MODULAR CONSTRAINTS IN $\mathcal{N}=4$ SYM / TYPE IIB HOLOGRAPHY

(WITH ARBITRARY CLASSICAL GAUGE GROUP)

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Based on

arXiv:2102.08305, Phys. Rev. Lett. 126 (2021) no.16, 161601

arXiv:2102.09537, JHEP 05 (2021) 089

arXiv:2109.08086, JHEP 11 (2021) 132

arXiv:2202.05784, *SciPost Phys.* 13 (2022) 4, 092

arXiv:2210.14038, JHEP 04 (2023) 114 (with Haitian Xie)

Earlier work with large N

Chester, MBG, Pufu, Wang, Wen arXiv:1912.13365 and arXiv:2008.02713

See also:

Binder, Chester, Pufu, Wang, arXiv:1902.06263, Chester, Pufu, arXiv:2003.08412, Chester, arXiv:1908.05207

Alday, Chester, Hanson, arXiv:2110.13106, Collier, Perlmutter arXiv:2201.05093, Wen, Zhang arXiv:2203.01890.

Outline

Supersymmetric localization will be used in order to determine the exact form of certain

INTEGRATED CORRELATION FUNCTIONS

in $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory (and generalisations to $SO(N)$ $USp(2N)$)

Integration averages over the spatial dependence

Non-local supersymmetric observables

But these correlators nevertheless contain a great deal of information:

- Determine SYM perturbation theory coefficients to all orders – BOTH PLANAR AND NON-PLANAR, FOR ALL VALUES OF N .
- Include detailed coefficients of infinite set of INSTANTON and ANTI-INSTANTON contributions.
- The large-N expansion reproduces known facts about low-energy expansion of HOLOGRAPHICALLY DUAL IIB superstring.
- Makes S-duality - **Modular Invariance** $SL(2,\mathbb{Z})$- manifest:

Montonen-Olive Duality of gauge theory $\leftrightarrow$ **S-Duality** of Type IIB

- The $1/N$ expansion is an asymptotic series and has an explicit S-dual completion $O(e^{-\sqrt{N}A})$

(p,q) String world-sheet instantons

Supersymmetric Localization

- $\mathcal{N}=2^*$ supersymmetric YM $\rightarrow \mathcal{N}=4$ in the limit in which the mass of hypermultiplet in adjoint rep. vanishes

$$\mathcal{N}=2^* \underset{m\to 0}{\rightarrow} \mathcal{N}=4$$

- The $\mathcal{N}=2^*$ partition function on S^4 is determined by supersymmetric localization. [Pestun arXiv:0712.2824]

- Localized partition function of $\mathcal{N}=2^*$ with $SU(N)$ gauge group is a $(N-1)$-dimensional integral over the Lie algebra $\mathfrak{su}(N)$. $SU(N)$ hermitian matrix model (integrate over VEV's of coulomb branch vector multiplet).

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{YM}} = \tau_1 + i\tau_2$$

classical localized action

$$Z_N(m,\tau,\bar\tau) = \int d^N a\, \delta(\sum_i a_i) \prod_{i<j}(a_i - a_j)^2\, e^{-\frac{8\pi^2}{g^2_{YM}}\sum_j a_j^2}\, \mathcal{Z}_{pert}(m, a_{ij})\, |\mathcal{Z}_{inst}(m, a_{ij}, \tau)|^2$$

mass parameter

Vandermonde determinant

perturbative factor

Nekrasov instanton partition function

$\mathcal{Z}_{pert}(m, a_i)$ is the one-loop determinant factor and is expressed in terms of a standard function (the **Barnes G-function**)

$\mathcal{Z}_{inst}(m, a_i)$ describes Coulomb branch instantons at the south pole and anti-instantons at the north pole of S^4. (Express as a sum of Young diagrams) [Nekrasov]

$m \to 0$ Limit

- The partition function of $\mathcal{N}=4$ SYM $Z_N(0,\tau,\bar\tau) = 1$

- But the $m=0$ limit of derivatives of $Z_N(m,\tau,\bar\tau)$ with respect to m may be nontrivial as we will see.

