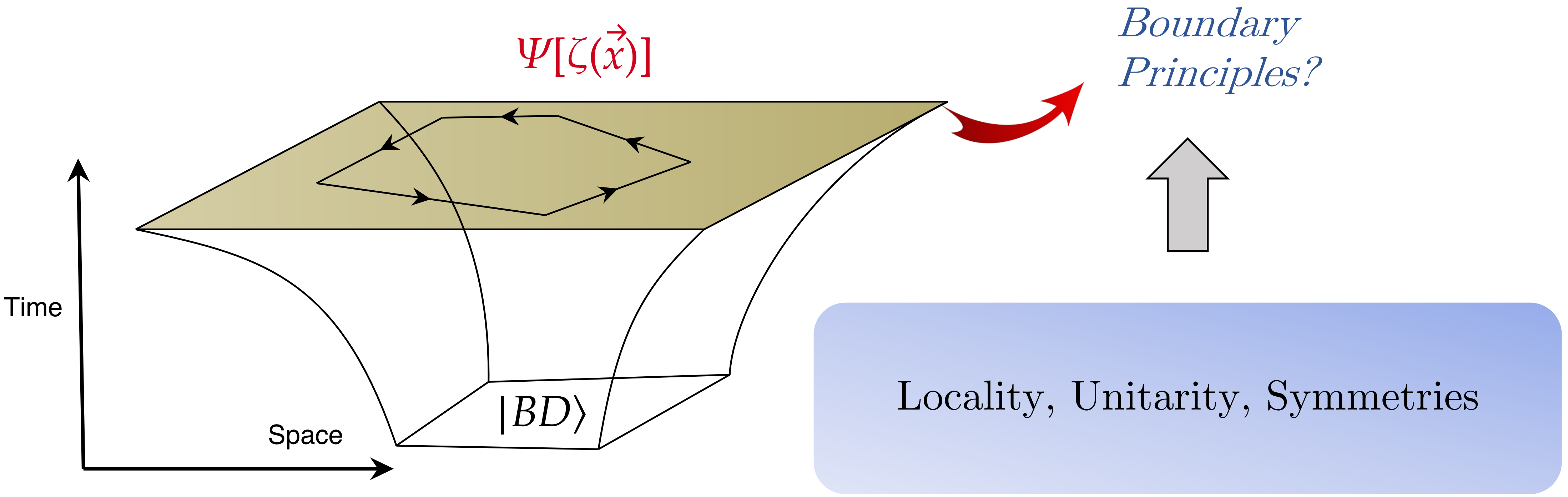
$$\lim_{(p_1+p_2)^2 \to 0} \mathcal{A}_4 = \frac{1}{(p_1+p_2)^2} \times \sum_\sigma \mathcal{A}_3(p_1, p_2, \sigma) \times \mathcal{A}_3(\sigma, p_3, p_4)$$

Locality — Unitarity — Lorentz Invariance

- Shifting the perspective: cosmological bootstrap aims at directly finding the boundary correlators without following the bulk time evolution

2017-2022: Arkani-Hamed, Baumann, Benincasa, Duaso Pueyo, Goodhew, Gorbenko, Jazayeri, Joyce, Lipstein, Lee McFadden, Meltzer, Melville, Pajer, Penedones, Pimentel, Sleight, Salehi-Vaziri, Stefanyszyn, Tarona ....

Earlier works: Bzowski et al (2011,2012, 2013), Raju (2012), Kundo et al (2013, 2015), Maldacena and Pimentel(2011)



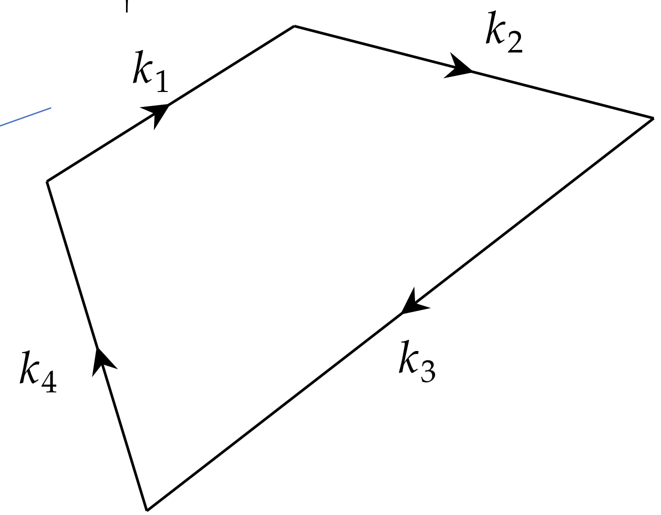
Bootstrap Elements

I. Observables

- The wave function of the universe is approximately Gaussian. (Perturbative) departures from Gaussianity can be systematically captured with a set of wave function coefficients.

$$\Psi[\eta_0, \phi] = \exp\left[-\sum_{n=2}^{+\infty} \frac{1}{n!} \int_{\mathbf{k}_1 \ldots \mathbf{k}_n} \psi_n(\{k\}, \{\mathbf{k}\}) (2\pi)^3 \delta_{\mathrm{D}}^{(3)}\left(\sum_{a=1}^{n} \mathbf{k}_a \right) \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_n) \right]$$

"external energies"



I. Observables

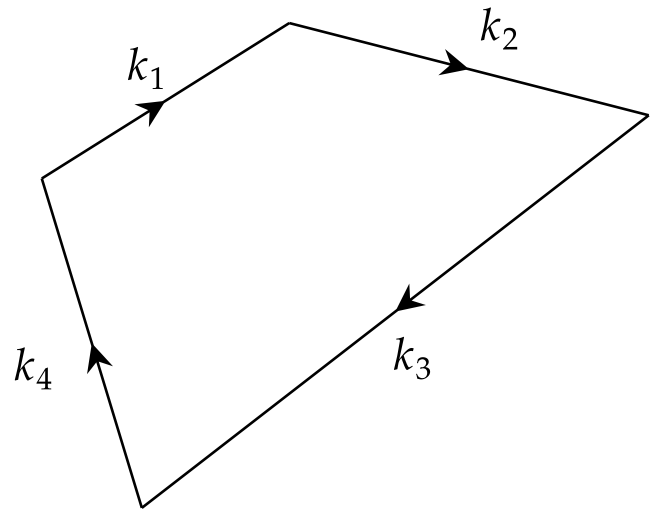
- The wave function of the universe is approximately Gaussian. (Perturbative) departures from Gaussianity can be systematically captured with a set of wave function coefficients.

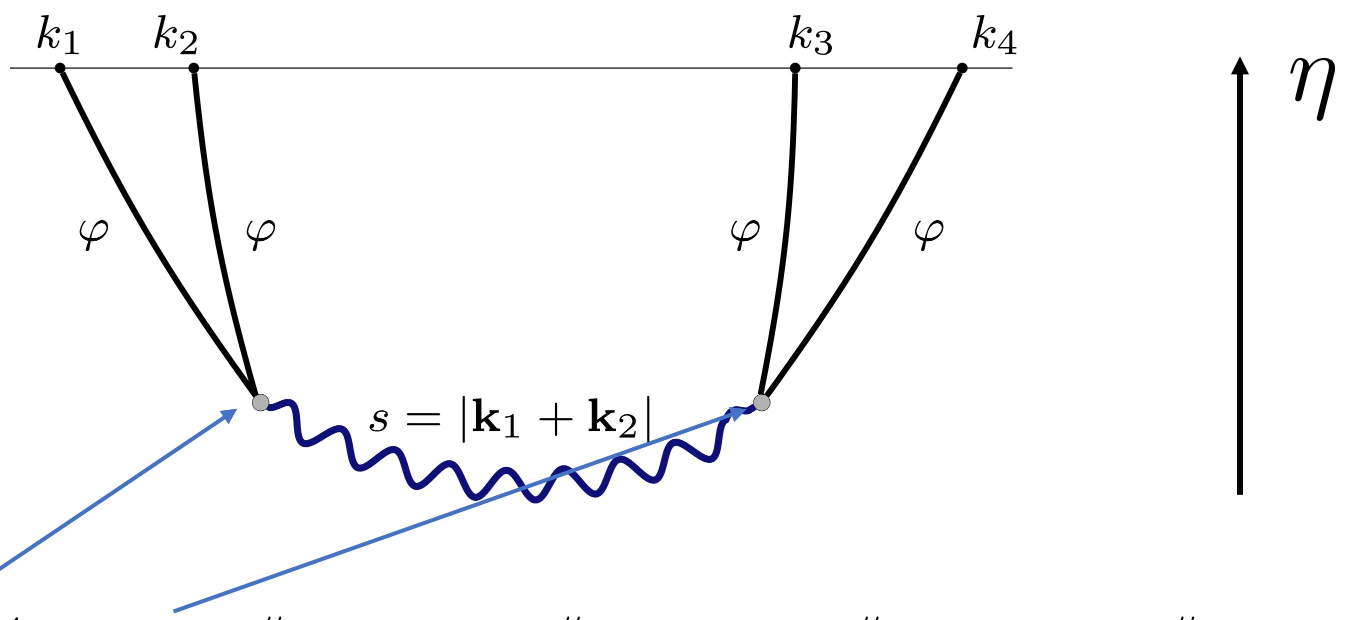
$$\Psi[\eta_0, \phi] = \exp\left[-\sum_{n=2}^{+\infty} \frac{1}{n!} \int_{\mathbf{k}_1 \ldots \mathbf{k}_n} \psi_n(\{k\}, \{\mathbf{k}\}) (2\pi)^3 \delta_{\mathrm{D}}^{(3)}\left(\sum_{a=1}^{n} \mathbf{k}_a \right) \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_n) \right]$$

$\psi_4(k_1, k_2, k_3, k_4, s, t)$ "internal energies"

$$s = |\mathbf{k}_1 + \mathbf{k}_2|\,, \quad t = |\mathbf{k}_1 + \mathbf{k}_3|\,, \quad u = |\mathbf{k}_1 + \mathbf{k}_4|$$

