Celestial Soft Algebras

Walker Melton

Harvard University

Jun 28th, 2023

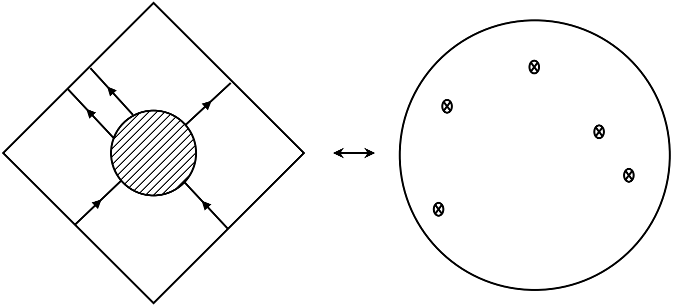
Outline

- Celestial Holography
- Celestial Soft Algebras for Gravity and Gauge Theory
- Dynamic and Central Extensions of Celestial Soft Algebras

Celestial Holography

Celestial Holography

Four-dimensional gravitational S-matrices have a holographic description as correlation functions on a 2D Euclidean sphere.



Celestial Holography

- We want examples where there is an independently defined 2D theory that computes 4D scattering amplitudes.
- In the absence of an independent boundary description, we would like to identify universal aspects of these dual theories starting from properties of gauge and gravitational amplitudes.
- We focus on a set of higher-spin currents that are present in the celestial dual due to universal low-energy features of guage and gravitational amplitudes.

Conformal Primary Wavefunction

A conformal primary wavefunction is a solution to the Klein-Gordon equation

$$\left(\partial^\mu \partial_\mu - m^2\right) \phi_{\Delta,m}(x^\mu | z, \bar{z}) \tag{1}$$

such that under Lorentz transformations

$$\phi_{\Delta,m}(\Lambda^\mu{}_\nu x^\nu | \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}) = |cw+d|^{2\Delta} \phi_{\Delta,m}(x^\mu | z, \bar{z}) \tag{2}$$

[a]

---

[a] Pasterski, S., Shao, S.H., Strominger, A. Flat space amplitudes and conformal symmetry of the celestial sphere.

Celestial Holography

Massless Conformal Primary Wavefunction

$$\phi^{\pm}_{\Delta}(x|z,\bar{z}) = \int d\omega \omega^{\Delta-1} e^{\pm i\omega \hat{q}(z,\bar{z})\cdot x}$$
$$\hat{q}^{\mu}(z,\bar{z}) = (1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}) \quad (3)$$

Celestial Amplitude

$$\mathcal{A}(1^{\epsilon_1,a_1}_{\Delta_1,J_1,z,\bar{z}} \cdots n^{\epsilon_n,a_n}_{\Delta_n,J_n,z_n,\bar{z}_n}) = \int \prod_j d\omega_j \omega_j^{\Delta_j-1} A^{a_1\cdots a_n}(\epsilon_1\omega_1 q(z_1,\bar{z}_1),\ldots)$$
$$= \langle \mathcal{O}^{\epsilon_1,a_1}_{\Delta_1,J_1}(z_1,\bar{z}_1)\cdots \mathcal{O}^{\epsilon_n,a_n}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle \quad (4)$$

Conformally Soft Theorems[1]

Energetically Soft Theorems

$$\lim_{\omega \to 0} \omega \langle \mathrm{out} | a_+(\omega \hat{q}) \mathcal{S} | \mathrm{in} \rangle = e \sum_k \frac{Q_k \cdot p_k \cdot \epsilon^+}{p_k \cdot \hat{q}} \langle \mathrm{out} | \mathcal{S} | \mathrm{in} \rangle + O(\omega) \tag{5}$$

Conformally Soft Theorems

$$\lim_{\Delta_1 \to 1} (\Delta_1 - 1) \mathcal{A}(1^+_{\Delta_1}, 2^{\epsilon_2}_{\Delta_2}, \ldots) = e \sum_k \frac{Q_k}{z_1 - z_k} \mathcal{A}(2^{\epsilon_2}_{\Delta_2}, \ldots) \tag{6}$$

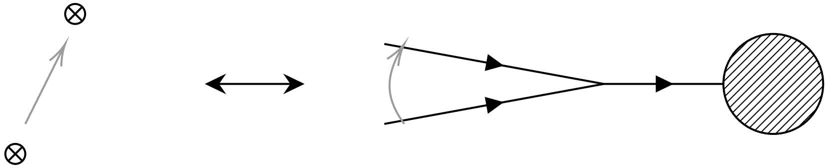
[1] Pate, M., Raclariu, A.M., Strominger, A. Conformally soft theorem in gauge theory. Phys. Rev. D 100, (2019).

