W algebras and ALE spaces

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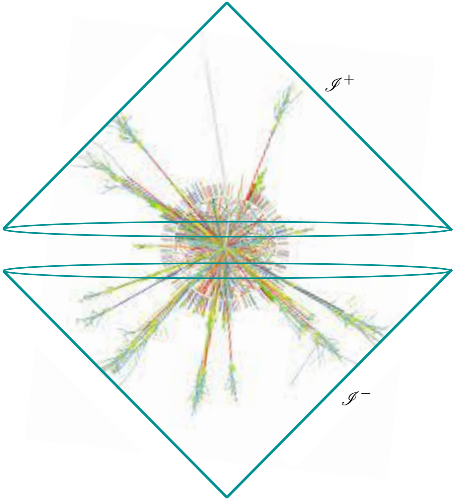
DAMTP, Cambridge

Nordita, 4 Jul 23

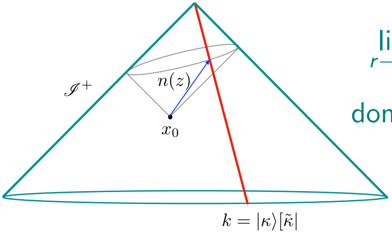
2208.13750 W. Bu, S. Heuveline & D.S.
2305.09451 + work in progress, R. Bittleston, S. Heuveline & D.S.

The S-matrix is perhaps the most natural observable of a gravitational theory in an asymptotically flat space-time



- diffeomorphism invariant
- for massless particles, initial & final states specified on $\mathscr{I}^\pm$
- 'naturally holographic'

In flat $\mathbb{R}^{1,3}$, we usually scatter momentum eigenstates $\sim e^{ik\cdot x} = e^{i\langle\kappa|x|\tilde{\kappa}]}$. These become localized on a generator as we approach $\mathscr{I}^+$



- let $x = x_0 + rn(z)$ with $n(z) = |\lambda(z)\rangle[\tilde{\lambda}(z)|$, then

$$\lim_{r\to\infty} e^{ik\cdot(x_0+rn(z))} = e^{ik\cdot x_0} \lim_{r\to\infty} e^{ir\langle\kappa\lambda(z)\rangle[\tilde{\kappa}\tilde{\lambda}(z)]}$$

dominated by points of stationary phase

$$\partial_z\langle\kappa\lambda(z)\rangle[\tilde{\kappa}\tilde{\lambda}(z)] = 0$$

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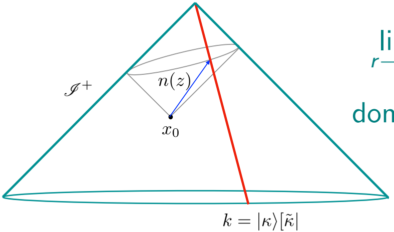

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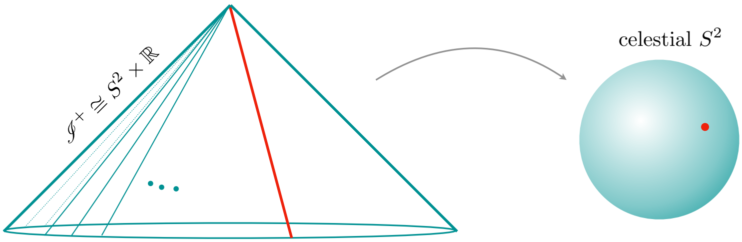
- implies $|\lambda(z)\rangle = |\kappa\rangle$ and $|\tilde{\lambda}(z)] = |\tilde{\kappa}]$ (Lorentzian)

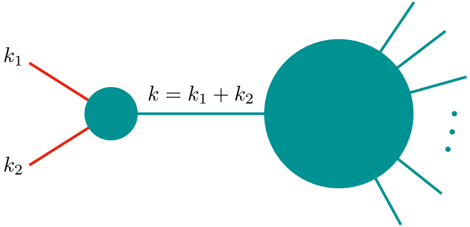


'Carrollian' 'Celestial'

- 3d Carrollian perspective more appropriate for dynamics on $\mathscr{I}$ (*eg* sequential bursts of gravitational radiation), but less well understood
- 2d celestial perspective closer to CFT; requires decomposing fields into modes along $\mathbb{R}$ direction (*eg* 'conformally soft' / Mellin modes)