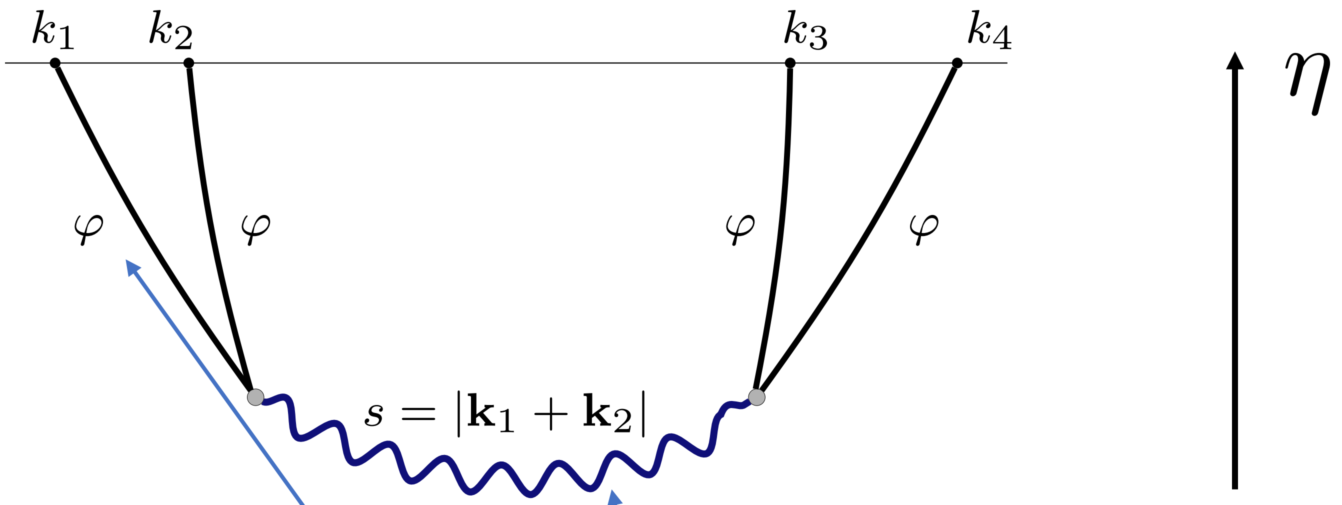
Dilation Sym. $\Rightarrow \psi_n(\lambda \mathbf{k}_1, ..., \lambda \mathbf{k}_n) = \lambda^3 \psi_n(\mathbf{k}_1, ..., \mathbf{k}_n)$





$$\psi_4 = \int_{-\infty(1-i\epsilon)}^{0} a^4(\eta)\, d\eta\, a^4(\eta')d\eta'\, \partial_\eta^{\#} K(k_1,\eta)\partial_\eta^{\#} K(k_2,\eta)\partial_\eta^{\#} K(k_3,\eta')\partial_\eta^{\#} K(k_4,\eta')$$

$$\times F_{\mathrm{L}}(\mathbf{k}_1,\mathbf{k}_2)\, F_{\mathrm{R}}(\mathbf{k}_3,\mathbf{k}_4)\, G(s,\eta,\eta')$$



$$\psi_4 = \int_{-\infty(1-i\epsilon)}^{0} a^4(\eta)\, d\eta\, a^4(\eta')d\eta'\, \partial_\eta^\# K(k_1,\eta)\partial_\eta^\# K(k_2,\eta)\partial_\eta^\# K(k_3,\eta')\partial_\eta^\# K(k_4,\eta')$$

$$\times F_{\rm L}(\mathbf{k}_1,\mathbf{k}_2)\, F_{\rm R}(\mathbf{k}_3,\mathbf{k}_4)\, G(s,\eta,\eta')$$

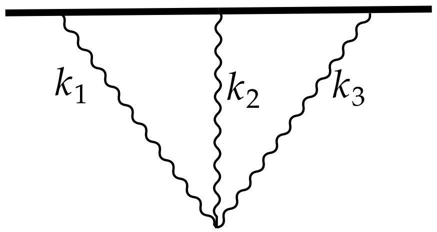
spatial derivatives

$$K(k,\eta) = \frac{\phi_k^+(\eta)}{\phi_k^+(\eta_0)}$$

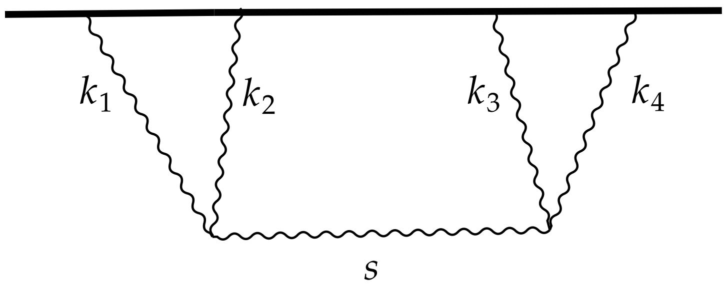
$$G(s,\eta,\eta') = i\left(\phi_s^-(\eta)\phi_s^+(\eta')\theta(\eta-\eta') + \eta \leftrightarrow \eta'\right) - i\frac{\phi_s^-(\eta_0)}{\phi_s^+(\eta_0)}\phi_s^+(\eta)\phi_s^+(\eta')$$

- Toy model: ϕ^3 in flat space (the wave function of the ground state in Minkowski)

contact diagram



Single exchange diagram



$$S = \int d\eta \, d^3\mathbf{x} \left(-\frac{1}{2}(\partial_\mu \phi)^2 - \lambda \phi^3 \right)$$

positive frequency mode function $\phi_k^+(\eta) = \frac{1}{\sqrt{2k}} e^{ik\eta}$

$$\psi_3(k_1, k_2, k_3) = i3!\lambda \int_{-\infty(1-i\epsilon)} d\eta \, e^{i(k_1+k_2+k_3)\eta} = \frac{6\lambda}{k_1+k_2+k_3} \, .$$

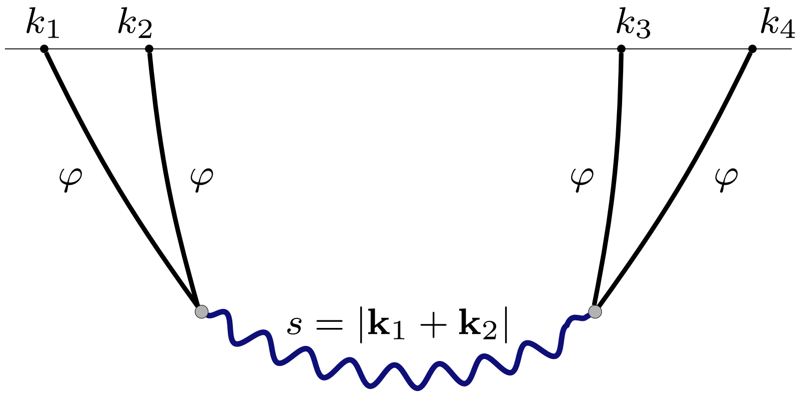
$$\psi_4 = \frac{-36\lambda^2}{(k_1+k_2+k_3+k_4)(k_1+k_2+s)(k_3+k_4+s)}$$

II. Analyticity (for Bunch-Davis)

- The bulk integral representation of the wave function coefficient defines an analytic function on the lower-half complex plane of external energies.

$$\lim_{\eta \to -\infty} K(k, \eta) \propto \exp(+ik\eta) \qquad \mathrm{Im}(k) < 0$$

- The only allowed singularities (at tree-level) for each diagram are when the total energy of the graph or any of its subgraphs goes to zero. The residue of these singularities are related to flat-space amplitudes Raju 2012



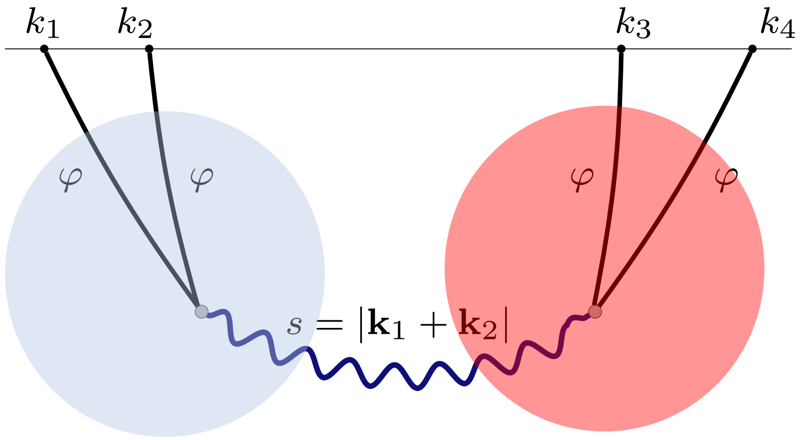
$$E_T = k_1 + k_2 + k_3 + k_4 \to 0$$

II. Analyticity (for Bunch-Davis)

- The bulk integral representation of the wave function coefficient defines an analytic function on the lower-half complex plane of external energies.

$$\lim_{\eta \to -\infty} K(k,\eta) \propto \exp(+ik\eta) \qquad\qquad \mathrm{Im}(k) < 0$$

- The only allowed singularities (at tree-level) for each diagram are when the total energy of the graph or any of its subgraphs goes to zero. The residue of these singularities are related to flat-space amplitudes Raju 2012

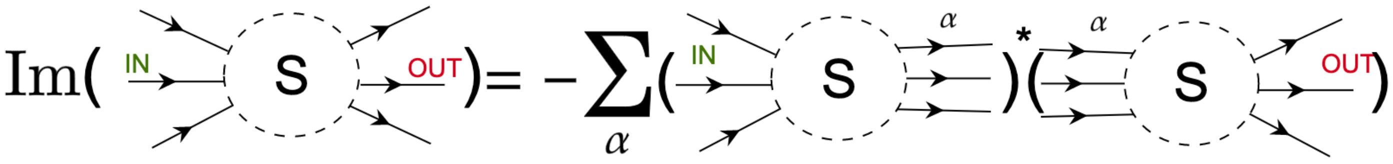


$$E_T = k_1 + k_2 + k_3 + k_4 \to 0$$
$$E_L = k_1 + k_2 + s \to 0$$
$$E_R = k_3 + k_4 + s \to 0$$

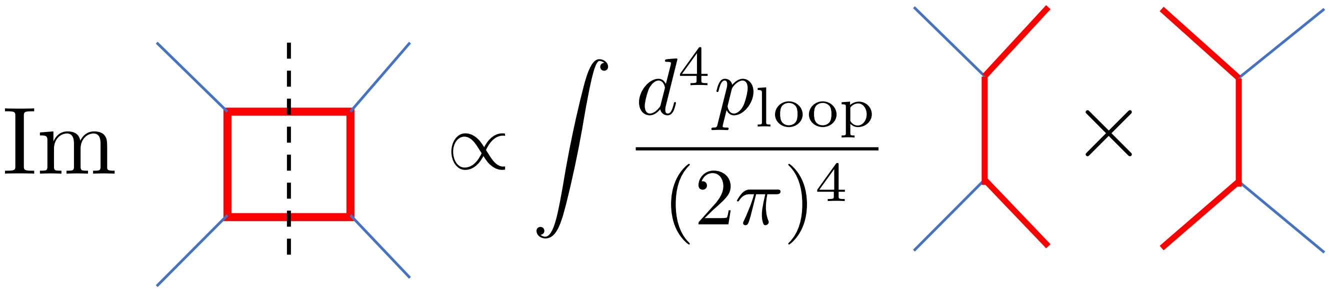
Also see R. Porto, D. Green 2020

III. Unitarity

- For scattering amplitudes, unitarity is encoded in the non-perturbative Optical Theorem:



