String-based vs. String-inspired in Quantum Gravity

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Amplifying Gravity at All Scales

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Content

- Worldline path integrals in quantum field theory.
- The field-theory limit of string theory and the worldline formalism.
- Gluon amplitudes, on-shell and off-shell.
- Berends-Giele currents in Yang-Mills theory and gravity.
- Graviton amplitudes.

The string-based formalism

1988-1992 Bern and Kosower: QCD amplitudes from strings

Polyakov path integral for string N-point functions

$$\left\langle V_1 \cdots V_N \right\rangle \sim \sum_{\mathrm{top}} \int \mathcal{D}h \int \mathcal{D}x(\sigma,\tau) V_1 \cdots V_N \, \mathrm{e}^{-S[x,h]}$$

$$\begin{aligned}
S[x,h] &= \text{worldsheet action} \\
\sum_{\mathrm{top}} &= \text{sum over worldsheet topologies} \\
\int \mathcal{D}h &= \text{integral over worldsheet metrics} \\
\int \mathcal{D}x &= \text{integral over worldsheet embeddings} \\
V_i &= \text{vertex operator representing particle } i
\end{aligned}$$

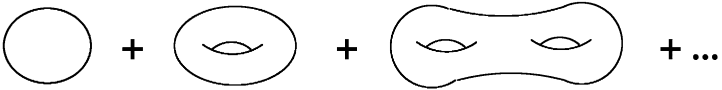
This is a first-quantized path integral describing a single string propagating and emitting/absorbing N particles.

E. g., for the closed string case

$$S[x,h] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu$$

($\frac{1}{2\pi\alpha'}$ is the string tension)

Sum over topologies (corresponds to loop expansion)



Infinite string tension limit $\alpha' \to 0$:
String theory amplitudes $\to$ field theory amplitudes

Bern-Kosower program:

1. Calculate the Polyakov path integral using worldsheet Green's functions,

$$\langle x^\mu(\sigma_1,\tau_1)x^\nu(\sigma_2,\tau_2)\rangle = G(\sigma_1,\tau_1;\sigma_2,\tau_2)\eta^{\mu\nu}$$

2. Infinite string tension limit $\rightarrow$ New parameter integral representations for the one-loop N - gluon amplitudes.
3. Rules for the direct construction of those integrals (Bern-Kosower rules) Z. Bern and D. A. Kosower, PRD 38, 1888 (1988); PRL 66, 1669 (1991); NPB 379, 451 (1992).

Similar rules for one-loop N - graviton amplitudes (from the closed string) Z. Bern, D. Dunbar, T. Shimada, PLB 312, 277 (1993)

The non-abelian master formula

Master formula for N - gluon amplitudes (Z. Bern and D. Kosower 1991)

$$\begin{aligned}\Gamma^{a_1\cdots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \quad &= \quad (-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T} \\ &\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-2}} d\tau_{N-1} \\ &\times \exp\left\{\sum_{i,j=1}^N \left[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Big|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}\end{aligned}$$

As it stands, this is a parameter integral representation for the (color-ordered) N - gluon vertex, with momenta k_i and polarizations ε_i, induced by a scalar loop, in D dimensions.
Here m and T are the loop mass and proper-time, τ_i the location of the ith gluon, and

$$G_{ij} = |\tau_i - \tau_j| - \frac{(\tau_i-\tau_j)^2}{T},\ \dot{G}(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T},\ \ddot{G}(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - \frac{2}{T}\,.$$

Bern-Kosower rules

Bern and Kosower found purely algebraic rules that

- change the loop scalar into a fermion or gluon (loop replacement rules).
- provide an easy way to include the missing one-particle irreducible diagrams (pinch rules).

The formalism was then used for the first calculation of the QCD one-loop five - gluon amplitudes (Z. Bern, L. Dixon, D.A. Kosower, PRL 70 (1993) 2677).

The string-inspired worldline path integral approach

M. J. Strassler, NPB 385 (1992) 145:

- Rederived the master formula and the loop replacement rules using worldline path integral representations of the gluonic effective actions. E.g. for the scalar loop

$$\Gamma[A] = \mathrm{tr} \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x(\tau)\, \mathcal{P} e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ig\dot{x}\cdot A(x(\tau))\right)}$$

  where $A_\mu = A_\mu^a T^a$ and $\mathcal{P}$ denotes path ordering.

- This also shows that the master formula and the loop replacement rules hold off-shell.

M. J. Strassler, SLAC-PUB-5978 (unpubl.): noted that the IBP generates automatically

- abelian field strength tensors $f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$ in the bulk and
- color commutators $[T^{a_i}, T^{a_j}]$ as boundary terms.
- Those fit together to produce full nonabelian field strength tensors

$$F_{\mu\nu} \equiv F_{\mu\nu}^a T^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) T^a + ig[A_\mu^b T^b, A_\nu^c T^c]$$

  in the low-energy effective action.

Thus we see the emergence of gauge invariant tensor structures at the integrand level.

Integration-by-parts algorithms

Removing all $\ddot{G}_{ij}$ by IBP can be done in many ways!

- M.J. Strassler 1992: started to investigate this ambiguity at the four-point level.
- C. S. 1998: found an algorithm that preserves the full permutation (Bose) symmetry. It leads to an unambiguous result that is called the Q representation.

$$
\begin{aligned}
\exp\Big\{\Big\}\Big|_{\rm multi-linear} &= (-i)^N P_N(\dot{G}_{ij}, \ddot{G}_{ij})\, \mathrm{e}^{\frac{1}{2}\sum G_{ij} k_i\cdot k_j} \\
&\stackrel{\rm part.int.}{\longrightarrow} (-i)^N Q_N(\dot{G}_{ij})\, \mathrm{e}^{\frac{1}{2}\sum G_{ij} k_i\cdot k_j}
\end{aligned}
$$

Advantages of the Q-representation

1. After the IBP, the integrand for the spinor loop can be obtained by the scalar loop one through the Bern-Kosower replacement rule:
   Replace every closed cycle $\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_k i_1}$ in Q_N by

$$\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_k i_1} - G_{F i_1 i_2} G_{F i_2 i_3} \cdots G_{F i_k i_1}$$

   (and multiply by a global factor of -2).