Scattering amplitudes become singular when the momenta of two massless particles become collinear

$$\mathcal{A}_n(k_1, k_2, \ldots, k_n) \xrightarrow{1\|2} \text{Split} \times \mathcal{A}_{n-1}(k, k_3, \ldots, k_n)$$

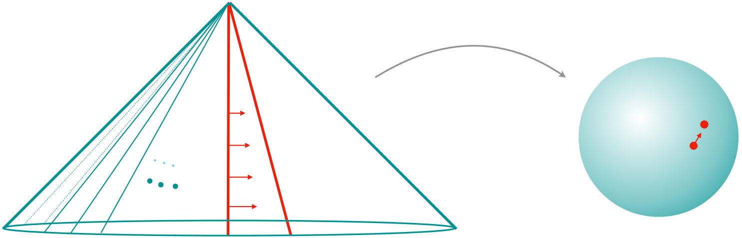


The splitting functions in Yang-Mills theory and gravity are well known

$$\text{Split}^+_{\text{YM}} = \frac{f^{\text{c}}_{\text{ab}}}{\langle 12 \rangle} \qquad\qquad \text{Split}^+_{\text{grav}} = \frac{[12]}{\langle 12 \rangle}$$

at tree level

On $\mathscr{I}$, taking the collinear limit corresponds to bringing the two points on the celestial sphere close together



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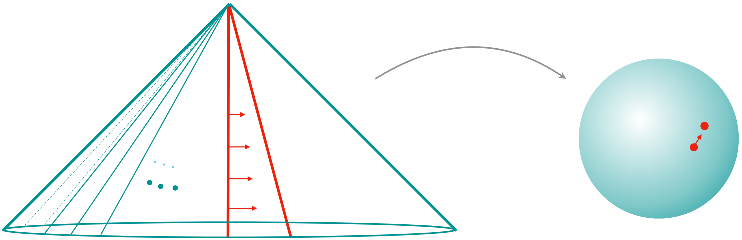

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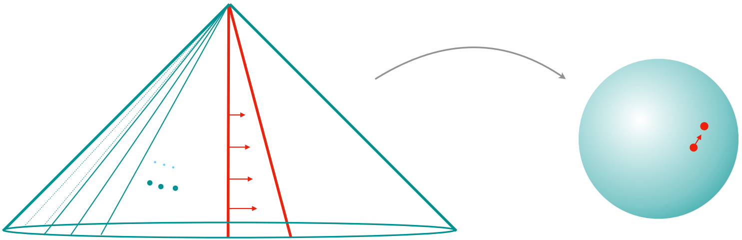

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- splitting functions give the space of on-shell linearised states the structure of an *algebra*
- assuming graviton modes $\leftrightarrow$ local conserved operators, we can read off the symmetry stucture of any purported CCFT dual [Fan, Fotopoulos, Taylor]

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- compactification of a string theory on $\mathbb{R}^{1,3} \times X$
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- self-dual theories
  - $\mathcal{N} = 2$ string / Ricci-flat Kähler [Plebanski;Ooguri,Vafa; Chalmers,Siegel]
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In this talk, we'll concentrate on self-dual Einstein gravity

Self-dual Einstein gravity may be described by the action [Chalmers,Siegel]

$$S[\tilde{\Phi}, \Phi] = \int \partial^{\dot{\alpha}\alpha}\tilde{\Phi}\,\partial_{\dot{\alpha}\alpha}\Phi + \frac{1}{2}\tilde{\Phi}\,\left\{\partial^{\dot{\alpha}}\Phi\,,\partial_{\dot{\alpha}}\Phi\right\}\;d^4x$$

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- this would be a hyperkähler metric in Riemannian signature