- In perturbation theory, the optical theorem is the consequence of Cutkosky rules



Underpinning principles: **reality of the couplings** + $\mathrm{Im}\frac{1}{p^2 - m^2 + i\,0^+} = -\pi\delta(p^2 - m^2)$

III. Unitarity

- Non-perturbative optical theorem for the wave function

$$U^\dagger(t)U(t) = 1 \Rightarrow \Psi_{\text{boundary}}\,?$$

- In perturbation theory still we have

$g^* = g$ reality of the couplings

$K(k,\eta) = K^*(e^{-i\pi}k,\eta)$ Hermitian analyticity

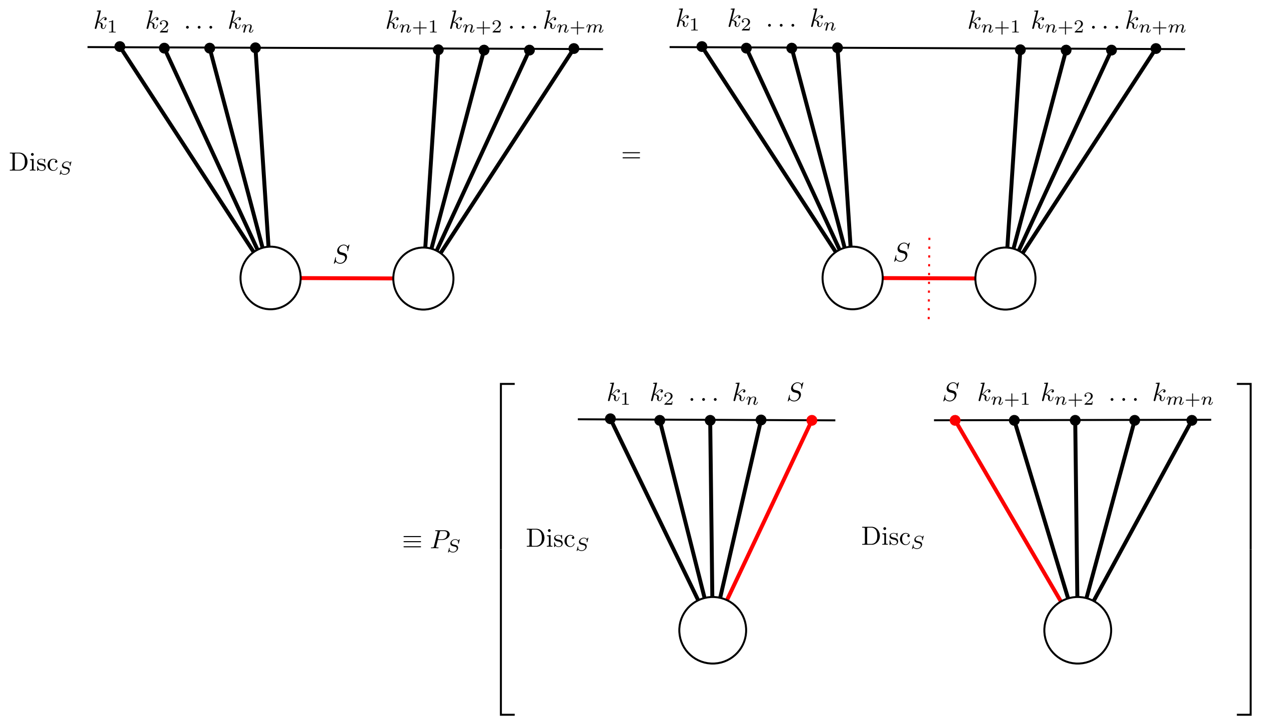
$\text{Im}\,G(s,\eta,\eta') = 2P(s,\eta_0)\text{Im}K(s,\eta)\text{Im}K(s,\eta')$ Image-factorization

Goodhew, SJ, Pajer 2020
Goodhew, SJ, Pajer, Lee 2021
Meltzer 2020

III. Unitarity

- These properties lead to a set of cutting rules for correlators similar to cutkosky rules in flat space. For example, for a tree-level exchange diagram one finds the following single-cut rule:

$$\psi_4(k_1, k_2, k_3, k_4, s) + \psi_4^*(-k_1, -k_2, -k_3, -k_4, s) =$$
$$2P_s\ (\psi_3(k_1, k_2, s) + \psi_3^*(-k_1, -k_2, s))\ (\psi_3(k_3, k_4, s) + \psi_3^*(-k_3, -k_4, s))$$



III. Unitarity

- These properties lead to a set of cutting rules for correlators similar to cutkosky rules in flat space. For example, for a tree-level exchange diagram one finds the following single-cut rule:

$$\psi_4(k_1,k_2,k_3,k_4,s)+\psi_4^*(-k_1,-k_2,-k_3,-k_4,s)=$$
$$2P_s\,\left(\psi_3(k_1,k_2,s)+\psi_3^*(-k_1,-k_2,s)\right)\,\left(\psi_3(k_3,k_4,s)+\psi_3^*(-k_3,-k_4,s)\right)$$

Valid under very general assumptions about free theory:
- Bunch davis initial condition
- Accelerating FLRW background No dS symmetry needed
- Any mass and spin

IV. Locality

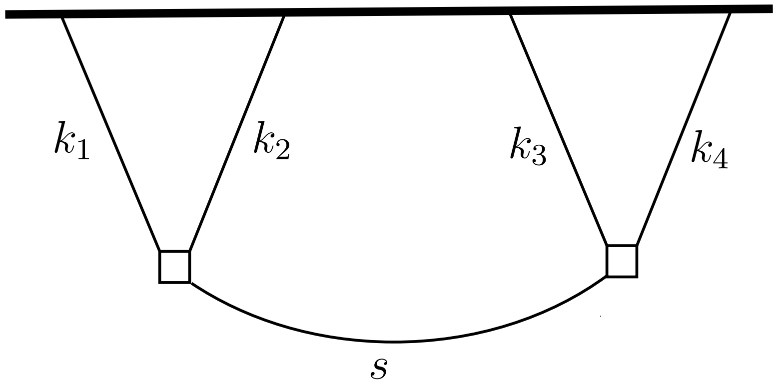
- Non-perturbative statement?

$$[\phi(\eta_1, \mathbf{x}_1), \phi(\eta_2, \mathbf{x}_2)] = 0 \ \ \text{(spacelike pairs)} \Rightarrow \Psi_{\text{boundary}}\,?$$

- In PT: only a strong version of locality and for massless fields

SJ, Pajer, Stefanyszyn 2021

$$\mathcal{L}_I = \sum_{n,m} \partial_\mu^n \, \phi^m \,, n \geq 0$$

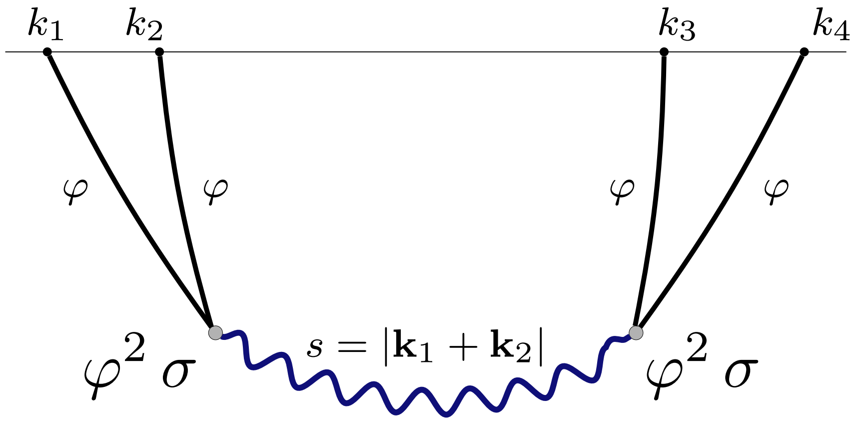


massless exchange

$$\psi_4(k_1, k_2, k_3, k_4, s) \text{ finite at } \ s = 0$$

IV. Locality

- For massive-exchange diagrams: a boundary differential equation for the four-point function. For example,



external legs: $m_\varphi^2 = 2H^2$ $\qquad \varphi_+ = \frac{H}{\sqrt{2k}}\eta\exp(ik\eta)$

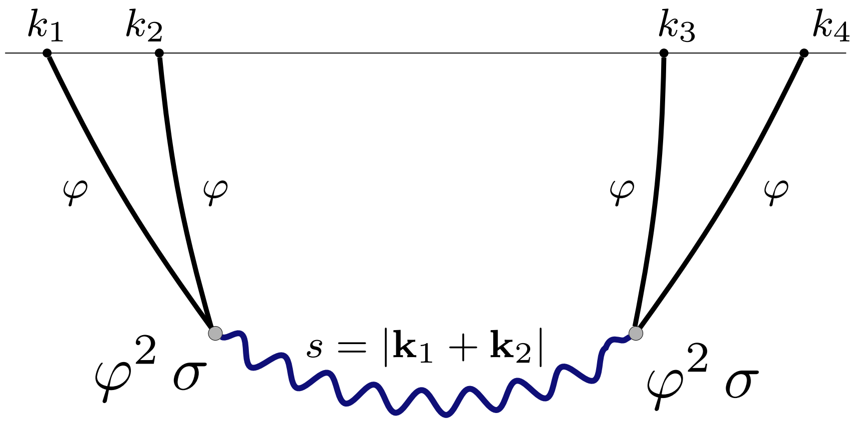
Internal line: $\mu = \sqrt{\frac{m^2}{H^2} - \frac{9}{4}}$ $\qquad \sigma_+ = \frac{\sqrt{\pi}H}{2} e^{-\pi(\mu/2 - i/4)} \times (-\eta)^{3/2} H_{i\mu}^{(1)}(-s\eta)$

$$\psi_4(k_1, k_2, k_3, k_4, s) = -\frac{4ig^2}{\eta_0^4}\int \frac{d\eta}{\eta^2}\frac{d\eta'}{\eta'^2} e^{i(k_1+k_2)\eta} e^{i(k_3+k_4)\eta'} G(s, \eta, \eta')$$

dS scale invariance: $\psi_4 = -\frac{4g^2}{\eta_0^4 s} F\left(u = \frac{s}{k_1 + k_2}, v = \frac{s}{k_3 + k_4}\right)$