THE $\mathcal{N} = 2^*$ PARTITION FUNCTION

$$Z_N(m,\tau,\bar{\tau}) = \int d^{N-1}a\,\mu(a_i)\,e^{-\frac{8\pi^2}{g_{YM}^2}\sum_j a_j^2}\,\mathcal{Z}_{pert}(m,a_{ij})\,|\mathcal{Z}_{inst}(m,a_{ij},\tau)|^2$$

PERTURBATIVE

$$Z_{pert}(m,a_{ij} = \prod_{i<j}\frac{a_{ij}^2 H^2(a_{ij})}{H(a_{ij}-m)H(a_{ij}+m)}$$

$$H(z) = e^{-(1+\gamma)z^2}\,G(1+iz)\,G(1-iz)$$

Barnes G-function defined by:

$$\log G(1+z) = \frac{z}{2}\log 2\pi - \left(\frac{z+(1+\gamma)z^2}{2}\right) + \sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k+1}z^{k+1}$$

INSTANTON TERMS:

The Nekrasov instanton partition function: contribution from instantons at North Pole and anti-instantons at South Pole of S^4

$$\mathcal{Z}_{inst}(m,\tau,a_{ij}) = \sum_{k=0}^{\infty} e^{2\pi i k\tau}\,\mathcal{Z}_{inst}^{(k)}(m,a_{ij})$$ Fourier sum (sum over instanton number)

Using Nekrasov's result we have for k INSTANTONS:

$$Z_{inst}^{(k)}(m,a_{ij}) = \frac{1}{k!}\left(\frac{2m^2}{m^2+1}\right)^k \oint \prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\prod_{i=1}^{N}\frac{(\phi_I-a_I)^2-m^2}{(\phi_I-a_i)^2+1}\prod_{I<J}^{k}\frac{\phi_{IJ}^2(\phi_{IJ}^2+4)(\phi_{IJ}^2-m^2)^2}{(\phi_{IJ}^2+1)((\phi_{IJ}-m)^2+1)((\phi_{IJ}+m)^2+1)}$$

where the integration contour circles the poles in a particular (and complicated) manner (sum of Young diagrams).

Integrated Four-Point Correlation Functions

- Superconformal primary of $\mathcal{N}=4$ stress tensor multiplet.

$$O_2(x,Y) = tr(\phi_{I_1}\,\phi_{I_2})\,Y^{I_1}\,Y^{I_2}$$ $I_1, I_2 = 1,\dots,6$ $\quad Y\cdot Y = 0$ $\quad$ Encodes $SU(4)$ quantum numbers

- Four-point correlator of superconformal primaries,

$$\langle O_2(x_1,Y_1)\,O_2(x_2,Y_2)\,O_2(x_3,Y_3)\,O_2(x_4,Y_4)\rangle = \frac{1}{x_{!2}^4 x_{34}^4}\left(\mathcal{T}_{N\,free}(U,V;Y_i) + \mathcal{I}_4(x_i;Y_i)\mathcal{T}_N(U,V)\right)$$

free correlator $\quad$ determined by symmetries $\quad$ [Eden, Petkou, Schubert, Sokatche

- Correlator is not supersymmetric but **integrated** correlator is **[Binder, Chester, Pufu, Wang** arXiv:1902.06263**]**

$$\mathcal{G}_N^i(\tau,\bar\tau) = \int dU\,dV\,\mu^i(U,V)\mathcal{T}_N(U,V,\tau,\bar\tau)$$ $\quad U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ cross ratios

where the measure $\mu^i(U,V)$ is designed to preserve supersymmetry,

Two examples of measures

$$\mathcal{G}_N^1(\tau,\bar\tau) = -\frac{8}{\pi}\int_0^\infty dr \int_0^\pi d\theta\,\frac{r\sin^2\theta}{U^2}\mathcal{T}_N(U,V,\tau,\bar\tau)$$ $\quad U = 1 + r^2 - 2r\cos\theta\,, \quad V = r^2$