The (classical) splitting function is closely related to the 3-pt vertex

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- in this limit $k_1 + k_2 = |z_2\rangle[\tilde z| + \cdots$ where $|\tilde z] = \begin{pmatrix}1\\ \tilde z_2 + t\tilde z_{12}\end{pmatrix}$
- to leading order in z_{12}, the state $[12]/\langle 12\rangle\, e^{i(k_1+k_2)\cdot x}$ becomes

$$\frac{\tilde z_{12}}{z_{12}}\, e^{i\omega\langle z_2|x|\tilde z_2] + t\omega\langle z_2|x^{\dot 1}\tilde z_{12}} = \frac{e^{i\omega\langle z_2|x|\tilde z_2]}}{z_{12}}\sum_{n=0}^{\infty}\frac{(i\omega t\langle z_2|x^{\dot 1})^n}{n!}(\tilde z_{12})^{n+1}$$

- For each particle we define conformally soft modes $w[p,q](z)$ via

$$\mathrm{Res}_{\Delta=k}\left(\int_0^\infty \frac{d\omega}{\omega}\,\omega^\Delta\,\frac{e^{-i\omega\langle z|x|\tilde{z}]}}{\omega^2\langle\alpha z\rangle^4}\right)=\frac{(-\mathrm{i})^{2-k}}{\langle\alpha z\rangle^4}\frac{\langle z|x|\tilde{z}]^{2-k}}{(2-k)!}$$

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$$w[p,q](z)\,w[r,s](0) \sim -\frac{ps-qr}{z}w[p+r-1,q+s-1](0)$$

- this is $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$, the loop algebra of Poisson algebra of holomorphic functions on $\mathbb{C}^2$ with Poisson bracket $\{f,g\} = \partial_u f\partial_v g - \partial_v f\partial_u g$

The *loop* algebra of $\mathfrak{ham}(\mathbb{C}^2)$ is closely related to twistor space [Penrose; Adamo,Mason,Sharma]

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- metric on $\mathbb{R}^4$ comes from pulling this back using incidence relations

$$\omega|_X = \mathrm{d}x^{\dot\alpha\alpha} \wedge \mathrm{d}x_{\dot\alpha}{}^{\beta}\lambda_\alpha\lambda_\beta = e^{\dot\alpha\alpha} \wedge e_{\dot\alpha\beta}\,\lambda_\alpha\lambda_\beta$$
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Classical sd gravity comes from deformations of the $\mathbb{C}$-str of twistor space

- $\bar\partial \mapsto \bar\partial + V$ for $V \in \Omega^{0,1}(\mathbb{PT}, T_{\mathbb{PT}})$ and the Weyl tensor on $\mathbb{R}^4$ is self-dual iff $(\bar\partial + V)^2 = 0$

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- we also get an Einstein metric if $V = \{h, \cdot\}$ where again

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The natural action on twistor space for self-dual gravity is thus [Mason,Wolf]

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where $\Omega = \langle \lambda \mathrm{d}\lambda \rangle \wedge \omega$

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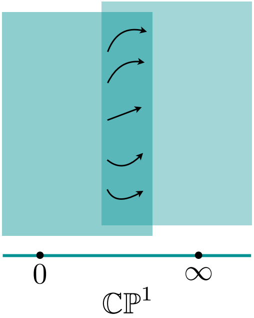
- h ($\tilde h$) represents the positive (negative) helicity graviton
- after gauge fixing and imposing some components of the eom, this reduces to Chalmers-Siegel action on $\mathbb{R}^4$ [Bittleston,Sharma,DS]

We can think of $V = \{h, \ \}$ as a Hamiltonian vector field defined on the overlap $U_0 \cap U_\infty$, telling us how to glue the two patches together



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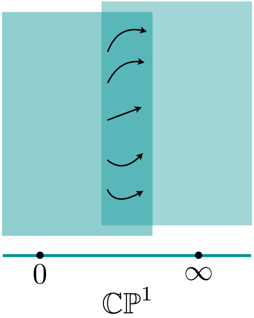

- in this picture h must be holomophic on the fibres of $\mathbb{PT} \to \mathbb{CP}^1$, but can be meromorphic on $\mathbb{CP}^1$ itself

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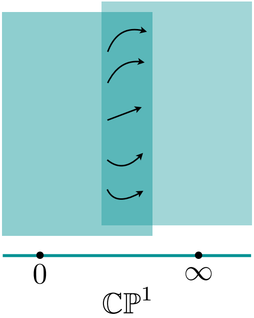