IV. Locality

- For massive-exchange diagrams: a boundary differential equation for the four-point function. For example,



$$\mathcal{O}_p(\eta)G_p(\eta,\eta') = \delta(\eta-\eta')$$

$$\mathcal{O}_k(\eta)K_k(\eta) = 0$$

⇨

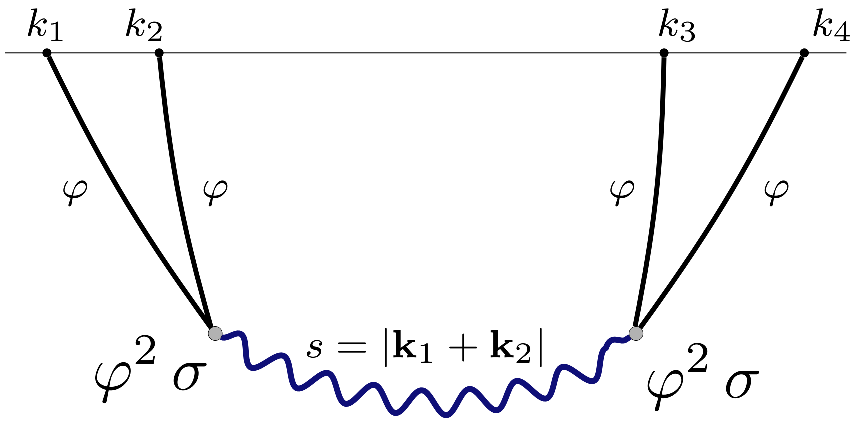
$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + (\mu^2 + \frac{1}{4})\right] F(u,v) = \frac{g^2 uv}{u+v}$$

Bulk Local EOM's

Boundary differential equation

IV. Locality

- For massive-exchange diagrams: a boundary differential equation for the four-point function. For example,



$$\mathcal{O}_p(\eta) G_p(\eta, \eta') = \delta(\eta - \eta')$$

$$\mathcal{O}_k(\eta) K_k(\eta) = 0$$

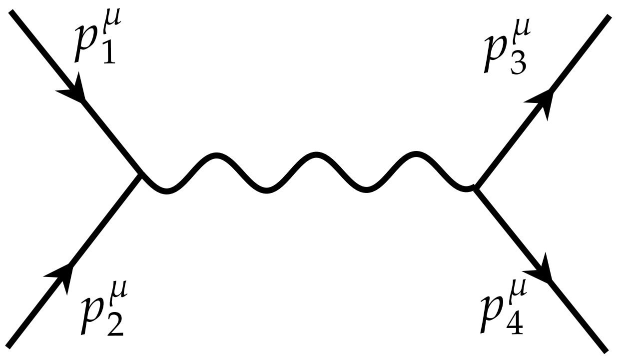
$$\Rightarrow \quad \left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + (\mu^2 + \frac{1}{4})\right] F(u,v) = \frac{g^2 uv}{u+v}$$

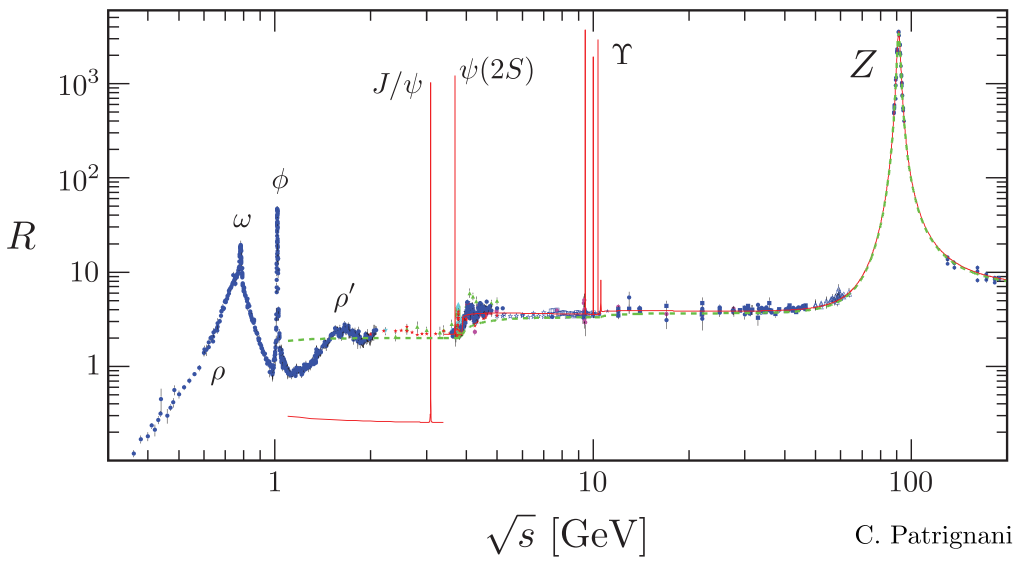
Bulk Local EOM's

Boundary differential equation

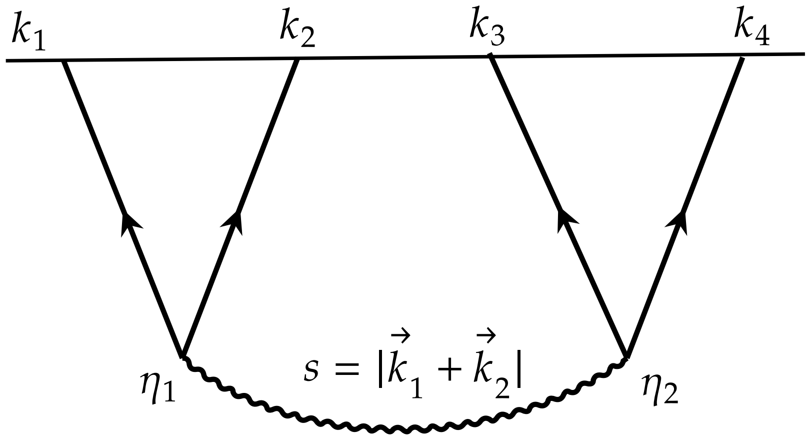
+ a similar PDE with $u, \partial_u \to v, \partial_v$