2. Each such "τ-cycle" comes together with a "Lorentz-cycle" $Z_n(i_1 i_2 \ldots i_n)$ defined as (Strassler)

$$\begin{aligned} Z_2(ij) &\equiv \frac{1}{2}\mathrm{tr}\left(f_i f_j\right) = \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j \\ Z_n(i_1 i_2 \ldots i_n) &\equiv \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big) \quad (n \geq 3) \end{aligned}$$

This motivates the definition of a "bicycle" as the product of the two:

$$\dot{G}(i_1 i_2 \cdots i_n) := \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} Z_n(i_1 i_2 \cdots i_n)$$

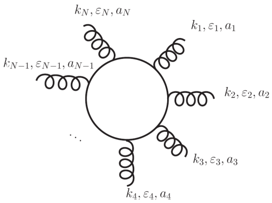
Example: N=4

$$
\begin{aligned}
Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22} \\
Q_4^4 &= \dot G(1234) + \dot G(1243) + \dot G(1324) \\
Q_4^3 &= \dot G(123)T(4) + \dot G(234)T(1) + \dot G(341)T(2) + \dot G(412)T(3) \\
Q_4^2 &= \dot G(12)T(34) + \dot G(13)T(24) + \dot G(14)T(23) + \dot G(23)T(14) + \dot G(24)T(13) + \dot G(34)T(12) \\
Q_4^{22} &= \dot G(12)\dot G(34) + \dot G(13)\dot G(24) + \dot G(14)\dot G(23)
\end{aligned}
\tag{1}
$$

Apart from the bicycles, there are also "tails" $T(a)$, $T(ab)$,

$$
\begin{aligned}
T(a) &\equiv \sum_{i\neq a} \dot G_{ai}\varepsilon_a\cdot k_i \\
T(ab) &\equiv \sum_{\substack{i\neq a, j\neq b\\ (i,j)\neq(b,a)}} \dot G_{ai}\varepsilon_a\cdot k_i \dot G_{bj}\varepsilon_b\cdot k_j + \frac{1}{2}\dot G_{ab}\varepsilon_a\cdot\varepsilon_b\Big[\sum_{i\neq a,b}\dot G_{ai}k_a\cdot k_i - \sum_{j\neq b,a}\dot G_{bj}k_b\cdot k_j\Big]
\end{aligned}
$$

The QCD N-gluon vertices



One-loop off-shell 1PI N-gluon functions ("vertices") $\Gamma_{s\,\mu_1\ldots\mu_N}^{a_1 a_2 \cdots a_N}[k_1, \ldots, k_N]$

$s = 0, \frac{1}{2}, 1$ for scalar, spinor, gluon loop.

- Building blocks for higher-loop amplitudes.
- Input for the Dyson-Schwinger equations.
- Important for the RG group.
- IR properties of QCD.
- ...

Ball-Chiu decomposition of the three-gluon vertex

J. S. Ball and T. W. Chiu 1980:

$$\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \quad = \quad & f^{abc}\Big\{A(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + B(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1+k_2)_{\mu_3} \\
& -C(k_1^2,k_2^2;k_3^2)[(k_1k_2)g_{\mu_1\mu_2} - k_{1\mu_2}k_{2\mu_1}](k_1-k_2)_{\mu_3} \\
& +\frac{1}{3}S(k_1^2,k_2^2,k_3^2)(k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} + k_{1\mu_2}k_{2\mu_3}k_{3\mu_1}) \\
& +F(k_1^2,k_2^2;k_3^2)[(k_1k_2)g_{\mu_1\mu_2} - k_{1\mu_2}k_{2\mu_1}][k_{1\mu_3}(k_2k_3) - k_{2\mu_3}(k_1k_3)] \\
& +H(k_1^2,k_2^2,k_3^2)\Big(-g_{\mu_1\mu_2}[k_{1\mu_3}(k_2k_3) - k_{2\mu_3}(k_1k_3)] + \frac{1}{3}(k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} - k_{1\mu_2}k_{2\mu_3}k_{3\mu_1})\Big) \\
& +[\text{cyclic permutations of } (k_1,\mu_1),(k_2,\mu_2),(k_3,\mu_3)]\Big\}
\end{aligned}$$

- Universal tensor decomposition, valid for scalar, spinor and gluon loop, and also for higher loop corrections. Only the coefficient functions A, B, C, F, H, S change.
- From an analysis of the Ward identities.
- A, B, C: two-point kinematics, not transversal.
- F, H: three-point kinematics, transversal.
- At tree-level, $A = 1$, the other functions vanish. $S = 0$ even at one-loop.

Ball-Chiu from the master formula

N. Ahmadiniaz, C. Schubert, NPB 869 (2013) 417:
For $N = 3$, the master formula yields

$$\Gamma_0^{a_1 a_2 a_3}[k_1, \varepsilon_1; k_2, \varepsilon_2; k_3, \varepsilon_3] = (-ig)^3 \mathrm{tr}(T^{a_1} T^{a_2} T^{a_3}) \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T}$$
$$\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \, (-i)^3 P_3 \, e^{(G_{12} k_1 \cdot k_2 + G_{13} k_1 \cdot k_3 + G_{23} k_2 \cdot k_3)} ,$$

where

$$P_3 = \dot{G}_{1i} \varepsilon_1 \cdot k_i \dot{G}_{2j} \varepsilon_2 \cdot k_j \dot{G}_{3k} \varepsilon_3 \cdot k_k - \ddot{G}_{12} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{3k} \varepsilon_3 \cdot k_k$$
$$- \ddot{G}_{13} \varepsilon_1 \cdot \varepsilon_3 \dot{G}_{2j} \varepsilon_2 \cdot k_j - \ddot{G}_{23} \varepsilon_2 \cdot \varepsilon_3 \dot{G}_{1i} \varepsilon_1 \cdot k_i ,$$

(repeated indices $i, j, k, \ldots$ are to be summed). To remove the term involving $\ddot{G}_{12} \dot{G}_{31}$, add the total derivative

$$-\frac{\partial}{\partial \tau_2} \Big(\dot{G}_{12} \varepsilon_1 \cdot \varepsilon_2 \dot{G}_{31} \varepsilon_3 \cdot k_1 e^{(G_{12} k_1 \cdot k_2 + G_{13} k_1 \cdot k_3 + G_{23} k_2 \cdot k_3)} \Big) .$$

In the abelian case this total derivative term would integrate to zero, but here due to the color ordering it produces (one half of) the term

$$\mathrm{tr}\,(T^{a_1}[T^{a_2}, T^{a_3}]) \varepsilon_3 \cdot f_1 \cdot \varepsilon_2 \dot{G}_{12} \dot{G}_{21} \, e^{G_{12} k_1 \cdot (k_2 + k_3)} .$$

This term involves only a two-point integral, with "pinched" momenta $k_2 + k_3$,

The three-gluon vertex in the Q-representation

At this stage have

$$
\Gamma_0 = \frac{g^3}{(4\pi)^{\frac{D}{2}}} \mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])(\Gamma_0^{\rm bulk} + \Gamma_0^{\rm bound})
$$

$$
\begin{aligned}
\Gamma_0^{\rm bulk} &= -\int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \, (Q_3^3 + Q_3^3) \exp\left\{ \sum_{i,j=1}^{3} \frac{1}{2} G_{ij} k_i \cdot k_j \right\} \\
\Gamma_0^{\rm bound} &= -\int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1 \dot{G}_{12} \dot{G}_{21} \left[\varepsilon_3 \cdot f_1 \cdot \varepsilon_2 \, e^{G_{12} k_1 \cdot (k_2 + k_3)} + \mathrm{cycl.} \right]
\end{aligned}
$$

$$
\begin{aligned}
Q_3^3 &= \dot{G}_{12} \dot{G}_{23} \dot{G}_{31} \, \mathrm{tr}\,(f_1 f_2 f_3) \\
Q_3^2 &= \frac{1}{2} \dot{G}_{12} \dot{G}_{21} \, \mathrm{tr}\,(f_1 f_2) \dot{G}_{3i} \varepsilon_3 \cdot k_i + 2\,\mathrm{perm.}
\end{aligned}
$$

This is not yet Ball-Chiu: Q_3^3 corresponds to the form factor H, but Q_3^2 not to F; it is not even transversal.