Conformally Soft Theorems

Conformally Soft Gluons and Gravitons

$$
\begin{aligned}
R^{k,a} &= \lim_{\epsilon\to 0} \epsilon \mathcal{O}^{+a}_{k+\epsilon}(z,\bar{z}), k = 1, 0, \ldots \\
H^{\ell} &= \lim_{\epsilon\to 0} \epsilon \mathcal{G}^{+}_{\ell+\epsilon}(z,\bar{z}), k = 2, 1, 0, -1, \ldots
\end{aligned}
\tag{7}
$$

Celestial OPE



Operators becoming coincident $\leftrightarrow$ Particles becoming collinear

Celestial OPE[2]

Hard OPE

$$\mathcal{O}^{+a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{+b}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{if^{ab}{}_c}{z_{12}}\sum_{m=0}^{\infty} B(\Delta_1+m-1,\Delta_2-1)\frac{\bar{z}_{12}^m}{m!} \bar{\partial}^m \mathcal{O}^{+c}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2)+\mathcal{O}(z_{12}^0)$$

$$\mathcal{G}^{+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{G}^{+}_{\Delta_2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\sum_{m=0}^{\infty} B(\Delta_1+m-1,\Delta_2-1)\frac{\bar{z}_{12}^m}{m!} \times \bar{\partial}^m \mathcal{G}^{+}_{\Delta_1+\Delta_2}(z_2,\bar{z}_2)+\mathcal{O}(z_{12}^0) \quad (8)$$

[2] Pate, M., Raclariu, A.M., Strominger, A., Yuan, E.Y. Celestial Operator Products of Gluons and Gravitons. Reviews in Mathematical Physics, 33, no 9 (2021).

Celestial OPE

Soft OPE

$$\begin{aligned} R^{k,a} R^{\ell,b} &\sim -\frac{i f^{ab}_{\ \ c}}{z_{12}} \sum_{j=0}^{1-k} \frac{(2-k-\ell-j)!}{(1-k-j)!(1-\ell)!} \frac{\bar{z}_{12}^{j}}{j!} \bar{\partial}^{j} R^{k+l-1,c} \\ H^{k} H^{\ell} &\sim -\frac{\kappa}{2} \frac{\bar{z}_{12}}{z_{12}} \sum_{j=0}^{1-k} \frac{(2-k-\ell-j)!}{(1-k-j)!(1-\ell)!} \frac{\bar{z}_{12}^{j}}{j!} \bar{\partial}^{j} H^{k+l} \end{aligned} \tag{9}$$

Celestial Soft Algebras[3]

Modes of Soft Operators

$$\begin{aligned} R^{3-2q,a} &= \sum_n \sum_{m=1-q}^{q-1} \frac{s^{q,a}_{n,m}}{(q+m-1)!(q-m-1)!z^{n+2-q}\bar{z}^{m+1-q}} \\ H^{4-2q} &= \sum_n \sum_{m=1-q}^{q-1} \frac{\kappa w^{q}_{n,m}}{(q+m-1)!(q-m-1)!z^{n+3-q}\bar{z}^{m+1-q}} \end{aligned} \tag{10}$$

Gauge and Gravitational Soft Algebras

$$\begin{aligned} [s^{q,a}_{n,m}, s^{p,b}_{n',m'}] &= -if^{ab}_{\ \ c} s^{p+q-1,c}_{n+n',m+m'} \\ [w^{p}_{n,m}, w^{q}_{n',m'}] &= [m(q-1) - m'(p-1)] w^{p+q-2}_{n+n',m+m'} \end{aligned} \tag{11}$$

[3] Melton, W., Narayanan, S.A., Strominger, A. Deforming soft algebras for gauge theory. J. High Energ. Phys. 2023, 233 (2023).