Cosmological Phonon Collider

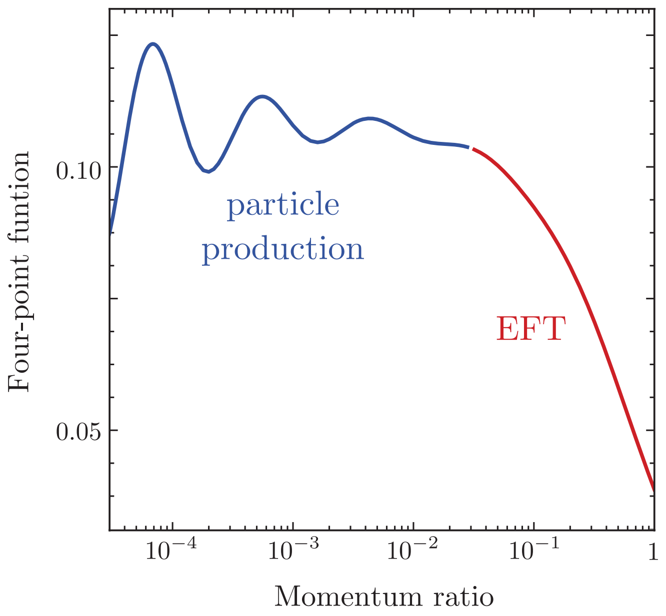




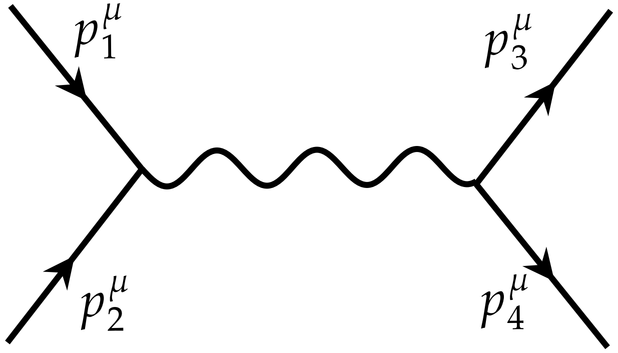
C. Patrignani et al

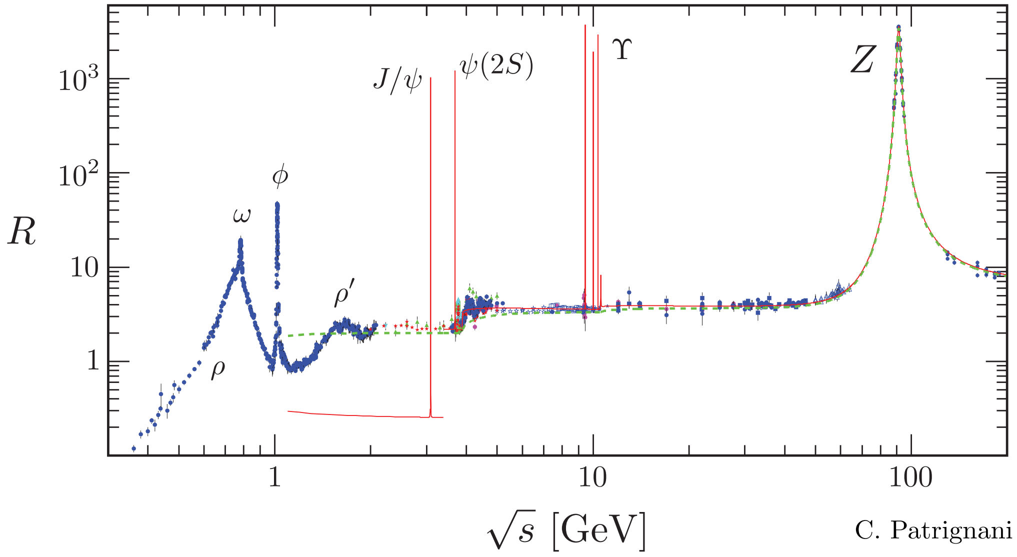


$$m \sim H \leq 10^{14}\,\text{Gev}$$

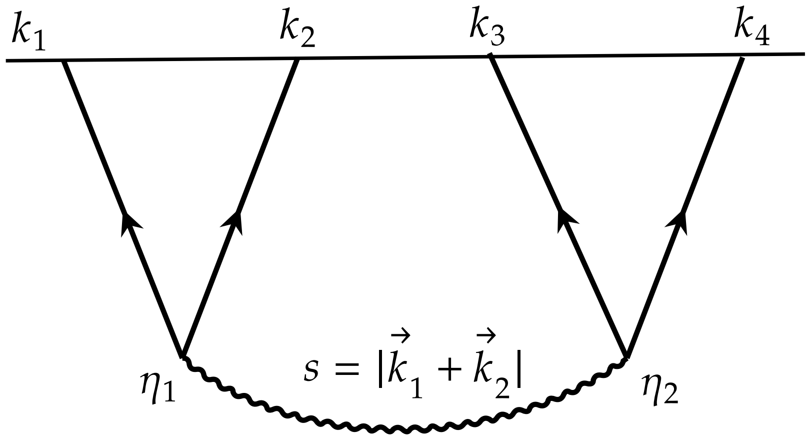


courtesy: Arkani-Hamed et al



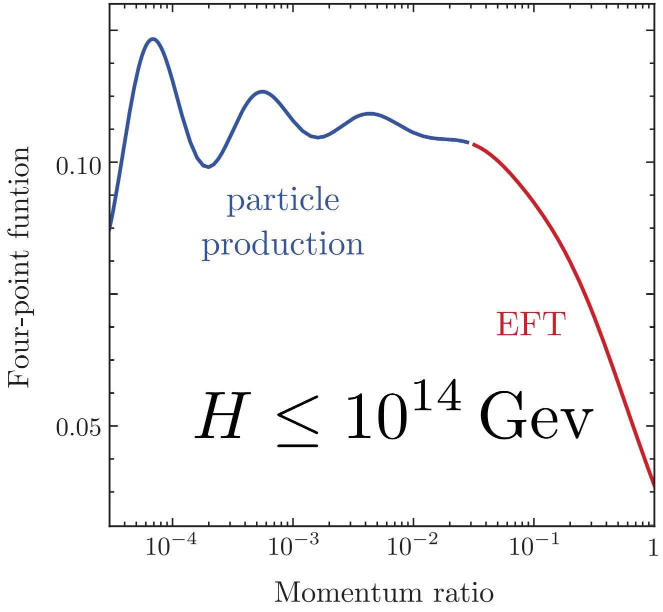


C. Patrignani et al



Cosmological Collider Physics

Chen, Wang 2009, Baumann Green 2011/ Noumi,Yamaguchi, Yokohama 2012,
Arkani-Hamed, Maldacena 2015/ Lee, Bauman, Pimentel 2016/
Arkani-Hamed, Baumann, Lee, Pimentel 2018 /+ many works



courtesy: Arkani-Hamed et al

Cosmological phonon collider

- Inflation can be seen as a phase of matter in which the time translation symmetry is spontaneously broken. The fluctuations around the vaccum can be described with a **Goldstone boson** that non-linearly realizes the broken time diffeomorphism

$$\phi = t + \pi(t, \mathbf{x}) \qquad \zeta \sim -H\pi$$

$$S_\pi = \int d\eta\, d^3\mathbf{x}\, a^2 \epsilon H^2 M_{\rm Pl}^2 \left[\frac{1}{c_s^2} \left(\pi'^2 - c_s^2 (\partial_i \pi)^2 \right) - \frac{1}{a} \left(\frac{1}{c_s^2} - 1 \right) \left(\pi' (\partial_i \pi)^2 + \frac{A}{c_s^2} \pi'^3 \right) + \dots \right]$$

(speed of sound) (large boost breaking interactions)

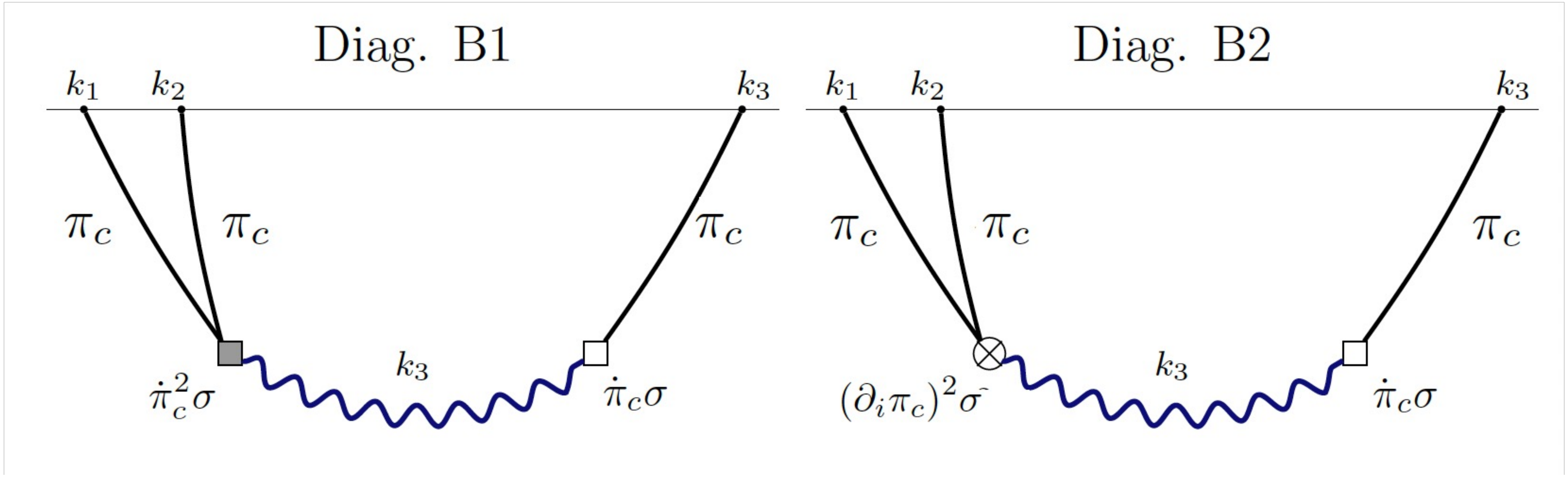
Cheung et al 2007

$$S_\sigma^{(2)} = \int d\eta d^3\boldsymbol{x}\, a^2 \left(\frac{1}{2}\sigma'^2 - \frac{1}{2}(\partial_i \sigma)^2 - \frac{1}{2} m^2 a^2 \sigma^2 \right)$$

(unit sound speed)

Cosmological phonon collider

$$S_{\pi\sigma} = \int d\eta d^3\mathbf{x}\, a^2 \left(\rho a \pi_c' \sigma + \frac{1}{\Lambda_1} \pi_c'^2 \sigma + \frac{c_s^2}{\Lambda_2} (\partial_i \pi_c)^2 \sigma \right) \qquad \pi_c = \sqrt{2\epsilon} H M_{\rm Pl} c_s^{-1} \pi$$



Cosmological phonon collider

$$S_{\pi\sigma} = \int d\eta d^3\mathbf{x}\, a^2 \left(\rho a \pi_c' \sigma + \frac{1}{\Lambda_1} \pi_c'^2 \sigma + \frac{c_s^2}{\Lambda_2} (\partial_i \pi_c)^2 \sigma \right) \qquad \pi_c = \sqrt{2\epsilon} H M_{\mathrm{Pl}} c_s^{-1} \pi$$