Second integration-by-parts

To make Q_3^2 transversal, add another total derivative:

$$-\frac{r_3\cdot\varepsilon_3}{r_3\cdot k_3}\frac{1}{2}\mathrm{tr}(f_1 f_2)\frac{\partial}{\partial\tau_3}\Bigl(\dot G_{12}\dot G_{21}e^{(\cdot)}\Bigr)\,.$$

Here r_3 is a *reference momentum* such that $r_3\cdot k_3\neq 0$. This transforms Q_3^2 into

$$\begin{aligned} S_3^2 \quad := \quad & \dot G_{12}\dot G_{21}\frac{1}{2}\mathrm{tr}(f_1 f_2)\dot G_{3k}\frac{r_3\cdot f_3\cdot k_k}{r_3\cdot k_3} + \dot G_{13}\dot G_{31}\frac{1}{2}\mathrm{tr}(f_1 f_3)\dot G_{2j}\frac{r_2\cdot f_2\cdot k_j}{r_2\cdot k_2} \\ & + \dot G_{23}\dot G_{32}\frac{1}{2}\mathrm{tr}(f_2 f_3)\dot G_{1i}\frac{r_1\cdot f_1\cdot k_i}{r_1\cdot k_1}\,. \end{aligned}$$

which is transversal. With the cyclic choice of reference vectors

$$r_1 = k_2 - k_3\,,\, r_2 = k_3 - k_1\,,\, r_3 = k_1 - k_2$$

S_3^2 becomes the Ball-Chiu form factor F. The boundary terms match with the form factors A, B, C.

Loop replacement rules for the three-gluon vertex

Scalar to Spinor Loop:

$$\begin{aligned} \dot{G}_{ij}\dot{G}_{ji} &\rightarrow \dot{G}_{ij}\dot{G}_{ji} - G_{Fij}G_{Fji} \\ \dot{G}_{12}\dot{G}_{23}\dot{G}_{31} &\rightarrow \dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31} \end{aligned}$$

where $G_{Fij} = \mathrm{sign}(\tau_i - \tau_j)$.

Scalar to Gluon Loop:

$$\begin{aligned} \dot{G}_{ij}\dot{G}_{ji} &\rightarrow \dot{G}_{ij}\dot{G}_{ji} - 4G_{Fij}G_{Fji} \\ \dot{G}_{12}\dot{G}_{23}\dot{G}_{31} &\rightarrow \dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - 4G_{F12}G_{F23}G_{F31} \end{aligned}$$

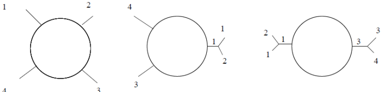
The generated integrand for the gluon loop corresponds to the background field method with quantum Feynman gauge.

The four-gluon vertex

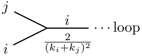
$N = 4$ is much more challenging - at four points, a priori one can construct 138 tensors!
N. Ahmadiniaz, C. Schubert, Int. J. Mod. Phys. E **25** (2016) 1642004: Decomposition of the four-gluon vertex in terms of 19 tensors, of which only 14 have the full four-point kinematics.

On-shell N-gluon matrix elements

When computing the on-shell N-gluon matrix elements, we have to use the full connected amplitude, not just the irreducible one. Following Bern and Kosower 1991, the additional one-particle-reducible terms can be obtained from Q_N by the following procedure:
(i) Draw all possible ϕ^3 1-loop diagrams D_i with N legs, labelled $1, \ldots, N$ (following the ordering of the color trace). Diagrams where the loop is a tadpole or isolated on an external leg can be omitted. E.g. at the four-point level there are single and double pinches only:



(ii) A diagram will contribute if each vertex except the ones attached directly to the loop corresponds to a possible pinch. A vertex with labels $i < j$ can be pinched if Q_N is linear in $\dot{G}_{ij}$. The pinching replaces this $\dot{G}_{ij}$ by a factor of $2/(k_i + k_j)^2$, removes the vertex and transfers the label i to the ingoing leg.



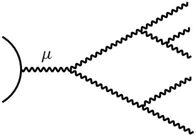
The τ_j - integration is omitted and the index j replaced by i in all $G_{kl}, \dot{G}_{kl}$. The pinching can thus be represented by a pinch operator $\mathcal{D}_{ij}$,

$$\mathcal{D}_{ij} f(\dot{G}) \equiv \frac{\partial}{\partial \dot{G}_{ij}} f(\dot{G}) \Big|_{\substack{\dot{G}_{ij}=0 \\ \dot{G}_{jk} \to \dot{G}_{ik}}} .$$

(N. Ahmadiniaz, F.M. Balli, C. Lopez-Arcos, A. Quintero Velez and C. S., PRD 104 (2021) L941702).
The trees are to be "pruned" recursively starting with the outermost vertices.

Berends-Giele Currents

Returning to the Bern-Kosower formalism, without the pinch rules we would have to construct the reducible contributions attaching off-shell currents to the loop:



Such currents were recognized as central objects in Yang-Mills theory since the eighties:

- They are naturally written in terms of multi-particle polarizations (F.A. Berends and W.T. Giele, NPB 306 (1988) 759) and then are called Berends-Giele currents.
- They are instrumental in the perturbiner approach where tree-level amplitudes are constructed directly from the field equations (A.A. Rosly and K.G. Selivanov, PLB 399 (1997) 135, S. Mizera and B. Skrzypek, JHEP 10 (2018) 018).
- They are important building blocks for amplitudes obeying color-kinematics duality (Z. Bern, J.J.M. Carrasco and H. Johansson, PRD 78, 085011 (2008)). This requires a specific gauge, BCJ gauge.

Multiparticle polarizations and field strength tensors

Multi-particle polarization tensors:

$$
\begin{aligned}
\varepsilon_{12}^{\mu} &= \tfrac{1}{2}\left[\varepsilon_2\cdot k_1\varepsilon_1^{\mu} - \varepsilon_{1\rho} f_2^{\rho\mu} - (1\leftrightarrow 2)\right]\\
\varepsilon_{123}^{\mu} &= \tfrac{1}{2}\Big[\,(k_3\cdot\varepsilon_{12})\,\varepsilon_3^{\mu} - (k_{12}\cdot\varepsilon_3)\,\varepsilon_{12}^{\mu} + \varepsilon_{12\nu} f_3^{\nu\mu}\\
&\quad -\varepsilon_{3\nu} f_{12}^{\nu\mu}\Big] - k_{123}^{\mu}\tfrac{1}{4}\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot(k_2-k_1)
\end{aligned}
$$

etc.

Multi-particle field-strength tensors:

$$
\begin{aligned}
f_{12}^{\mu\nu} &= \varepsilon_2\cdot k_1 f_1^{\mu\nu} - (f_1 f_2)^{\mu\nu} - (1\leftrightarrow 2)\\
f_{123}^{\mu\nu} &= k_{123}^{\mu}\varepsilon_{123}^{\nu} - k_{12}\cdot k_3\varepsilon_{12}^{\mu}\varepsilon_3^{\nu}\\
&\quad -k_1\cdot k_2(\varepsilon_1^{\mu}\varepsilon_{23}^{\nu} + \varepsilon_{13}^{\mu}\varepsilon_2^{\nu}) - (\mu\leftrightarrow\nu)
\end{aligned}
$$

etc.