Extensions of Gauge Soft Algebras

Yang-Mills and Neutral Scalar Theory

$$S = \int d^4x \left(-\frac{1}{4} \mathrm{Tr}\ F^2 - \partial^\mu \bar{\phi} \partial_\mu \phi - \frac{\mu}{4} (\phi \mathrm{Tr}\ F_+^2 + \bar{\phi} \mathrm{Tr}\ F_-^2) \right) \qquad (12)$$

Extended Soft Algebra

$$\begin{aligned} [s^{q,a}_{n,m}, s^{r,b}_{n',m'}] &= -if^{ab}{}_c s^{q+r-1,c}_{n+n',m+m'} \\ &\quad - \mu \delta^{ab} (m'(q-1) - m(r-1)) \bar{\sigma}^{q+r-2}_{n+n',m+m'} \\ [s^{q,a}_{n,m}, \sigma^{r}_{n',m'}] &= -\mu (m'(q-1) - m(r-1)) \bar{s}^{q+r-2,a}_{n+n',m+m'} \\ \lim_{\epsilon \to 0} \epsilon \bar{\Phi}^+_{2-2q+\epsilon} &= \sum_{m=1-q}^{q-1} \frac{\bar{\sigma}^q_{n,m}}{(q+m-1)!(q-m-1)! z^{n+1-q} \bar{z}^{m+1-q}} \end{aligned} \qquad (13)$$

Central Terms From Sourced Backgrounds

Soft Amplitudes in a Background

$$\begin{aligned} \delta S &= -\frac{\mu}{4}\int d^4x\phi_0(x)\mathrm{Tr}\ F^2 \\ \phi_0(p^2) &= \sum_j a_j(p^2)^j \\ \langle R^{k,a}(z_1,\bar{z}_1)R^{j,b}(z_2,\bar{z}_2)\rangle &= \frac{2\mu\delta^{kj}\delta^{ab}a_{-1-k}}{z_{12}^2|z_{12}|^{2(k-1)}} \end{aligned} \tag{14}$$

Central Extensions to Gauge Algebra

$$\begin{aligned} [s^{q,a}_{n,m}, s^{r,b}_{n',m'}] &= -if^{ab}{}_c s^{q+r-1,c}_{n+n',m+m'} \\ &\quad + (c_1\delta^{q1}n + c_{3/2}\delta^{q3/2}m)\delta^{pq}\delta_{n+n',0}\delta_{m+m',0} \\ c_1 \propto a_{-2},\ & c_{3/2} \propto a_{-1} \end{aligned} \tag{15}$$

Soft Algebras and Form Factors

- For certain local theories on twistor space, it is known that *conformal blocks* of the soft algebra are in a one-to-one correspondence with local operators of the theory.
- A conformal block is a set of correlation functions consistent with the OPE of the soft algebra.
- The conformal block gives bulk scattering amplitudes in the presence of a local operator in the bulk.[4]

[4]Costello, K., Paquette, N.M. Celestial holography meets twisted holography: 4d amplitudes from chiral correlators. J. High Energ. Phys. 2022, 193 (2022).

Sourced Backgrounds as Conformal Blocks

$$\int [D\phi][DA]e^{-S}(1+\alpha\phi(0))\mathcal{O}(\phi,A) \approx \int [D\phi][DA]e^{-S+\alpha\phi(0)}\mathcal{O}(\phi,A) \quad (16)$$

$$\Box\phi_0(x) = \alpha\delta^{(4)}(x) \implies \phi_0(p) = \frac{\alpha}{p^2} \quad (17)$$

$$\langle R^{k,a}R^{\ell,b}\rangle = \frac{2\mu\alpha\delta^{k0}\delta^{\ell0}\bar{z}_{12}}{z_{12}} \quad (18)$$

Sourced Backgrounds as Conformal Blocks

- For YM coupled to a canonical scalar, no operator deforms leading soft theorem.
- For YM coupled to scalar with higher-order kinetic term, ϕF^2 can deform the leading soft theorem.
- $\phi_0(p) \propto 1/p^4$ gives conformal block for $\phi(0)$:

$$\langle R^{k,a} R^{\ell,b} \rangle = \frac{2\mu\alpha\delta^{k1}\delta^{\ell1}}{z_{12}^2} \tag{19}$$

Conclusions

- Celestial amplitudes organize a tower of soft theorems into a simple algebra.
- Near universality of conformally soft theorems implies that these algebras should be present in explicit examples of celestial holography.
- Scattering amplitudes in the presence of sources or local operators inserted in the bulk are naturally described by conformal blocks of the soft algebra.