$$\mathcal{G}_N^2(\tau,\bar\tau) = -\frac{96}{\pi}\int_0^\infty dr \int_0^\pi d\theta\,\frac{r\sin^2\theta}{U^2}\,\bar D_{1111}(U,V)\,\left(\mathcal{T}_N(U,V,\tau,\bar\tau) + \mathcal{T}_{free}(U,V)\right)$$

box diagram

RELATION TO LOCALISED $\mathcal{N}=2^*$ PARTITION FUNCTION

- Correlators are obtained by four derivatives acting on $Z_N(m,\tau,\bar\tau)$ the partition function of the $\mathcal{N}=2^*$ theory on S^4

$$\mathcal{G}^1_N(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau\,\partial_m^2\,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ Considered in this talk $\Delta_\tau = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$

$$\mathcal{G}^2_N(\tau,\bar\tau) = \partial_m^4\,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ Briefly considered in this talk

- Equality with integrated correlators on R^4 shown in [Binder, Chester, Pufu, Wang, arXiv:1902.06263]

  Uses supersymmetric Ward identities and accounts for operator mixing on S^4.

- Analysis of $\mathcal{G}_N \equiv \mathcal{G}^1_N$ is complicated.
  - Consider the exact perturbation expansion for many values of N.
  - Consider the exact 1-instanton contribution for many values of N.
  - Generalise to the k-instanton contribution.

    Only Young diagrams in the Nekrasov partition function with a single rectangular $k=p\times q$ block contribute (up to "partial transpositions").[Chester, MBG, Pufu, Wang, Wen]

LEADS TO A REMARKABLY SIMPLE CONJECTURED EXPRESSION FOR $\mathcal{G}_N(\tau,\bar\tau)$

2 Dim. Lattice Representation

$$\mathcal{G}_N(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B_N(t)dt$$

where $B_N(t) = \dfrac{\mathbb{Q}_N(t)}{(1+t)^{2N+1}}$ and $\mathbb{Q}_N(t)$ is a rational polynomial of order $(2N-1)$.

- $SL(2,\mathbb{Z})$ invariance is manifest: $\tau \to \dfrac{a\tau+b}{c\tau+d}$ $\quad a,b,c,d\in\mathbb{Z}$, $\quad ad-bc=1$

  [Montonen-Olive]

  Relates theories at different values of coupling constant – holographic image of S-duality in type IIB superstring.

- It is important that $B_N(t) = \dfrac{1}{t} B_N(1/t)$, as well as $\displaystyle\int_0^\infty B_N(t)dt = \frac{N(N-1)}{4}$ and $\displaystyle\int_0^\infty B_N(t)\frac{1}{\sqrt{t}}dt = 0$.

e.g. $\quad SU(2)$: $\quad B_2(t) = \dfrac{9t^3-30t^2+9t}{(1+t)^5}$ $\qquad SU(3)$: $\quad B_3(t) = \dfrac{18t^5-99t^4+126t^3-99t^2+18t}{(1+t)^7}$

- General N: obtained by experimental algebraic calculations but recently proved [Dorigoni, MBG, Wen, Xie 2210.140384]

$$\mathbb{Q}_N(t) = -\frac{1}{4}N(N-1)(1-t)^{N-1}(1+t)^{N+1}\left[(3+(8N+3t-6)t)\,P_N^{(1,-2)}(z) + \frac{3t^2-8Nt-3}{t+1}P_N^{(1,-1)}(z)\right]$$

where $z = \dfrac{1+t^2}{1-t^2}$ and $P_N^{(\alpha,\beta)}(z)$ is a Jacobi polynomial.