BCJ gauge and generalized Jacobi identities

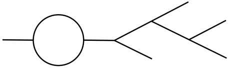
The multi-particle polarizations are subject to generalized gauge transformations. To construct currents in BCJ gauge, they must obey the generalized Jacobi identities

$$\varepsilon^{\mu}_{123} + \varepsilon^{\mu}_{213} = 0\,, \quad \varepsilon^{\mu}_{123} + \varepsilon^{\mu}_{312} + \varepsilon^{\mu}_{231} = 0\,, \quad \textit{etc.}$$

(C.R. Mafra and O. Schlotterer, JHEP 03, 090 (2016)).

Multi-particle polarizations from pinching

Clearly the Bern-Kosower pinching procedure must hold the information on the Berends-Giele currents. It turns out that to obtain the currents, it is sufficient to look at the maximal pinch of the N-gluon amplitude, defined by the consecutive pinching of $N-2$ adjacent legs. It corresponds to the Bern-Kosower diagram



(which in the original Bern-Kosower rules was actually discarded, since it is absorbed by the gluon wave-function renormalization).

Only single-cycle terms contribute to it, thus in its calculation we can replace Q_N by $\tilde{Q}_N \equiv Q_N^2 + Q_N^3 + \ldots Q_N^N$. It turns out that the $(N-1)$ - field-strength tensor $f^{\mu\nu}_{12\cdots(N-1)}$ can be harvested through

$$\mathcal{D}_{1(N-1)} \cdots \mathcal{D}_{13}\mathcal{D}_{12}\tilde{Q}_N \quad = \quad \frac{1}{2} f^{\mu\nu}_{12\cdots(N-1)} f_{N\nu\mu} \dot{G}^2_{1N}$$

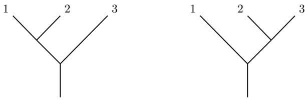
and (less obviously) the $(N-2)$ - polarization tensor $\varepsilon_{12\cdots(N-2)}$ directly from the $(N-2)$ - tail:

$$\mathcal{D}_{1(N-2)} \cdots \mathcal{D}_{13}\mathcal{D}_{12} T(1,2,\ldots,N-2) \quad = \quad \varepsilon_{12\cdots(N-2)} \cdot k_{N-1} \dot{G}_{1(N-1)} + \varepsilon_{12\cdots(N-2)} \cdot k_N \dot{G}_{1N}$$

(N. Ahmadiniaz, F.M. Balli, C. Lopez-Arcos, A. Quintero Velez and C. S., PRD 104 (2021) L941702)

BCJ gauge comes for free

It turns out that these polarization and field strength tensors automatically fulfill the generalized Jacobi identities. This can be shown using the natural mapping between the Bern-Kosower pinch diagrams and the Lie-bracketing algebra for N ordered legs,



$[[1,2],3]$ $\qquad$ $[1,[2,3]]$

etc.

The proof does not involve any specific properties of the integrand, i.e. it would work with any symmetric polynomial in the $\dot{G}_{ij}$.

Constructing the tree-level N-gluon amplitude (1)

N. Ahmadiniaz, F.M. Balli, O. Corradini, C. Lopez-Arcos, A. Quintero Velez and C. S., NPB 975 (2022) 115690

To compute the N-gluon tree-level amplitude:

1. Use the above to calculate the generalized polarization tensor ε_{N-1} in BCJ gauge. (in the above paper we calculate them up to multiplicity five).

2. Sum over all pinch diagrams to this order to construct the color-stripped Berends-Giele currents $A^\mu_{12\cdots(N-1)}$:

$$
\begin{aligned}
A^\mu_1 &= \varepsilon^\mu_1 ,\\
A^\mu_{12} &= \frac{\varepsilon^\mu_{[1,2]}}{s_{12}} ,\\
A^\mu_{123} &= \frac{\varepsilon^\mu_{[[1,2],3]}}{s_{12}s_{123}} + \frac{\varepsilon^\mu_{[1,[2,3]]}}{s_{23}s_{123}} ,\\
A^\mu_{1234} &= \frac{\varepsilon^\mu_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{\varepsilon^\mu_{[[1,[2,3]],4]}}{s_{123}s_{1234}s_{23}} + \frac{\varepsilon^\mu_{[[1,2],[3,4]]}}{s_{12}s_{1234}s_{34}} + \frac{\varepsilon^\mu_{[1,[[2,3],4]]}}{s_{1234}s_{23}s_{234}} + \frac{\varepsilon^\mu_{[1,[2,[3,4]]]}}{s_{1234}s_{234}s_{34}} ,\\
&\ \vdots
\end{aligned}
$$

The denominators can be read off from the pinch diagram.

Constructing the tree-level N-gluon amplitude (2)

3 From this we can get the colour-ordered partial amplitude of N gluons through the Berends-Giele formula

$$A^{\mathrm{tree}}(1,2,\ldots,N) = s_{12\cdots(N-1)} A^{\mu}_{12\cdots(N-1)} A_{N\mu}\,.$$

The factor $s_{12\cdots(N-1)}$ is inserted to cancel the final off-shell propagator, and the factor $A_{N\mu} = \varepsilon_{N\mu}$ puts the final gluon on-shell.

4 The color-dressed Berends-Giele currents $\mathcal{A}^{\mu}_{12\cdots(N-1)}$ are obtained from the color-stripped ones $A^{\mu}_{12\cdots(N-1)}$ by summing over all inequivalent orderings ($(2N-5)!!$ terms in total), and supplying color factors that (by color-kinematics duality) have the same Lie bracketing structure in color space. E. g.

$$\begin{aligned} \varepsilon^{\mu}_{[[1,2],3]} &\longrightarrow \varepsilon^{\mu}_{[[1,2],3]} c^{a}_{[[1,2],3]}\,, \\ c^{a}_{[[1,2],3]} &= \tilde{f}_{a_1 a_2}{}^{b} \tilde{f}_{b a_3}{}^{a} \end{aligned}$$

($\tilde{f}_{ab}{}^{c} \equiv i\sqrt{2}\, f_{ab}{}^{c}$).