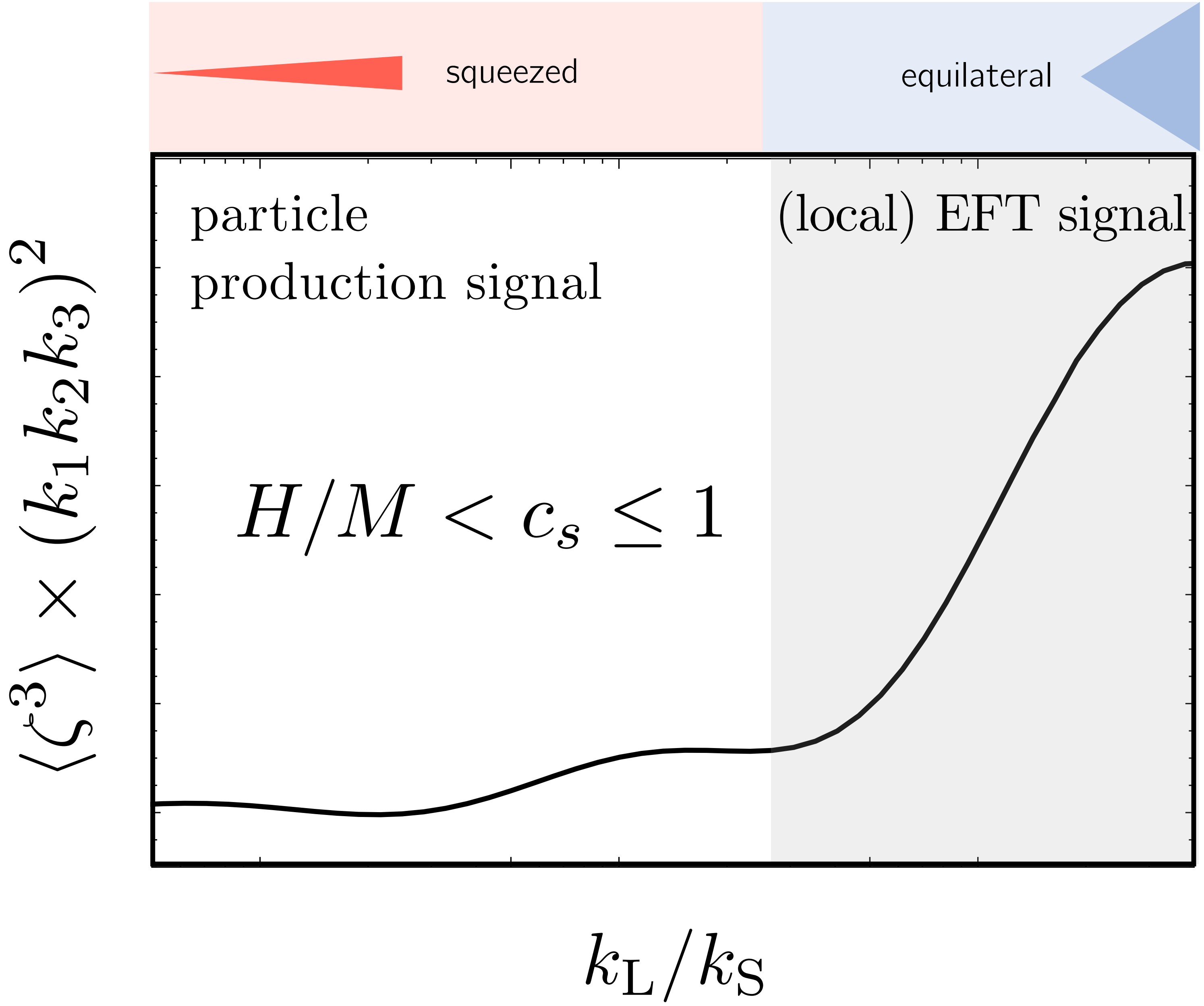
$$E = c_s |\mathbf{k}|$$

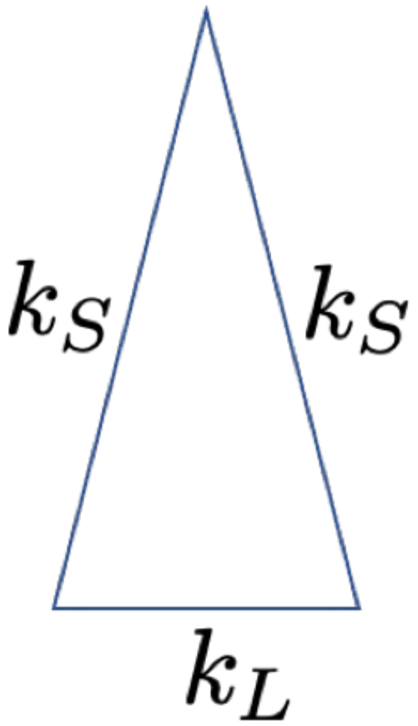
modified dispersion relation

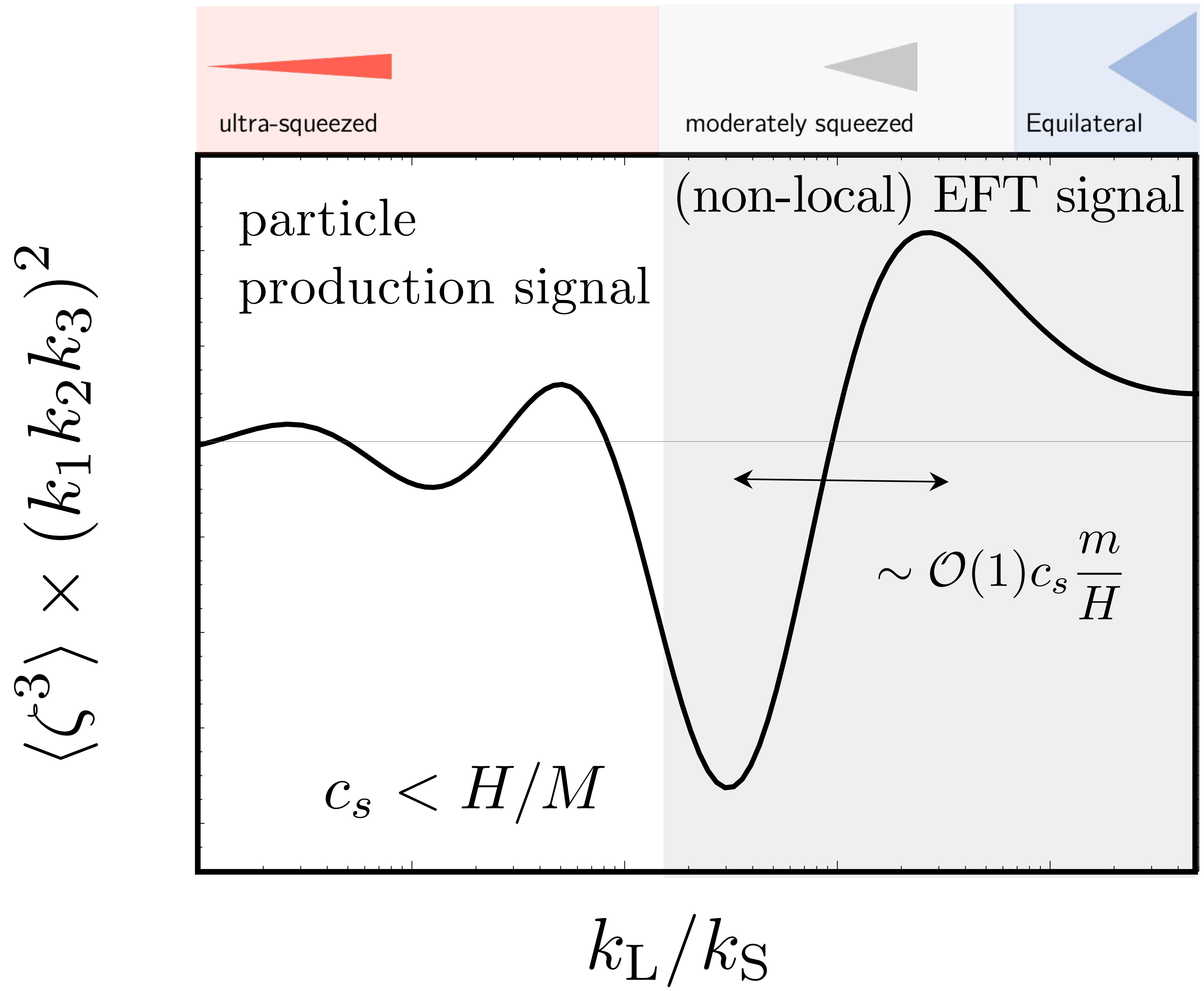
mode function for free fields

$$\pi_c^{\pm}(k,\eta) = \frac{iH}{\sqrt{2c_s^3 k^3}} (1 \pm i c_s k\eta) \exp(\mp i c_s k \eta)\,,$$

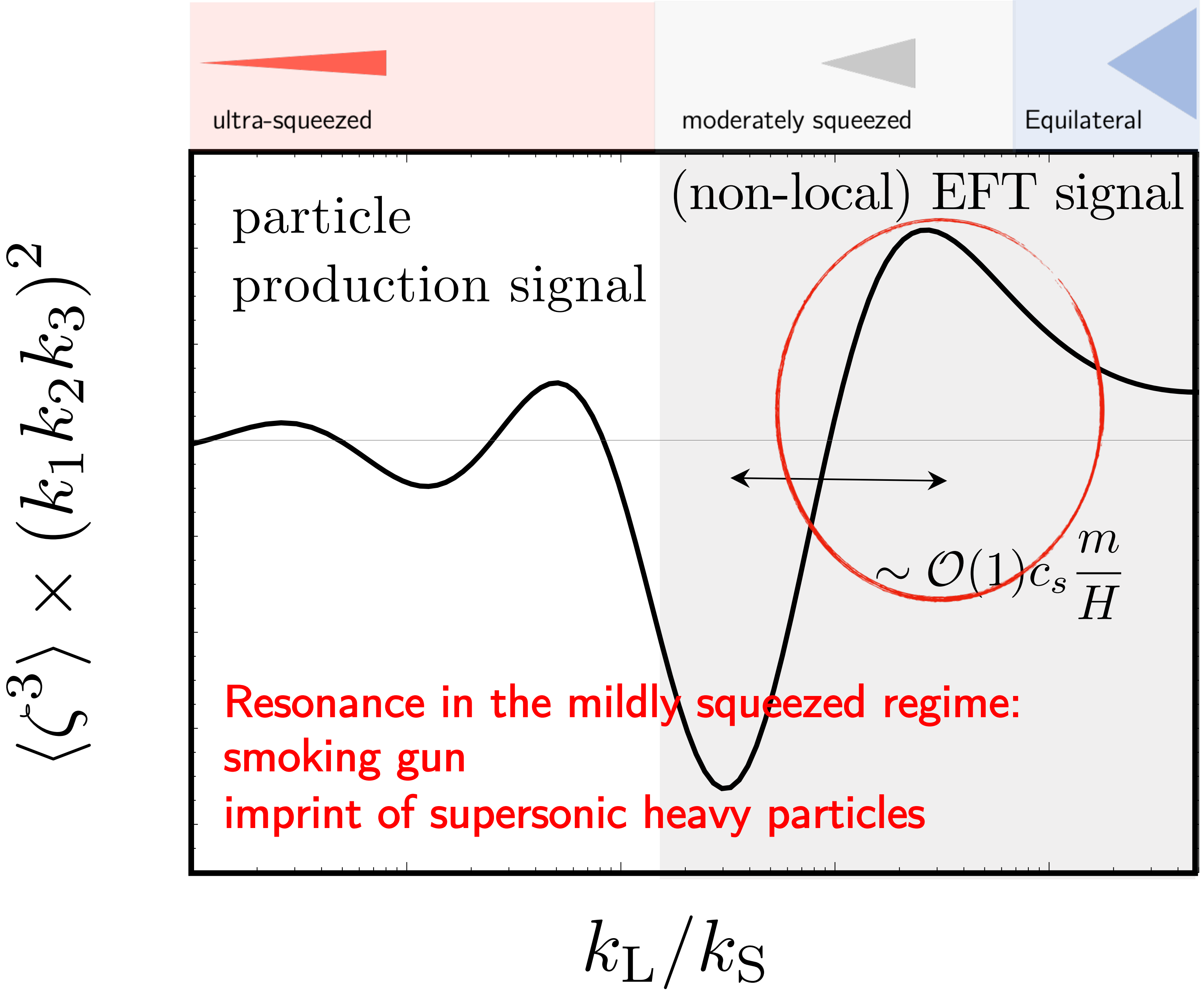
$$\sigma_+(k,\eta) = \frac{\sqrt{\pi} H}{2} \exp(-\pi\mu/2)\, \exp(i\pi/4) (-\eta)^{3/2} H_{i\mu}^{(1)}(-k\eta)$$


squeezed
equilateral
particle production signal
(local) EFT signal
$H/M < c_s \leq 1$
$\langle \zeta^3 \rangle \times (k_1 k_2 k_3)^2$
k_L / k_S


k_S
k_S
k_L



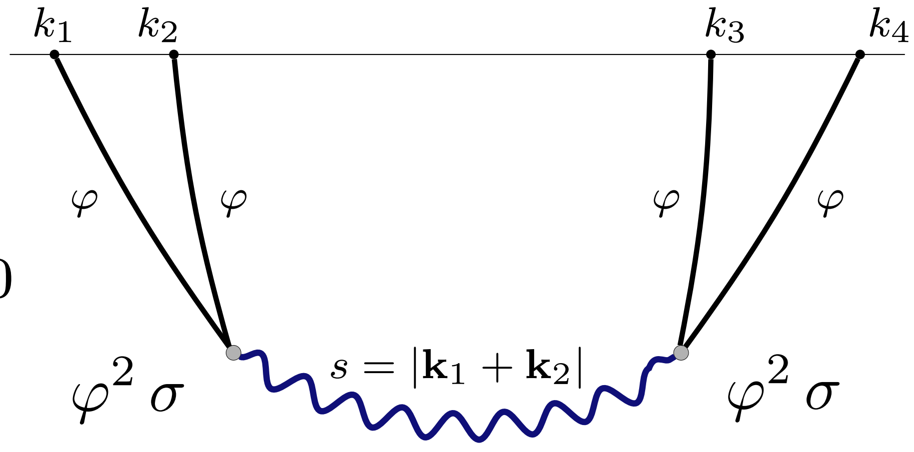
ultra-squeezed
moderately squeezed
Equilateral
particle
production signal
(non-local) EFT signal
$\sim \mathcal{O}(1) c_s \frac{m}{H}$
$c_s < H/M$
$\langle \zeta^3 \rangle \times (k_1 k_2 k_3)^2$
$k_\mathrm{L}/k_\mathrm{S}$



ultra-squeezed
moderately squeezed
Equilateral
particle production signal
(non-local) EFT signal
$\sim \mathcal{O}(1) c_s \frac{m}{H}$
Resonance in the mildly squeezed regime:
smoking gun
imprint of supersonic heavy particles
$\langle \zeta^3 \rangle \times (k_1 k_2 k_3)^2$
$k_{\rm L}/k_{\rm S}$