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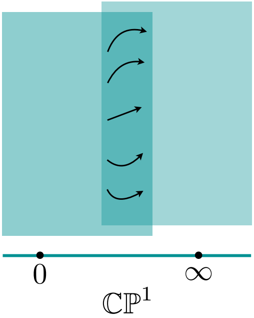

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Taking the twistor Poisson bracket of two such elements gives $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$

- on $\mathbb{R}^4$, this amounts to considering all $\mathbb{C}$-structures simultaneously

It's important to note that $\mathfrak{ham}(\mathbb{C}^2)$ is **not** $w_{1+\infty}$, nor its wedge subalgebra $w_\wedge$

$$\left.\begin{array}{l} w_{1+\infty} \\ w_\wedge \\ \mathfrak{ham}(\mathbb{C}^2) \end{array}\right\} \text{ has } p,q \ \in \ \begin{cases} \mathbb{Z} & \mathrm{p}+q=0\,|\,2 \\ \mathbb{N}_0 & \mathrm{p}+q=0\,|\,2 \\ \mathbb{N}_0 & \end{cases}$$

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The distinctions are important when considering deformations

$w_\wedge$ possesses a family of deformations, called $W(\mu)$, with relations

$$[w[p,q], w[r,s]] = \sum_{\ell\geq 0} \mathfrak{q}^{2\ell} f_\ell(p,q,r,s)\, w[p+r-2\ell-1, q+s-2\ell-1]\,,$$

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- the structure constants involve a hypergeometric function

$$f_\ell(p,q,r,s) \;=\; R_\ell(p,q,r,s)\,{}_4F_3\left[\begin{matrix}-1/2-2\sigma, 3/2+2\sigma, -\ell/2, (1-\ell)/2\\ 1/2-m, 1/2-n, m+n+3/2-\ell\end{matrix};1\right]$$

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These are the **only** deformations of $w_\wedge$ [Pope,Romans,Shen;Fairlie]

- it's possible that $\mathcal{L}w_\wedge$ may have further deformations [Strominger]

For example, in the theory of a free complex fermion $\int \bar{\psi}\bar{\partial}\psi \, d^2z$, a copy of $\mathcal{L}W_{1+\infty} = \mathcal{L}W(-1/4)$ is realised by modes of the currents [Pope,Romans,Shen]

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J &= \bar{\psi}\psi \,, \\
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Other 2d quantum theories (*eg* $SU(\infty)$ Toda) realise $W(\mu)$ algebras with different values of μ

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$$S_{\mathfrak{q}}[\Phi, \tilde{\Phi}] = \int \tilde{\Phi} \left(\Box\Phi + \frac{1}{2} \left\{ \partial^{\dot{\alpha}}\Phi, \partial_{\dot{\alpha}}\Phi \right\}_{\mathfrak{q}} \right) d^4x$$

where $\{\ ,\ \}_{\mathfrak{q}}$ is the Moyal bracket defined via

$$\{f, g\}_{\mathfrak{q}} = \mathfrak{q}^{-1} (f \star g - g \star f) \qquad\qquad f \star g = f \, \exp\left[\mathfrak{q} \left(\epsilon^{\dot{\alpha}\dot{\beta}} \overleftarrow{\partial_{\dot{\alpha}}} \overrightarrow{\partial_{\dot{\beta}}} \right) \right] g$$

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- deformed algebra comes from deformed splittting function

$$\mathrm{Split}^{+}_{\mathfrak{q}} = \frac{[ij]_{\mathfrak{q}}}{\langle ij \rangle} = \frac{\sinh(\mathfrak{q}[ij]\langle i\alpha\rangle\langle j\alpha\rangle)}{\mathfrak{q}\langle i\alpha\rangle\langle j\alpha\rangle}$$

It's interesting to compare this to AdS/CFT

- in free CFT, there's an ∞-dimensional space of local operators whose OPEs tell us they transform in reps of *eg* a $\mathfrak{hs}$ algebra
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In flat space, there is no (fixed) scale to compare to α'

- it may be fruitful to consider celestial holography in an *asymptotically* flat space, even though it's not yet understood in flat space itself