Perturbation Expansion

'**t Hooft expansion** $a = \frac{g_{YM}^2 N}{4\pi^2} = \frac{N}{\pi}\tau_2^{-1}$

First non-planar contribution

$$\mathcal{G}_{N,0}(\tau_2) = (N^2-1)\Big[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32} + \frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024} + \frac{32207175\,\zeta(15)\left(1+\frac{55}{13}N^{-2}+\frac{332}{143}N^{-4}\right)a^7}{2048} + \mathcal{O}(a^8)\Big]\,,$$

- Coefficients are RATIONAL MULTIPLES OF ODD ZETA VALUES.
- Recall the UNINTEGRATED CORRELATOR has very complicated dependence on cross ratios involving polylogs,

e.g. $L=1,\ 2$

$$f^{(L)}(z,\bar z) = \sum_{r=0}^{L} \frac{(-1)^r(2L-r)!}{r!(L-r)!L!}\log^r(z\,\bar z)\left(\mathrm{Li}_{2L-r}(z) - \mathrm{Li}_{2L-r}(\bar z)\right)$$

$z\bar z = U \qquad (1-z)(1-\bar z) = V$

- The **INTEGRATED CORRELATOR is much simpler.** The coefficients can be compared with calculations from Feynman diagrams. [Belokurov and Usyukina, 1983] [Usyukina, 1991] [Wen and Zhang 2022]
- **NON-PLANAR CORRECTIONS** BEGIN AT FOUR LOOPS – as is known from Feynman perturbation theory. Interesting pattern of non-planarity determined to arbitrary order.

[Eden, Heslop, Korchemsky, Sokatchev] [Boels, Kniehl, Tarasov, Yang] [Fleury and Pereira]

Differential Recurrence Relation

- Using the differential recurrence relation of Jacobi functions we find that

$$t\,\frac{d^2}{dt^2}(t\,B_N(t)) \;= N(N-1)B_{N+1}(t) - 2(N^2-1)B_N(t) + N(N+1)B_{N-1}(t)$$

recall $\mathcal{G}_N(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}}\, B_N(t)dt$

- From which one can show that the integrated correlator satisfies a

Laplace difference equation:

$$(\Delta_\tau - 2)\,\mathcal{G}_N(\tau,\bar\tau) = N^2\left[\mathcal{G}_{N+1}(\tau,\bar\tau) - 2\mathcal{G}_N(\tau,\bar\tau) + \mathcal{G}_{N-1}(\tau,\bar\tau)\right] - N\left[\mathcal{G}_{N+1}(\tau,\bar\tau) - \mathcal{G}_{N-1}(\tau,\bar\tau)\right]$$

where $\Delta_\tau = \tau_2^2\left(\partial^2_{\tau_1} + \partial^2_{\tau_2}\right)$ is the hyperbolic laplacian.

- Since $\mathcal{G}_1 = 0$ this equation determines $\mathcal{G}_N(\tau,\bar\tau)$ for all $N>2$ In terms of $\mathcal{G}_2(\tau,\bar\tau)$.

- Solutions can be expressed in terms of NON-HOLOMORPHIC EISENSTEIN SERIES

Non-Holomorphic Eisenstein Series

$$E(s,\tau,\bar\tau) = \frac{1}{\pi^s}\sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} = \sum_{(m,n)\neq(0,0)} \frac{1}{\Gamma(s)}\int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} t^{s-1}dt \quad = \sum_{k\in\mathbb{Z}} \mathcal{F}_k(s;\tau_2)\, e^{2\pi i k\tau_1} \qquad s\in\mathbb{C}$$

↑ Fourier modes

Modular function $SL(2,\mathbb{Z}):\ \tau \to \frac{a\tau+b}{c\tau+d}$

$a,b,c,d\in\mathbb{Z} \quad ad-bc=1$

- Zero mode (perturbative)

$$\mathcal{F}_0(s;\tau_2) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\,\Gamma(s-\frac12)\zeta(2s-1)}{\pi^s\Gamma(s)}\,\tau_2^{1-s}$$

TWO POWER-BEHAVED TERMS

τ_2^s ↑ $(4\pi/g^2_{_{YM}})^s$ — PERTURBATIVE (singular)

τ_2^{1-s} ↑ $(g^2_{_{YM}}/4\pi)^{s-1}$ — PERTURBATIVE

- Non-zero modes (instantons)

divisor sum ↙ Bessel ↙

$$\mathcal{F}_k(s;\tau_2) = \frac{4}{\Gamma(s)}\,|k|^{s-\frac12}\,\sigma_{1-2s}(|k|)\sqrt{\tau_2}\,K_{s-\frac12}(2\pi|k|\tau_2)\,, \quad k\neq 0 \qquad \sigma_r(k)=\sum_{d|k} d^r$$

$$\underset{\tau_2\to\infty}{\sim} (\dots)\, e^{-2\pi|k|\tau_2}$$

characteristic of INSTANTON or ANTI-INSTANTON

- **LAPLACE EIGENVALUE EQUATION** $(\Delta_\tau - s(s-1))\,E(s;\tau,\bar\tau) = 0$