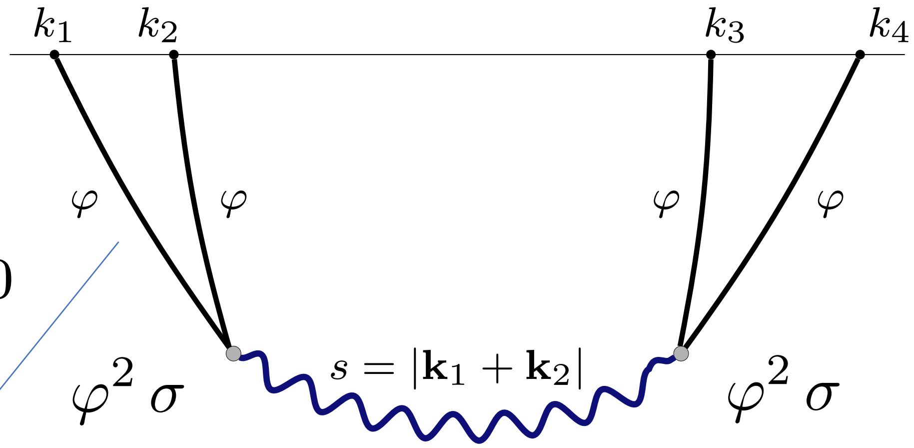
- A major simplification occurs in that, based on Feynman rules for the individual diagrams, one can see that the single exchange diagrams for massless field can be related to the de Sitter invariant four-point function of a conformally coupled field

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = \hat{W}(k_1, k_2, k_3, \partial_{k_i}) \lim_{k_4\to 0}$$



- A major simplification occurs in that, based on Feynman rules for the individual diagrams, one can see that the single exchange diagrams for massless field can be related to the de Sitter invariant four-point function of a conformally coupled field

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = \hat{W}(k_1, k_2, k_3, \partial_{k_i}) \lim_{k_4\to 0}$$

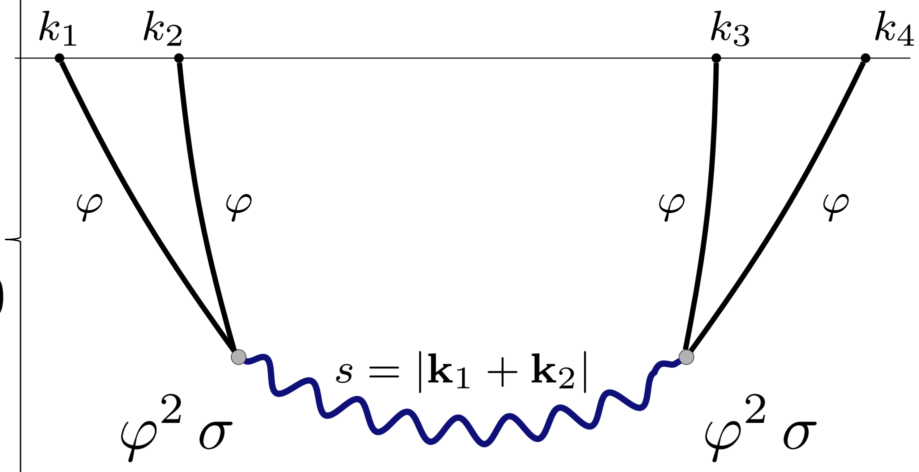


$$\varphi_\pm(k,\eta) = -\frac{H}{\sqrt{2k}}\eta\,\exp(\mp ik\eta)$$

(relativistic dispersion relation)

- The breaking of boost manifests itself both in the weight-shifting operators (boost breaking vertices) and also in the argument of the four-point function (different speeds of propagation)

$$\hat{W}(k_1, k_2, k_3, \partial_{k_i}) \lim_{k_4 \to 0} \left[\quad \right]$$

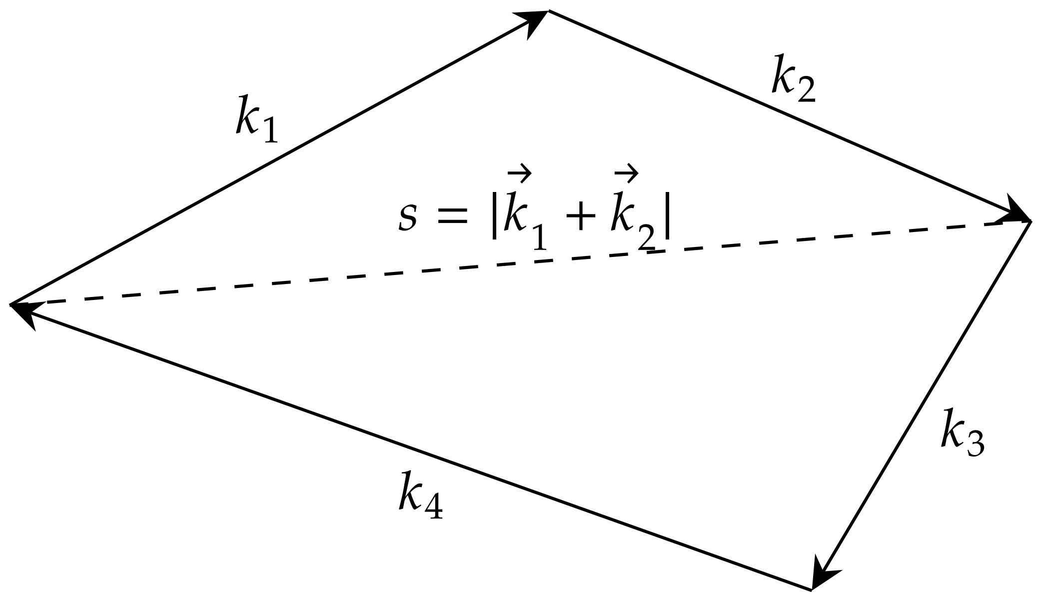


$$k_i (i = 1, 2, 3) \to c_s k_i$$

- Therefore, the seed four-point function should be anaytically continued beyond the region allowed by triangle inequality. This fact hugely complicates the computation

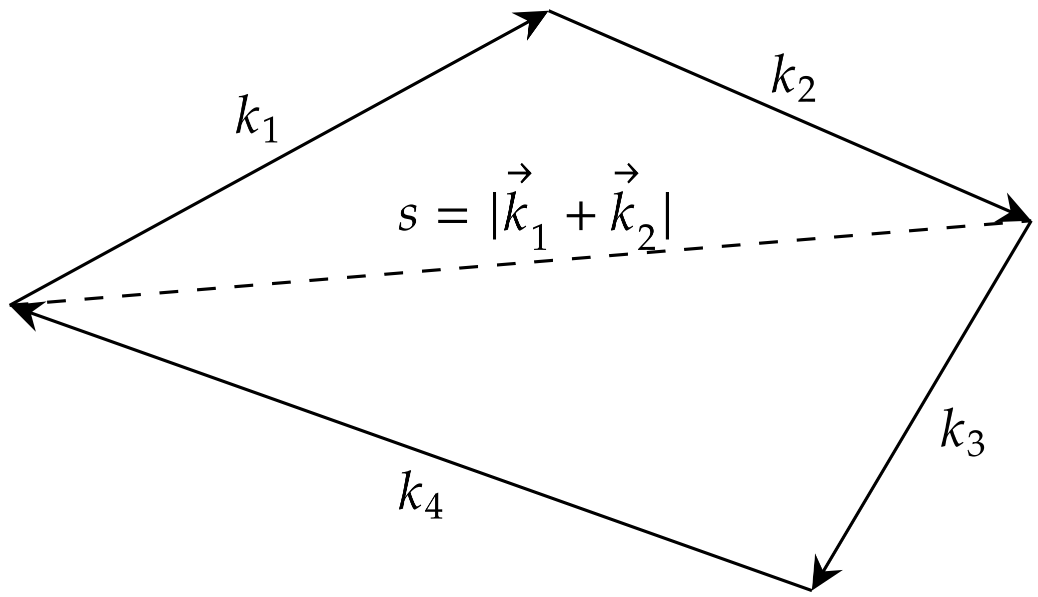
$$u = \frac{s}{k_1 + k_2} \leq 1$$

$$v = \frac{s}{k_3 + k_4} \leq 1$$



- Therefore, the seed four-point function should be anaytically continued beyond the region allowed by triangle inequality. This fact hugely complicates the computation

$$\begin{cases} k_1, k_2, k_3 \to c_s k_1, c_s k_2, c_s k_3 \\ k_4 \to 0 \\ s = |\mathbf{k}_4 + \mathbf{k}_3| \to 0 \end{cases}$$