We begin on the twistor space $\mathbb{PT}/\mathbb{Z}_2$ of $\mathbb{R}^4/\mathbb{Z}_2$, where

$$\mathbb{Z}_2 \ : \ (\mu^{\dot\alpha}, \lambda_\alpha) \mapsto (-\mu^{\dot\alpha}, +\lambda_\alpha)$$

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- Mabuchi gravity and Burns space arise from a sector of sd conformal gravity; interesting to have a version for sd Einstein gravity
- no full string realization as yet, though closely related to $\mathcal{N} = 2$ string and B-model in presence of D3-branes [Bittleston,Heuveline,DS wip]

Our defect couples electrically to $\tilde{h}$, so action becomes

$$S[\tilde{h}, h] = \int_{\mathbb{PT}/\mathbb{Z}_2} \Omega \wedge \tilde{h} \left(\bar{\partial} h + \frac{1}{2}\{h, h\} \right) - \frac{\pi^2}{2} \int_{\mu^{\dot{\alpha}}=0} \langle \lambda \mathrm{d} \lambda \rangle \, c^2(\lambda) \wedge \tilde{h}$$

- coupling requires choice of weight $+2$ function $c(\lambda) = c\langle \alpha\lambda \rangle \langle \beta\lambda \rangle$

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- this is solved by

$$h = \frac{c^2(\lambda)}{2} \frac{[\hat{\mu} \, \mathrm{d}\hat{\mu}]}{[\mu \, \hat{\mu}]^2} \qquad \text{where} \qquad \hat{\mu}^{\dot{\alpha}} = (\overline{-\mu^{\dot{1}}}, \overline{\mu^{\dot{0}}})$$

- note that this solution respects the $\mathbb{Z}_2$ action

This background h means the $\bar{\partial}$-operator is deformed to

$$\bar{\nabla} = \bar{\partial} + \{h,\ \} = \bar{\partial} - c^2(\lambda)\frac{[\hat{\mu}\,\mathrm{d}\hat{\mu}]}{[\mu\,\hat{\mu}]^3}\,\hat{\mu}^{\dot{\alpha}}\frac{\partial}{\partial\mu^{\dot{\alpha}}}$$

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$$\lambda_\alpha \qquad \text{and} \qquad X^{\dot{\alpha}\dot{\beta}} = X^{(\dot{\alpha}\dot{\beta})} = \mu^{\dot{\alpha}}\mu^{\dot{\beta}} - c^2(\lambda)\frac{\hat{\mu}^{\dot{\alpha}}\hat{\mu}^{\dot{\beta}}}{[\mu\,\hat{\mu}]^2}$$

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Redefining $(X, Y, Z) = (X^{\dot{0}\dot{0}}, X^{\dot{1}\dot{1}}, X^{\dot{0}\dot{1}})$, the deformed twistor space is

$$\mathcal{PT} = \{XY = (Z - c(\lambda))(Z + c(\lambda))\} \subset \begin{array}{c}\mathcal{O}(2)\oplus\mathcal{O}(2)\oplus\mathcal{O}(2)\\ \downarrow \\ \mathbb{CP}^1\end{array}$$

- $\mathcal{PT}$ is the twistor space of Eguchi-Hanson space [Eguchi,Hanson;Hitchin;Tod,Ward]; sending the defect coupling $c \to 0$ returns to $\mathbb{PT}/\mathbb{Z}_2$

Eguchi-Hanson space itself may be recovered from the incidence relations

$$X^{\dot{\alpha}\dot{\beta}} = x^{\dot{\alpha}\alpha} x^{\dot{\beta}\beta} \left(\lambda_\alpha \lambda_\beta - \frac{4c^2 \langle \alpha\lambda \rangle^2}{x^4} \beta_\alpha \beta_\beta \right)$$

- these define holomorphic sections of $\mathcal{PT} \to \mathbb{CP}^1$; in particular the *rhs* above obeys the constraint $X^{\dot{\alpha}\dot{\beta}} X_{\dot{\alpha}\dot{\beta}} = -2c^2(\lambda)$