Expression for Integrated Correlator

Formal Infinite sum of Eisenstein series (with integer-index)

$$\mathcal{G}_N(\tau,\bar\tau) = \frac{N(N-1)}{8} + \frac{1}{2}\sum_{s=2}^{\infty} c_N(s)\, E(s,\tau,\bar\tau)$$

where the coefficients are given by $B_N(t) = \sum_{s=2}^{\infty} c_N(s)\,\frac{t^{s-1}}{\Gamma(s)}$ with $B_N(t) = \frac{1}{t} B_N(1/t)$

e.g. for $SU(2)$ $$c_2(s) = \frac{(-1)^s}{2}(s-1)(1-2s)^2\,\Gamma(s+1)$$

- Perturbative terms.: Infinite sum of $\tau_2^s = (4\pi/g_{YM}^2)^s$ terms $=$ infinite sum of $\tau_2^{1-s} = (g_{YM}^2/4\pi)^{s-1}$ terms!
  after Borel resummation
- Instanton Contributions

e.g. $k=1$ in $SU(2)$

$$\mathcal{G}_{2,k=1}(\tau,\bar\tau) = e^{2\pi i\tau}\left[12y^2 - 3\sqrt{\pi}e^{4y}y^{3/2}(1+8y)\,\mathrm{erfc}\,(2\sqrt{y})\right]$$

$$\underset{g_{YM}^2\to 0}{\sim} e^{2\pi i\tau}\left[-\frac{3}{8} + \frac{9}{32y} - \frac{135}{512y^2} + \frac{315}{1024y^3} + \cdots\right]$$

$$y = \pi\tau_2 = \frac{4\pi^2}{g_{YM}^2}$$

General $k = \hat m n$ and N

$$\mathcal{G}_{N,k}(\tau,\bar\tau) = \frac{1}{2}\sum_{\substack{\hat m\neq 0,\, n\neq 0\\ \hat m n = k}} e^{2\pi(-|k|\tau_2 + ik\tau_1)}\int_0^\infty \exp\left[-\left(\frac{|\hat m|}{\sqrt t} - |n|\sqrt t\right)^2\pi\tau_2\right]\sqrt{\frac{\tau_2}{t}}\,B_N(t)dt\,.$$

LARGE-N EXPANSION

't Hooft Expansion

SUPPRESSES INSTANTONS so duality is not manifest

$$\mathcal{G}_N(\tau,\bar\tau) \sim \sum_{g=0}^{\infty} N^{2-2g}\,\mathcal{G}^{(g)}(\lambda)$$

$\lambda = g^2_{_{YM}}\,N = 4\pi\tau_2^{-1}\,N$
't Hooft coupling

1. Small-λ expansion

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n$$

Radius of convergence $|\lambda| \le \pi^2$

Proportional to N^2
PLANAR DIAGRAMS

BOREL SUM $= \lambda \int_0^{\infty} dw\, w^3 \frac{{}_1F_2\left(\frac{5}{2};2,4;-\frac{w^2\lambda}{\pi^2}\right)}{4\pi^2\,\sinh^2(w)}\,.$

2. Large-λ expansion

$$\mathcal{G}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\lambda^{n+1/2}}$$

Divergent sum

- Asymptotic series that is **not Borel summable**. Requjres non-perturbative completion (resurgence)

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \frac{489\mathrm{Li}_3(e^{-2\sqrt{\lambda}})}{16\lambda^{3/2}} + \cdots\right]$$

- The behaviour $e^{-2\sqrt{\lambda}}$ is characteristic of a **WORLD-SHEET INSTANTON** in string theory since $e^{-2\sqrt{\lambda}} = e^{-2L^2/\alpha'}$.