Constructing the tree-level N-gluon amplitude (3)

E.g. For $N = 5$:

$$\mathcal{A}^{a\mu}_{1234} = \frac{c^a_{[[[1,2],3],4]}\varepsilon^\mu_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{c^a_{[[[1,2],4],3]}\varepsilon^\mu_{[[[1,2],4],3]}}{s_{12}s_{124}s_{1234}} + \frac{c^a_{[[[1,3],4],2]}\varepsilon^\mu_{[[[1,3],4],2]}}{s_{13}s_{134}s_{1234}} + \frac{c^a_{[[[2,3],4],1]}\varepsilon^\mu_{[[[2,3],4],1]}}{s_{23}s_{234}s_{1234}}$$
$$+ \frac{c^a_{[[[1,3],2],4]}\varepsilon^\mu_{[[[1,3],2],4]}}{s_{13}s_{123}s_{1234}} + \frac{c^a_{[[[1,4],2],3]}\varepsilon^\mu_{[[[1,4],2],3]}}{s_{14}s_{124}s_{1234}} + \frac{c^a_{[[[1,4],3],2]}\varepsilon^\mu_{[[[1,4],3],2]}}{s_{14}s_{134}s_{1234}} + \frac{c^a_{[[[2,3],1],4]}\varepsilon^\mu_{[[[2,3],1],4]}}{s_{23}s_{123}s_{1234}}$$
$$+ \frac{c^a_{[[[2,4],1],3]}\varepsilon^\mu_{[[[2,4],1],3]}}{s_{24}s_{124}s_{1234}} + \frac{c^a_{[[[2,4],3],1]}\varepsilon^\mu_{[[[2,4],3],1]}}{s_{24}s_{234}s_{1234}} + \frac{c^a_{[[[3,4],1],2]}\varepsilon^\mu_{[[[3,4],1],2]}}{s_{34}s_{134}s_{1234}} + \frac{c^a_{[[[3,4],2],1]}\varepsilon^\mu_{[[[3,4],2],1]}}{s_{34}s_{234}s_{1234}}$$
$$+ \frac{c^a_{[[1,2],[3,4]]}\varepsilon^\mu_{[[1,2],[3,4]]}}{s_{12}s_{34}s_{1234}} + \frac{c^a_{[[1,3],[2,4]]}\varepsilon^\mu_{[[1,3],[2,4]]}}{s_{13}s_{24}s_{1234}} + \frac{c^a_{[[1,4],[2,3]]}\varepsilon^\mu_{[[1,4],[2,3]]}}{s_{14}s_{23}s_{1234}}.$$

5 From this we get the total tree-level N - gluon amplitude,

$$A^{\text{tree}}_N = s_{12\cdots(N-1)}\mathcal{A}^\mu_{12\cdots(N-1)}\mathcal{A}_{N\mu}.$$

Pinch terms from multi-particle polarizations (1)

Turning the logic around, it seems plausible that the multiparticle polarizations obtained in this way hold the full information on the pinch contributions. This leads us to conjecture that the complete effect of the pinching procedure in the Bern-Kosower formalism may be taken into account simply by adding, to the un-pinched integrand, all possible terms where some cycles and/or tails are replaced by generalized ones, defined in the following way:

- Generalized Lorentz cycle:

$$\mathcal{Z}_k(I_1,\ldots,I_k) \quad \equiv \quad \left(\tfrac{1}{2}\right)^{\delta_{k2}} \mathrm{tr}\left(\prod_{i=1}^{k} \mathcal{F}_{I_i}\right)$$

  which now uses the full Berends-Giele currents $\mathcal{F}_{I_i}$.

- Generalized tail:

$$\mathcal{T}_k(I_1,\ldots,I_k) \quad \equiv \quad T(k_{I_1},\mathcal{A}_{I_1};\ldots;k_{I_k},\mathcal{A}_{I_k})$$

  with the Berends-Giele polarizations $\mathcal{A}_{I_i}$.

Pinch terms from multi-particle polarizations (2)

For example, the three-gluon amplitude, whose un-pinched integrand is $Q_3^3 + Q_3^2$,

$$
\begin{aligned}
Q_3^3 &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{31}\,\mathrm{tr}\,(f_1 f_2 f_3) \\
Q_3^2 &= \frac{1}{2}\dot{G}_{12}\dot{G}_{21}\,\mathrm{tr}\,(f_1 f_2)\dot{G}_{3i}\varepsilon_3\cdot k_i + 2\,\mathrm{perm.}
\end{aligned}
$$

should have the pinch contribution

$$\dot{G}(1,23) + \dot{G}(2,31) + \dot{G}(3,12)$$

where, e.g.,

$$\dot{G}(1,23) = \dot{G}_{12}\dot{G}_{21}\frac{1}{2}\mathrm{tr}\,\big(f_1\mathcal{F}_{23}\big)$$

and this is indeed what the Bern-Kosower pinch rules produce.
Similarly, at the four-point level the prediction for the single-pinch terms would be

$$\dot{G}(1,2,34) + \dot{G}(1,23)T(4) + \dot{G}(1,2)\mathcal{T}(34) + \mathrm{perm.}$$

and for the double pinches,

$$\dot{G}(1,234) + \dot{G}(12,34) + \mathrm{perm.} \tag{2}$$

which we have again found to be in agreement with the application of the pinch rules.

The Bern-Dunbar-Shimada formalism(1)

Z. Bern, D. Dunbar, T. Shimada, PLB 312, 277 (1993)
Master formula for the irreducible one-loop N-graviton amplitudes with a massless scalar loop:

$$
\begin{aligned}
\Gamma[k_1, h_1; \cdots; k_n, h_n] \quad &= \quad -(-\frac{\kappa}{4})^n \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \int_0^T d\tau_1 \cdots \int_0^T d\tau_n \\
&\times \exp\biggl\{ \sum_{i,j=1}^n \Bigl[\frac{1}{2} G_{ij} k_i \cdot k_j - i(\dot{G}_{ij}\varepsilon_i + \dot{\bar{G}}_{ij}\bar{\varepsilon}_i) \cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i \cdot \varepsilon_j \\
&+ \frac{1}{2}\ddot{\bar{G}}_{ij}\bar{\varepsilon}_i \cdot \bar{\varepsilon}_j + \frac{1}{2} H_{ij}(\varepsilon_i \cdot \bar{\varepsilon}_j + \varepsilon_j \cdot \bar{\varepsilon}_i) \Bigr] \biggr\} \Big|_{\varepsilon_1 \ldots \varepsilon_n \bar{\varepsilon}_1 \ldots \bar{\varepsilon}_n}
\end{aligned}
$$

Here we have used that on-shell the graviton polarisations can be chosen so as to factorize, $h_i^{\mu\nu} = \varepsilon_i^\mu \bar{\varepsilon}_i^\nu$. In the absence of the terms with H_{ij} this would, after the expansion of the exponent, lead to a prefactor polynomial that simply factorizes into two copies of the one of the gluonic case,

$$
\exp\Bigl\{\cdot\Bigr\}\Big|_{\varepsilon_1 \ldots \varepsilon_n \bar{\varepsilon}_1 \ldots \bar{\varepsilon}_n} = P_n(\dot{\bar{G}}_{ij}, \ddot{\bar{G}}_{ij}) P_n(\dot{G}_{ij}, \ddot{G}_{ij}) \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^n G_{ij} k_i \cdot k_j}
$$

At the string level, this comes from the factorisation of the closed string modes into left-movers and right-movers. The additional terms involving H_{ij} stem from the fact that the left- and right-movers are coupled through the zero mode of the string.

The Bern-Dunbar-Shimada formalism(2)

The integration-by-parts can be done independently for the left- and right-movers, except for derivatives hitting the universal exponent where the following identities have to be used,

$$
\begin{aligned}
\frac{\partial}{\partial \bar{\tau}_k} \dot{G}_{ij} &= \frac{1}{2}(\delta_{ki} H_{ij} - \delta_{kj} H_{ij}) \\
\frac{\partial}{\partial \tau_k} \dot{\bar{G}}_{ij} &= \frac{1}{2}(\delta_{ki} H_{ij} - \delta_{kj} H_{ij}) \\
\frac{\partial}{\partial \bar{\tau}_k} \ddot{G}_{ij} &= 0 \\
\frac{\partial}{\partial \tau_k} \ddot{\bar{G}}_{ij} &= 0
\end{aligned}
$$

The H_{ij} are to be treated as constants in the integration-by-parts.