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The fibres of $\mathcal{PT} \to \mathbb{CP}^1$ have a weight 2 symplectic (2,0)-form

$$\omega = \frac{1}{2}\mathrm{d}\mu^{\dot{\alpha}} \wedge \mathrm{d}\mu_{\dot{\alpha}} = \frac{X^{\dot{\alpha}\dot{\beta}}\,\mathrm{d}X_{\dot{\gamma}\dot{\alpha}} \wedge \mathrm{d}X^{\dot{\gamma}}{}_{\dot{\beta}}}{8c^2(\lambda)} = \frac{\mathrm{d}X \wedge \mathrm{d}Z}{2X}$$

- evaluating ω on the incidence relations leads to the space-time metric

$$ds^2 = dx^{\dot{\alpha}\alpha} \odot dx_{\dot{\alpha}\alpha} + \frac{16c^2}{x^6}\langle\beta|x\,dx|\alpha\rangle^{\odot 2}$$

- this is the Eguchi-Hanson metric in Kerr-Schild form

[Sparling,Tod;Burnett-Stuart;Berman,Chacón,Luna,White]

The CCA comes from the ring of holomorphic functions on a fibre of $\mathcal{PT} \to \mathbb{CP}^1$

$$\mathcal{O}_{\mathcal{M}_\lambda} = \mathbb{C}[X, Y, Z]/\mathscr{I} \quad \text{where the ideal } \mathscr{I} = \text{span}\{XY - Z^2 + c^2(\lambda)\}$$

- this ring has a natural basis

$$V[2p, 2q] = X^p Y^q\,, \qquad V[2p+1, 2q+1] = X^p Y^q Z\,, \qquad p, q \in \mathbb{N}_0$$

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- the ideal involves only the quadratic Casimir so $\{\mathcal{O},\mathscr{I}\} \subset \mathscr{I}$

Taking Poisson brackets of our basis gives the algebra

$$\big[V[2p,2q],V[2r,2s]\big] = 4(ps-qr)\,V[2(p+r-1)+1,2(q+s-1)+1],$$

$$\begin{aligned}\big[V[2p,2q],V[2r+1,2s+1]\big]&\\ &= 2(p(2s+1)-q(2r+1))\,V[2(p+r),2(q+s)]\\ &+4c^2(\lambda)\,(ps-qr)\,V[2(p+r-1),2(q+s-1)]\end{aligned}$$

$$\begin{aligned}\big[V[2p+1,2q+1],V[2r+1,2s+1]\big]&\\ &= ((2p+1)(2s+1)-(2q+1)(2r+1))\,V[2(p+r)+1,2(q+s)+1]\\ &+4c^2(\lambda)\,(ps-qr)\,V[2(p+r)-1,2(q+s)-1]\end{aligned}$$

- when $c=0$ this is $w_\wedge$; new terms at $c\neq 0$ are non-trivial in Lie algebra cohomology
- again the twistor space pieces together these algebras over $\mathbb{CP}^1$, giving us the corresponding loop algebra

To relate this to CCAs, consider scattering the external states

$$\delta\Phi = \frac{1}{\langle\alpha\kappa\rangle^4}\cos\left(\sqrt{(k\cdot x)^2 - \frac{4c^2\langle\alpha|kx|\beta\rangle^2}{x^4}}\right) \qquad k = |\kappa\rangle[\tilde{\kappa}|$$

in terms of Kerr-Schild coordinates $x^{\dot{\alpha}\alpha}$ on the Eguchi-Hanson background

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- they may be obtained from Penrose transfom of twistor states

$$\delta h(X,\lambda) \sim \bar{\delta}(\langle\lambda\kappa\rangle)\cos(\sqrt{[\tilde{\kappa}|X|\tilde{\kappa}]})$$

  using the incidence relations

$$X^{\dot{\alpha}\dot{\beta}} = x^{\dot{\alpha}\alpha}x^{\dot{\beta}\beta}\left(\lambda_\alpha\lambda_\beta - \frac{4c^2\langle\alpha\lambda\rangle^2}{x^4}\beta_\alpha\beta_\beta\right)$$