- Similar analysis for terms with higher powers of $1/N^2$

BUT SINCE WE KNOW THE EXACT FUNCTION WE CAN DETERMINE ITS LARGE-N $SL(2,\mathbb{Z})$ COMPLETION ANALYTICALLY

Generating function for N-dependence

Dorigoni, MBG, Wen, Xie 2210.140384

$$\mathcal{B}_{SU}(z;t) := \sum_{N=1}^{\infty} B_{SU(N)}(t) z^N = \frac{3tz^2\left[(t-3)(3t-1)(t+1)^2 - z(t+3)(3t+1)(t-1)^2\right]}{2(1-z)^{\frac{3}{2}}\left[(t+1)^2-(t-1)^2 z\right]^{\frac{7}{2}}}$$

Note branch cuts

Inverse

$$B_{SU(N)}(t) = \oint_C \frac{\mathcal{B}_{SU}(z;t)}{z^{N+1}}\frac{dz}{2\pi i}$$

Distort contour $= \oint_{C'} \frac{\mathcal{B}_{SU}(z;t)}{z^{N+1}}\frac{dz}{2\pi i} + \oint_{C_\infty} \frac{\mathcal{B}_{SU}(z;t)}{z^{N+1}}\frac{dz}{2\pi i}$

vanishes

Integral around C' $\quad B_{SU(N)}(t) = \int_1^{\frac{(t+1)^2}{(t-1)^2}} \frac{\text{Disc}\mathcal{B}_{SU}(z;t)}{z^{N+1}}\frac{dz}{2\pi i}$

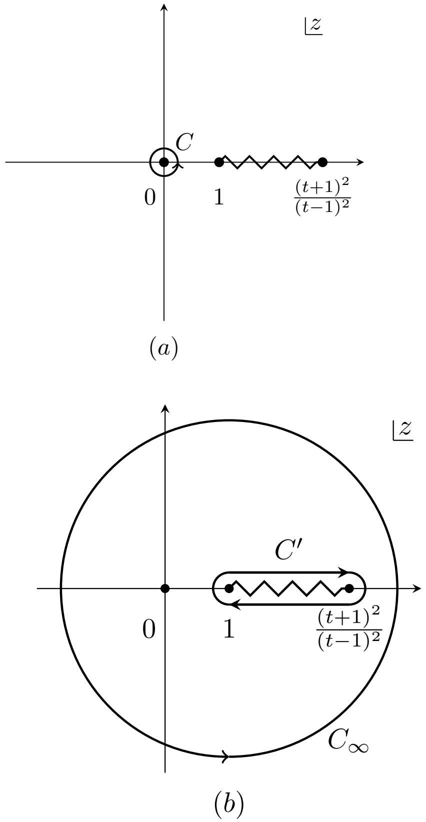
$$= \int_1^{\infty} \dots \quad + \quad \int_{\infty}^{\frac{(t+1)^2}{(t-1)^2}} \dots$$

generates series in ½-integer powers of $\frac{1}{N}$ — $B^{(1)}_{SU(N)}(t)$

non-perturbative in N — $B^{(2)}_{SU(N)}(t)$

$$\sum_{(m,n)\in\mathbb{Z}^2} \int_0^{\infty} e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B^{(2)}_{SU(N)}(t)\,dt$$

Dominated at large N by saddle point of t integral



Modular invariant Large-N Expansion

Fixed - g_{YM}^2 Expansion INSTANTONS NOT SUPPRESSED – S-duality is manifest.