After the removal of the $\ddot{G}_{ij}$, $\ddot{\bar{G}}_{ij}$, the inclusion of the reducible contributions can be achieved by a pinching procedure that is a doubling-up of the one for the gluon case above,

$$
\dot{G}_{ij} \dot{\bar{G}}_{ij} \to \frac{4}{s_{ij}}
$$

After the recursive removal of all trees attached to the loop one has at hand a parameter integral representation for the full on-shell N-graviton matrix element with a scalar loop. Representations for other spins in the loop (Weyl fermion, vector, gravitino, graviton) can again be obtained from this by certain loop replacement rules that are essentially independent applications of the QCD rules to the left- and right-mover parts.

Constructing the tree-level N-graviton amplitude (1)

N. Ahmadiniaz, F.M. Balli, O. Corradini, C. Lopez-Arcos, A. Quintero Velez and C. S., NPB 975 (2022) 115690

We define a "double pinch operator"

$$\bar{\mathcal{D}}_{ij}\mathcal{D}_{ij}\bar{f}_n(\dot{\bar{G}})f(\dot{G}) = \left(\frac{\partial}{\partial\dot{\bar{G}}_{ij}}\bar{f}(\dot{\bar{G}})\Big|_{\substack{\dot{\bar{G}}_{ij}=0\\ \dot{\bar{G}}_{jk}\to\dot{\bar{G}}_{ik}}}\right)\left(\frac{\partial}{\partial\dot{G}_{ij}}f(\dot{G})\Big|_{\substack{\dot{G}_{ij}=0\\ \dot{G}_{jk}\to\dot{G}_{ik}}}\right)$$

and the "maximal pinch" as in the Yang-Mills case. It turns out that terms involving the left-right correlator H_{lj} do not survive the maximal pinching, so that a maximal pinch always factorizes. As in the Yang-Mills case, the gravitational Berends-Giele currents of multiplicity N can be obtained from the $(N-2)$ - tail:

$$\begin{aligned}\bar{\mathcal{D}}_{1(n-1)}\mathcal{D}_{1(n-1)}\bar{\mathcal{D}}_{1(n-2)}\mathcal{D}_{1(n-2)}\cdots\bar{\mathcal{D}}_{13}\mathcal{D}_{13}\bar{\mathcal{D}}_{12}\mathcal{D}_{12}\,\bar{T}(1,2,\ldots,n-2)T(1,2,\ldots,n-2)&\\ =\bar{\varepsilon}^{\mu}_{12\cdots(n-2)}\varepsilon^{\nu}_{12\cdots(n-2)}k_{(n-1)\mu}k_{(n-1)\nu}\,.&\end{aligned}$$

They are just simply squares of the Yang-Mills multi-particle polarisation tensors derived above:

Constructing the tree-level N-graviton amplitude (2)

Up to $N = 5$:

$$\mathcal{G}_1^{\mu\nu} = \bar{\varepsilon}_1^{\mu} \varepsilon_1^{\nu},$$

$$\mathcal{G}_{12}^{\mu\nu} = \frac{\bar{\varepsilon}_{[1,2]}^{\mu} \varepsilon_{[1,2]}^{\nu}}{s_{12}},$$

$$\mathcal{G}_{123}^{\mu\nu} = \frac{\bar{\varepsilon}_{[[1,2],3]}^{\mu} \varepsilon_{[[1,2],3]}^{\nu}}{s_{12} s_{123}} + \frac{\bar{\varepsilon}_{[[1,3],2]}^{\mu} \varepsilon_{[[1,3],2]}^{\nu}}{s_{13} s_{123}} + \frac{\bar{\varepsilon}_{[[2,3],1]}^{\mu} \varepsilon_{[[2,3],1]}^{\nu}}{s_{23} s_{123}},$$

$$\begin{aligned}
\mathcal{G}_{1234}^{\mu\nu} = {} & \frac{\bar{\varepsilon}_{[[[1,2],3],4]}^{\mu} \varepsilon_{[[[1,2],3],4]}^{\nu}}{s_{12} s_{123} s_{1234}} + \frac{\bar{\varepsilon}_{[[[1,2],4],3]}^{\mu} \varepsilon_{[[[1,2],4],3]}^{\nu}}{s_{12} s_{124} s_{1234}} + \frac{\bar{\varepsilon}_{[[[1,3],4],2]}^{\mu} \varepsilon_{[[[1,3],4],2]}^{\nu}}{s_{13} s_{134} s_{1234}} + \frac{\bar{\varepsilon}_{[[[2,3],4],1]}^{\mu} \varepsilon_{[[[2,3],4],1]}^{\nu}}{s_{23} s_{234} s_{1234}} \\
& + \frac{\bar{\varepsilon}_{[[[1,3],2],4]}^{\mu} \varepsilon_{[[[1,3],2],4]}^{\nu}}{s_{13} s_{123} s_{1234}} + \frac{\bar{\varepsilon}_{[[[1,4],2],3]}^{\mu} \varepsilon_{[[[1,4],2],3]}^{\nu}}{s_{14} s_{124} s_{1234}} + \frac{\bar{\varepsilon}_{[[[1,4],3],2]}^{\mu} \varepsilon_{[[[1,4],3],2]}^{\nu}}{s_{14} s_{134} s_{1234}} + \frac{\bar{\varepsilon}_{[[[2,3],1],4]}^{\mu} \varepsilon_{[[[2,3],1],4]}^{\nu}}{s_{23} s_{123} s_{1234}} \\
& + \frac{\bar{\varepsilon}_{[[[2,4],1],3]}^{\mu} \varepsilon_{[[[2,4],1],3]}^{\nu}}{s_{24} s_{124} s_{1234}} + \frac{\bar{\varepsilon}_{[[[2,4],3],1]}^{\mu} \varepsilon_{[[[2,4],3],1]}^{\nu}}{s_{24} s_{234} s_{1234}} + \frac{\bar{\varepsilon}_{[[[3,4],1],2]}^{\mu} \varepsilon_{[[[3,4],1],2]}^{\nu}}{s_{34} s_{134} s_{1234}} + \frac{\bar{\varepsilon}_{[[[3,4],2],1]}^{\mu} \varepsilon_{[[[3,4],2],1]}^{\nu}}{s_{34} s_{234} s_{1234}} \\
& + \frac{\bar{\varepsilon}_{[[1,2],[3,4]]}^{\mu} \varepsilon_{[[1,2],[3,4]]}^{\nu}}{s_{12} s_{34} s_{1234}} + \frac{\bar{\varepsilon}_{[[1,3],[2,4]]}^{\mu} \varepsilon_{[[1,3],[2,4]]}^{\nu}}{s_{13} s_{24} s_{1234}} + \frac{\bar{\varepsilon}_{[[1,4],[2,3]]}^{\mu} \varepsilon_{[[1,4],[2,3]]}^{\nu}}{s_{14} s_{23} s_{1234}},
\end{aligned}$$

Constructing the tree-level N-graviton amplitude (3)

Following Berends-Giele, the tree-level N - graviton amplitude now reads

$$\mathcal{M}_N^{\mathrm{tree}} = s_{12\cdots(N-1)}\mathcal{G}^{\mu\nu}_{12\cdots(N-1)}\mathcal{G}_{N\mu\nu}\,.$$

Not surprisingly, the previous expression takes the well-known double-copy form of the tree-level gravity amplitudes,

$$\mathcal{M}_N^{\mathrm{tree}} = \sum_{\Gamma} \frac{\bar{n}_\Gamma n_\Gamma}{\prod_{e\in\Gamma} s_e}$$

which is equivalent to the KLT formula (Z. Bern, J.J.M. Carrasco, M. Chiodaroli, H. Johansson, R. Roiban, arXiv:1909.01358).
We have also checked our result up to degree $N = 5$ for particular polarisations.