Expanding $\cos(\sqrt{[\tilde{\kappa}|X|\tilde{\kappa}]}) = \sum_{m=0}^{\infty}[\tilde{\kappa}|X|\tilde{\kappa}]^m/(2m)!$ we have

$$\begin{aligned}\frac{[\tilde{\kappa}|X|\tilde{\kappa}]^m}{(2m)!} &= \sum_{p+q=m} W[2p,2q]\frac{(-)^{p+q}\tilde{\kappa}_{\dot{0}}^{2p}\tilde{\kappa}_{\dot{1}}^{2q}}{(2p)!(2q)!} \\ &\quad + \sum_{p+q=m-1} W[2p+1,2q+1]\frac{(-)^{p+q+1}\tilde{\kappa}_{\dot{0}}^{2p+1}\tilde{\kappa}_{\dot{1}}^{2q+1}}{(2p+1)!(2q+1)!}\end{aligned}$$

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- the Ws are polys in (X,Y,Z) given in terms of the previous basis by

$$W[2p,2q] = \sum_{\ell=0}^{\min(p,q)} (2c(\lambda))^{2\ell}\, C_0(p,q,\ell)\, V[2p-2\ell,2q-2\ell]$$

$$W[2p+1,2q+1] = \sum_{\ell=0}^{\min(p,q)} (2c(\lambda))^{2\ell}\, C_1(p,q,\ell)\, V[2p-2\ell+1,2q-2\ell+1]$$

with coefficients

$$C_0(p,q,\ell) = \frac{[p]_\ell\,[q]_\ell\,[p+q]_\ell}{\ell!\,[2(p+q)]_{2\ell}}\,, \qquad C_1(p,q,\ell) = \frac{[p]_\ell\,[q]_\ell\,[p+q+1]_\ell}{\ell!\,[2(p+q+1)]_{2\ell}}$$

In terms of the scattering basis, the previous algebra takes the form

$$\big[W[p,q],W[r,s]\big]$$
$$=\sum_{\ell\geq 0}(2c(\lambda))^{2\ell}\,R_{2\ell+1}(p,q,r,s)\,\psi_{2\ell+1}\left(\frac{p+q}{2},\frac{r+s}{2}\right)W[p+r-2\ell-1,q+s-2$$

where

$$\psi_{2\ell+1}(m,n)=(-)^{\ell}\frac{[\ell+1/2]_\ell}{4^{2\ell}\,[m-1/2]_\ell\,[n-1/2]_\ell\,[m+n-1/2-\ell]_\ell}$$

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- we call this algebra $W(\infty)$ (not to be confused with $W_\infty = W(0)$!)
- it can be identified as the scaling limit

$$W(\infty) = \lim_{\substack{\mathfrak{q}\to 0\\ \mu\to\infty}} W(\mu)\,, \qquad \mathfrak{q}\sqrt{\mu} \quad \text{fixed}$$

  of the $W(\mu)$ algebras, where the fixed value $\mathfrak{q}\sqrt{\mu}$ is determined by the radius c of the Eguchi-Hanson core

To summarize the story so far:

- gravitational scattering in flat space has a CCA $\mathcal{L}\mathfrak{ham}(\mathbb{C}^2)$ at the classical level
- instead scattering on the orbifold $\mathbb{Z}_2$ (*ie* allowing only even parity states) reduces this to $\mathcal{L}w_\wedge = \mathcal{L}\mathfrak{ham}(\mathbb{C}^2)^{\mathbb{Z}_2}$

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We've checked that the above algebras, obtained via twistor space, agree with an explicit calculation of the gravitational splittting function on Eguchi-Hanson space. This space-time calculation seems much harder...

Going beyond the scaling limit requires turning on non-commutativity

- twistor action for non-commutative sd gravity is

$$S_{\mathfrak{q}}[\tilde{h}, h] = \int_{\mathbb{PT}} \Omega \wedge \tilde{h} \left(\bar{\partial} h + \frac{1}{2}[h, h]_{\mathfrak{q}} \right)$$

using the Moyal bracket defined with $\mathfrak{q}(\lambda) = \mathfrak{q}\langle\alpha\lambda\rangle\langle\beta\lambda\rangle$ as

$$[f, g]_{\mathfrak{q}} = \sum_{k=0}^{\infty} \frac{\mathfrak{q}^{2k}(\lambda)}{2^{2k}\,(2k+1)!} \partial_{\dot{\alpha}_1} \ldots \partial_{\dot{\alpha}_{2k+1}} f \;\partial^{\dot{\alpha}_1} \ldots \partial^{\dot{\alpha}_{2k+1}} g$$