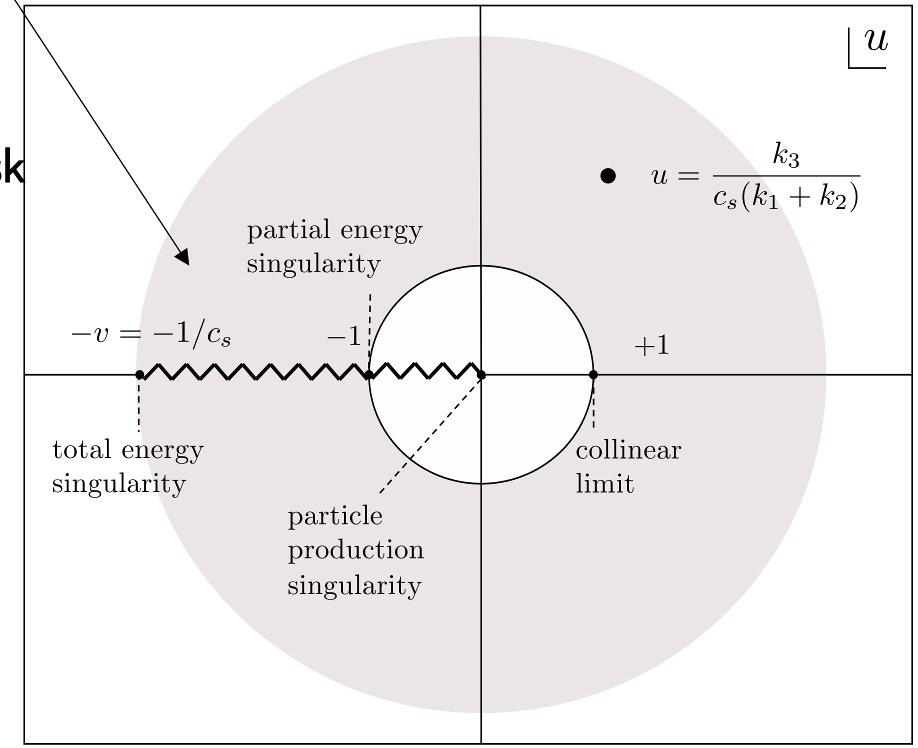
$$u \to \frac{k_3}{c_s(k_1 + k_2)}, v \to \frac{1}{c_s}$$

u and v might be outside the unit disk

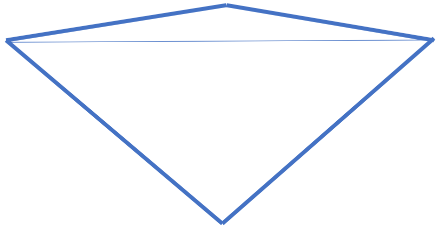
$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + (\mu^2 + \frac{1}{4})\right] F(u,v) = \frac{g^2 uv}{u+v}$$

$$\hat{F}(u,v) = \boxed{\sum_{m,n=0}^{\infty} (a_{mn} + b_{mn}\log(u))\, u^{-m} \left(\frac{u}{v}\right)^n} + \sum_{\pm\pm} \beta_{\pm\pm} f_\pm(u) f_\pm(v)$$

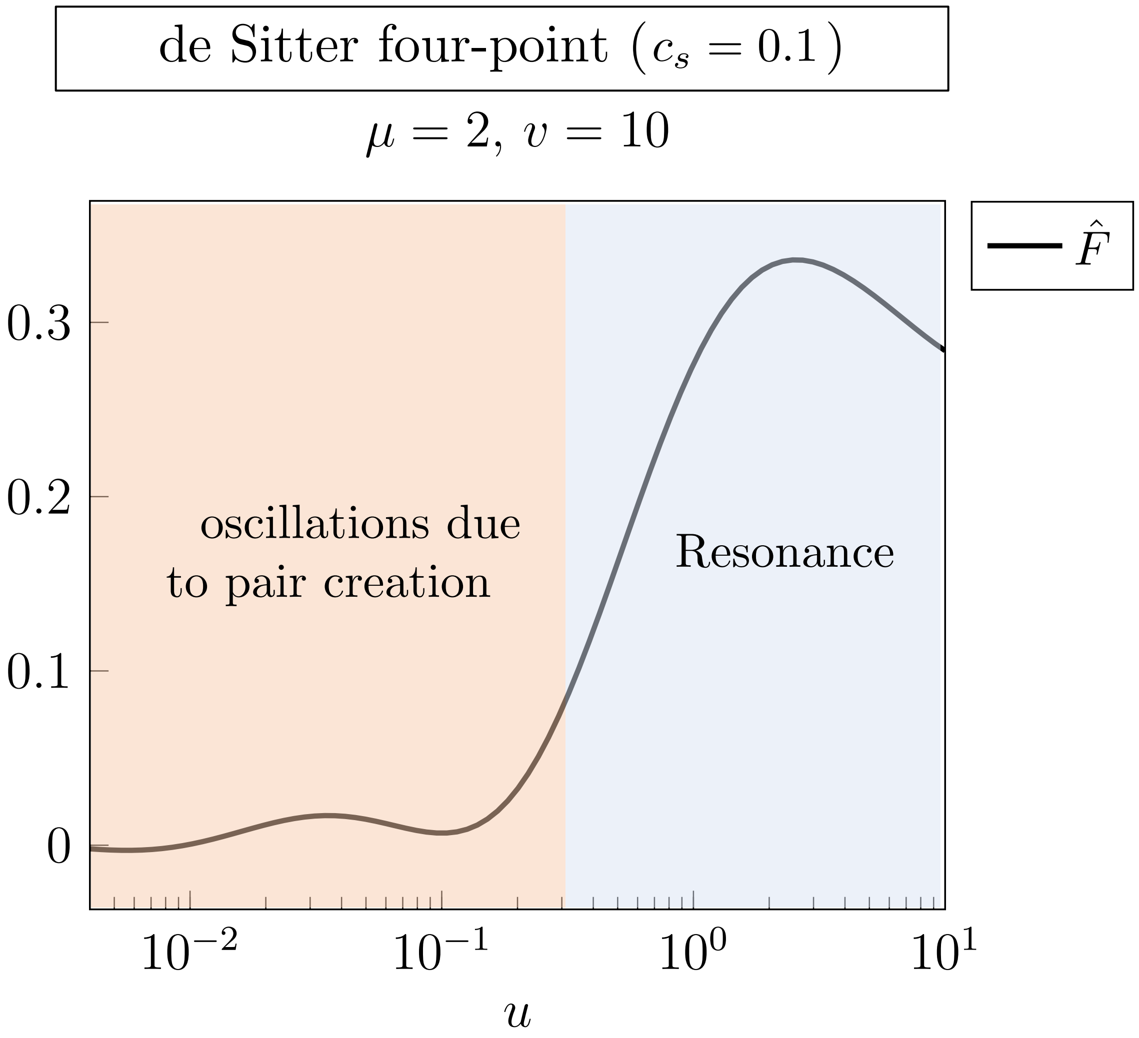
convergent
outside the unique disk

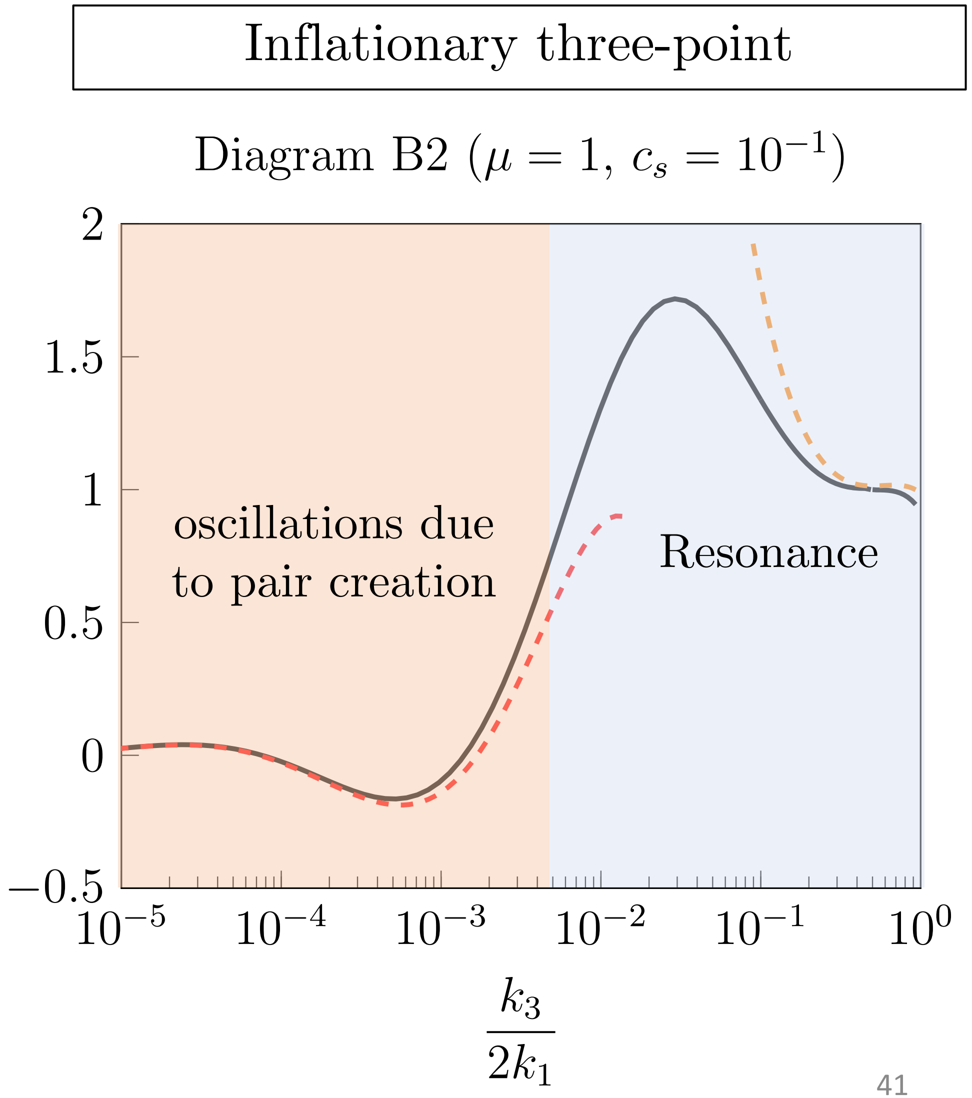


Fixed by the cutting rule and analyticity at u=1



• Low speed collider $m < H/c_s$





Final Remarks

- Cosmological bootstrap offers a powerful set of tools for computing cosmological correlators by shifting our focus from the bulk of spacetime to its boundary.
- On the theoretical frontier: (i) loop diagrams (ii)UV-IR relations at the level of correlators and positivity bounds for EFT operators in Cosmology (iii)non-perturbative bootstrap methods, etc.
- Further to Cosmological Phonon Collider: more general diagrams with multiple particle exchanges, potentially with larger non-Gaussianity, incorporating boost-bteaking massive spinning fields, etc.