Worldline approach to gravity

In gravity, the differences between the string-based and string-inspired approaches become much more pronounced than in gauge theory:

- On-shell vs. Off-shell.
- Worldline Lagrangians in gravity involve various ghost fields.
- No left-right structure $\longrightarrow$ proliferation of Wick contractions.

Path integrals in curved space

To include background gravity, naively,

$$S_0 = \frac{1}{4}\int_0^T d\tau \dot{x}^2 \quad \rightarrow \quad \frac{1}{4}\int_0^T d\tau \dot{x}^\mu g_{\mu\nu}(x(\tau))\dot{x}^\nu$$

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$$

$\rightarrow$ Graviton vertex operator $\varepsilon_{\mu\nu} \int_0^T d\tau \, \dot{x}^\mu \dot{x}^\nu \, \mathrm{e}^{ik\cdot x}$

But this leads to ill-defined expressions involving $\delta(0), \delta^2(\tau_i - \tau_j), \ldots$

This was to be expected already from the nonrelativistic QM case!

If you like excitement, conflict and controversy...then you will love the history of quantization on curved spaces...

L. Schulman, *Techniques and Application of Path Integration*, Wiley, 1981.

Non-relativistic particle in curved space

$$S[x] = \int_0^T dt \left(\frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + V(x) \right)$$

Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu + V(x)$$

Ordering ambiguity $\Rightarrow$ one parameter family of quantum hamiltonians,

$$\hat{H} = -\frac{1}{2} \nabla^2 + V(x) + \xi R$$

ξ cannot be determined from first principles (it is, in fact, a free parameter of the Standard Model through the coupling of the Higgs to gravity).

The work of DeWitt

B.S. DeWitt, RMP 29, 377 (1957):
Calculating the transition amplitude $\langle x_f\,, t_f | x_i\,, t_i \rangle$ both using the Schrödinger equation and the path integral

$$\begin{aligned} \langle x_f\,, t_f \mid x_i\,, t_i \rangle &= \langle x_f \mid \mathrm{e}^{-\frac{1}{\hbar}(t_f - t_i)\hat{H}} \mid x_i \rangle \\ \langle x_f\,, t_f \mid x_i\,, t_i \rangle &= \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x\ \mathrm{e}^{-S} \end{aligned}$$

DeWitt found that an additional term was needed in the worldline Lagrangian,

$$\hat{H} \to \hat{H} + \xi R$$

$\xi = -\frac{1}{6}$ (later corrected to $\xi = -\frac{1}{8}$).

Longstanding controversy:

- Is this term really needed, or are we just not constructing the path integral properly?
- What is its coefficient?

The worldline formalism in curved space

1993-2006 F. Bastianelli, P. van Nieuwenhuizen and collaborators:
Systematic study of the curved space path integral from the point of view of 1D quantum field theory (σ-model).

F. Bastianelli & P. van Nieuwenhuizen, NPB 389 (1993) 53
F. Bastianelli, K. Schalm and P. van Nieuwenhuizen, PRD 58: 044022 (1998)
K. Schalm and P. van Nieuwenhuizen, PLB 446, 247 (1999)
F. Bastianelli and O. Corradini, PRD 60: 044014 (1999)
F. Bastianelli, O. Corradini, & P. van Nieuwenhuizen PLB 490 (2000) 154; PLB 494, 161 (2000)
F. Bastianelli, O. Corradini, and A. Zirotti, JHEP 0401:023, 2004

Summarized in
F. Bastianelli and P. van Nieuwenhuizen, *Path integrals and anomalies in curved space*, Cambridge University Press 2006.

Two basic problems of the worldline path integral in curved space

1. Nontrivial path integral measure
   T.D. Lee and C.N. Yang, Phys. Rev. 128 (1962) 885

$$Dx \to \mathcal{D}x \quad = \quad Dx \prod_{0 \le \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))}$$

2. Spurious UV divergences of the path integral

The measure

Exponentiate the nontrivial path integral measure,

$$\begin{aligned} \mathcal{D}x &= Dx \prod_{0\le\tau<T} \sqrt{\det g_{\mu\nu}(x(\tau))} \\ &= Dx \int_{PBC} DaDbDc\, \mathrm{e}^{-S_{gh}[x,a,b,c]} \end{aligned}$$

with Faddeev-Popov type ghost action

$$S_{gh}[x,a,b,c] = \int_0^T d\tau\, \frac{1}{4} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)$$

(a bosonic, b, c fermionic)

$$\sqrt{\det g_{\mu\nu}(x(\tau))} = \frac{\det g_{\mu\nu}(x(\tau))}{\sqrt{\det g_{\mu\nu}(x(\tau))}}$$

$$\begin{aligned} \int_{PBC} Da\, \mathrm{e}^{-\int_0^T d\tau\, \frac{1}{4} g_{\mu\nu}(x) a^\mu a^\nu} &= \frac{1}{\sqrt{\det g_{\mu\nu}(x(\tau))}} \\ \int_{PBC} DbDc\, \mathrm{e}^{-\int_0^T d\tau\, \frac{1}{4} g_{\mu\nu}(x) b^\mu c^\nu} &= \det g_{\mu\nu}(x(\tau)) \end{aligned}$$

Correlators

$$\begin{aligned} \langle a^\mu(\tau_1) a^\nu(\tau_2)\rangle &= 2\delta(\tau_1-\tau_2)\delta^{\mu\nu} \\ \langle b^\mu(\tau_1) c^\nu(\tau_2)\rangle &= -4\delta(\tau_1-\tau_2)\delta^{\mu\nu} \end{aligned}$$

Now,

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$$

leads to the new graviton vertex operator

$$V^h_{\mathrm{scal}}[k,\varepsilon] = \varepsilon_{\mu\nu}\int_0^T d\tau \left[\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu\right] \mathrm{e}^{ik\cdot x}$$

After regularization, the ghost field contributions will cancel all divergent terms.
For example, Wick contracting the vertex operator itself

$$\left\langle \dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\right\rangle = \delta^{\mu\nu}(\ddot{G}_B(\tau,\tau) + 2\delta(0) - 4\delta(0))$$

But $\ddot{G}_B(\tau,\tau) = 2\delta(0) - \frac{2}{T}$, so the $\delta(0)$'s cancel!