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- including the $\mathbb{CP}^1$ defect, the background h takes exactly the same form as before

$$h = \frac{c^2(\lambda)}{2} \frac{[\hat{\mu}\,\mathrm{d}\hat{\mu}]}{[\mu\,\hat{\mu}]^2}$$

However, the background $\bar{\nabla}$ operator now involves Moyal bracket

$$\begin{aligned}\bar{\nabla}_{\mathfrak{q}} &= \bar{\partial} + [h,\ ]_{\mathfrak{q}} \\ &= \bar{\partial} - c^2(\lambda)[\hat{\mu}\,\mathrm{d}\hat{\mu}] \sum_{k=0}^{\infty} \frac{(k+1)\mathfrak{q}^{2k}(\lambda)}{[\mu\,\hat{\mu}]^{2k+3}2^{2k}} \hat{\mu}^{\alpha_1} \ldots \hat{\mu}^{\dot{\alpha}_{2k+1}} \partial_{\dot{\alpha}_1} \ldots \partial_{\dot{\alpha}_{2k+1}}\end{aligned}$$

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- the non-commutative twistor space is instead defined by the ideal

$$X^{\dot{\alpha}\dot{\beta}} \star_{\mathfrak{q}} X_{\dot{\alpha}\dot{\beta}} = X^{\dot{\alpha}\dot{\beta}}X_{\dot{\alpha}\dot{\beta}} + \frac{3\mathfrak{q}^2(\lambda)}{4} = -2c^2(\lambda) + \frac{3\mathfrak{q}^2(\lambda)}{4}$$

In terms of the algebra, non-commutative $\mathfrak{sl}_2$ is still defined by relations

$$[X, Z]_{\mathfrak{q}} = 2X \qquad [Y, Z]_{\mathfrak{q}} = -2Y \qquad [X, Y]_{\mathfrak{q}} = 4Z$$

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The CCA of non-commutative self-dual gravity on Eguchi-Hanson space is thus $\mathcal{L}W(\mu)$

- to go from $w_\wedge$ (or $\mathfrak{ham}(\mathbb{C}^2)$) to a W algebra requires turning on non-commutativity $\mathfrak{q}$
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- to go away from the symplecton $W(-3/14)$ into the full family of $W(\mu)$-algebras requires turning on a finite Eguchi-Hanson scale
- *eg* at $c = 0$ we obtain the symplecton $W(-3/16)$; at $c^2 = -\mathfrak{q}^2/4$ we obtain $W_{1+\infty}$; at $c^2 = 3\mathfrak{q}^2/4$ we obtain W_∞

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- deforming the orbifold to the corresponding ALE space likewise deforms the CCA to a scaling limit of $\mathcal{L}W_\Gamma(\mu)$ algebras as $\mathfrak{q} \to 0$
- we expect the generic $W_\Gamma(\mu)$ to depend on $\mathrm{rk}(\Gamma)$ parameters corresponding to the sizes of the $\mathbb{CP}^1$ cores in the ALE space; accessing the generic case requires turning on non-commutativity

For example, the twistor space of the Gibbons-Hawking A_{k-1} space is

$$\mathcal{PT} = \left\{ XY = \prod_{i=1}^{k} (Z - c_i(\lambda)) \right\} \subset \begin{array}{c} \mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \\ \downarrow \\ \mathbb{CP}^1 \end{array}$$

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- taking Poisson brackets of these gives the algebra

$$\begin{aligned} &\big[w[kp+i, kq+i], w[kr+j, ks+j]\big] \\ &= 2k((p+i/k)(s+j/k) - (q+i/k)(r+j/k))\, w[k(p+r)+i+j-1, k(q+s \\ &- 2(ps-qr) \sum_{m=0}^{k+i+j-2} F_k(i+j, m)\, w[k(p+r-1)+m, k(q+s-1)+m]\,, \end{aligned}$$

where the F_k depend on the radii c_i and are defined recursively

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We're hopeful that today's story can be embedded in a full stringy top-down realization, analogous to Burns holography [Costello,Paquette,Sharma]

Thank You