Similarly, get cancellations of $\delta^2(\tau_i-\tau_j), \delta^3(\tau_i-\tau_j), \ldots$.

Renormalization

These cancellations leave integrals with finite ambiguities, e.g.,

$$\int_0^T \int_0^T d\tau_1 d\tau_2 \, \delta(\tau_1 - \tau_2)\theta(\tau_1 - \tau_2)\theta(\tau_2 - \tau_1)$$

$\Rightarrow$ need finite worldline counterterms to get the correct transition amplitude.
The curved space σ model behaves effectively like a UV divergent but renormalizable 1D QFT. Moreover, it turns out to be super-renormalizable $\Rightarrow$ only a small number of counterterms needed. Those are regularization dependent and in general noncovariant:

$$\Delta S_{\rm reg} = \int_0^T d\tau \, V_{\rm reg}$$

Counterterms

Time slicing:

$$V_{TS} \quad = \quad -\frac{1}{4}R - \frac{1}{12}g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma^{\lambda}_{\mu\alpha}\Gamma^{\rho}_{\nu\beta}$$

Mode regularization:

$$V_{MR} \quad = \quad -\frac{1}{4}R + \frac{1}{4}g^{\mu\nu}\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}$$

1D dimensional regularization:

$$V_{DR} \quad = \quad -\frac{1}{4}R \ .$$

Only known covariant regularization
H. Kleinert & A. Chervyakov, PLB 464 (1999) 257
Once a regularization scheme has been fixed and the counterterms have been determined, there are no further ambiguities.

Spinors coupling to gravity

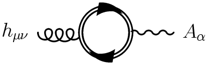
The ξ - term for the scalar coupling to gravity causes a proliferation of terms at higher orders.
For spin half fermions, the worldline coupling to gravity is linear.
Higher-valence vertices are generated only through delta functions arising from the Wick contractions. This suggests that the formalism might be very suitable for a direct numerical calculation of the worldline path integral (Worldline Monte Carlo).
Work along these directions has already started (O. Corradini and M. Muratori, JHEP 11 (2020) 169).

Some applications

- Anomalies (Chiral anomaly, conformal anomaly . . .)
  L. Alvarez-Gaumé 1983, L. Alvarez-Gaumé 1983 and E. Witten 1984, D. Friedan and P. Windey 1984, . . .
- One loop graviton self energy (scalar and spinor loop)
  F. Bastianelli and A. Zirotti, NPBB 642 (2002) 372.
- One loop graviton self energy (graviton loop)
  F. Bastianelli and R. Bonezzi, JHEP 07 (2013), 016; J. Phys. Conf. Ser. 1208 (2019) 1, 012009.
- One loop photon vacuum polarization in a generic gravitational background (scalar loop, semiclassical approximation)
  T. Hollowood and G. Shore, PLB 655 (2007) 67.

Some applications involving electromagnetic fields

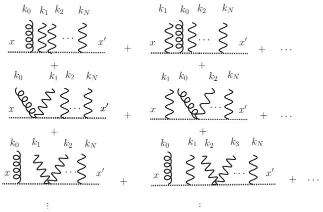
- Leading gravitational corrections to the Euler-Heisenberg Lagrangian
  F. Bastianelli, J.M. Dávila and C.S. 2009, JHEP 03 (2009) 086.
- F. Bastianelli, U. Nucamendi, C. Schubert and V.M. Villanueva, JHEP **0711** (2007) 099
  One-loop photon-graviton conversion in a magnetic field



  M. Ahlers, J. Jaeckel, and A. Ringwald, PRD 79, 075017 (2009)
  (leading source of dichroism in the standard model)

Amplitudes with N photons and one graviton (1)

N. Ahmadiniaz, F. Balli, F. Bastianelli, O. Corradini, J. M. Dávila, C.S., NPB 950 (2020) 114877:
Master formula for the scalar propagator dressed with N photons and one graviton:



$$\widetilde{D}^{(N,1)}(p,p';\varepsilon_1,k_1,\dots,\varepsilon_N,k_N;\epsilon,k_0) = (-ie)^N\left(-\frac{\kappa}{4}\right)\int_0^\infty dT e^{-T(m^2+p'^2)}\prod_{l=0}^{N}\int_0^T d\tau_l$$

$$\times \exp\Big\{(p'-p)\cdot\sum_{l=0}^{N}(-k_l\tau_l+i\varepsilon_l)+\sum_{l<l'=0}^{N}\Big(k_l\cdot k_{l'}|\tau_l-\tau_{l'}|+i(\varepsilon_{l'}\cdot k_l-\varepsilon_l\cdot k_{l'})\mathrm{sgn}(\tau_l-\tau_{l'})$$

$$+2\varepsilon_l\cdot\varepsilon_{l'}\,\delta(\tau_l-\tau_{l'})\Big)\Big\}\Big|_{\mathrm{m.l.}},$$

where $\epsilon_{\mu\nu} := \lambda_\mu\rho_\nu$, $\varepsilon_{0\mu} := \lambda_\mu+\rho_\mu$, 'm.l.' stands for 'multilinear' i.e. linear in all ε_l, $l = 1,\dots,N$ and linear in λ and ρ.

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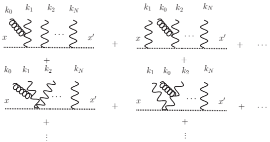
Amplitudes with N photons and one graviton (2)

The reducible contributions at the one-graviton level look like



and they can be generated from the line dressed just with N photons by the (on-shell) replacement rules

$$
\begin{aligned}
k_i \quad &\longrightarrow \quad k_i + k_0 \equiv k' \\
\epsilon_i \quad &\longrightarrow \quad \upsilon_i \equiv -\kappa \frac{\{\epsilon_0, f_i\} \cdot k'}{k'^2} = -\kappa \frac{\epsilon_0 \cdot f_i \cdot k_0 + f_i \cdot \epsilon_0 \cdot k_i}{2k_0 \cdot k_i}
\end{aligned}
$$

This rule can also be derived in the perturbiner approach (C. Lopez-Arcos, A. Quintero Velez and C.S, work in progress).

Amplitudes with N photons and one graviton (3)

N. Ahmadiniaz, F. Balli, F. Bastianelli, O.Corradini, J. M. Dávila, J. P. Edwards, C.S., Stefan Theisen (in preparation):
Master formula for the scalar loop dressed with N photons and one graviton, and explicit results for the low-energy limit (soft-graviton theorems....)

Classical black hole scattering

A classical version of the worldline formalism has shown to be very efficient for the calculation of classical black hole scattering.
G. Mogull, J. Plefka, J. Steinhoff, JHEP 02 (2021) 048; G.U. Jakobsen, G. Mogull, J. Plefka, Y. Xu, 2306.01714 [hep-th].
Here the counterterm problematics does not appear because they are of order $\hbar^2$.

To do list

- Find a direct construction of the tails (not through IBP).
- Prove the equivalence of pinching with multiparticle polarizations for Bern-Kosower and Bern-Dunbar-Shimada.
- Combining the worldline and perturbiner approaches.
- Form factor decomposition of the off-shell N - graviton vertices.
- Graviton amplitudes with a massive loop and application to gravitational pair creation.
- etc.

THANK YOU FOR YOUR ATTENTION!