- The $1/N$ expansion is holographically related to the low energy expansion of the dual IIB superstring amplitude in $AdS_5 \times S^5$.

$$N \to \frac{\tau_2 L^4}{\alpha'^2}, \qquad \tau_2 \to \frac{1}{g_s}$$

- Substitute the large-N expansion of $B^{(1)}_{SU(N)}(t)$ (determined by the differential recurrence relation) :

Supergravity $\quad R^4 \quad d^4R^4$

$$\mathcal{G}_N(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac12}}{2^4} E(\tfrac32;\tau,\bar\tau) + \frac{45}{2^8 N^{\frac12}} E(\tfrac52;\tau,\bar\tau)$$
$$+ \frac{3}{N^{\frac32}}\left[\frac{1575}{2^{15}} E(\tfrac72;\tau,\bar\tau) - \frac{13}{2^{13}} E(\tfrac32;\tau,\bar\tau)\right] + \frac{225}{N^{\frac52}}\left[\frac{441}{2^{18}} E(\tfrac92;\tau,\bar\tau) - \frac{5}{2^{16}} E(\tfrac52;\tau,\bar\tau)\right]$$
$$+ \frac{63}{N^{\frac72}}\left[\frac{3898125}{2^{27}} E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}} E(\tfrac72;\tau,\bar\tau) + \frac{73}{2^{22}} E(\tfrac32;\tau,\bar\tau)\right] + O(N^{-\frac92}),$$

Extends the earlier analysis in [Chester, MBG, Pufu, Wang, Wen]

- Series of **½-integer** index Eisenstein series.
- Close connection to well-established BPS terms in low energy expansion of IIB superstring in the flat space limit.
- Note the absence of terms with integer powers of $1/N$, such as the term of order d^6R^4.
  Such terms arise in the $1/N$ expansion of $\mathcal{G}^2_N(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)|_{m=0}$.

Summary of the large-N expansion

$\int_1^\infty \cdots \quad \to$ Series in half-integer powers of $1/N$

$\int_\infty^{\frac{(t+1)^2}{(t-1)^2}} \cdots \to$ Saddle point in t integral exponentially suppressed in N

$$\mathcal{G}_{SU(N)}(\tau,\bar\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{\frac{1}{2}-r} \sum_{m=0}^{\lfloor r/2 \rfloor} b_{r,m} E(\tfrac{3}{2}+\delta_r+2m;\tau,\bar\tau)$$ powers of $1/N$

$$\pm\, i \sum_{r=0}^{\infty} N^{2-\frac{r}{2}} \sum_{m=0}^{r} d_{r,m} D_N\big(\frac{r}{2}-2m;\tau,\bar\tau\big)$$ exponentially suppressed in N

where $$D_N(s;\tau,\bar\tau) = \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp\Big(-4\sqrt{N\pi}\,\ell\frac{|p+q\tau|}{\sqrt{\tau_2}}\Big)\frac{1}{\pi^s}\frac{\tau_2^s}{\ell^{2s}|p+q\tau|^{2s}}\,.$$ novel modular function

$$\sim \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp(-4\pi L^2 \ell\, T_{p,q})$$ uses AdS/CFT dictionary $\sqrt{g_{_{YM}}^2 N} = L^2/\alpha'$

- Tension of (p,q) strings
- Sum over ℓ (p,q) world-sheet instantons. (p,q) string world-sheets wrapping an equatorial S^2 in S^5.
- The $p=1\,,\ q=0$ term reproduces the resurgence expression in the 't Hooft limit.

Integrated Correlators for SO(N), USp(N)

[Dorigoni, MBG, Wen, arXiv:2202.05784]]
(See Alday, Chester and Hansen)

Goddard-Nuyts-Olive duality of magnetic monopoles and electric charges (c.f. Langlands)

(Correlators are not sensitive to global factors)

- Simply-laced Self-duality $SU(N)\,, \quad SO(2N)$ $\quad S: \quad \tau \to -\frac{1}{\tau} \qquad T: \quad \tau \to \tau+1$

  generate $SL(2,\mathbb{Z}) \; : \quad \tau \to \frac{a\tau+b}{c\tau+d}$ $\quad a,b,c,d \in \mathbb{Z}$, $\quad ad-bc=1$

- Non simply-laced $SO(2N+1)\,, \quad USp(2N)$ $\quad \hat{S}: \quad \tau \to -\frac{1}{2\tau} \qquad T: \quad \tau \to \tau+1$

  $\left(\frac{\text{long roots}}{\text{short roots}}\right)^2 = 2$

  $\hat{S}\,T\,\hat{S}$ and T generate $\Gamma_0(2): \quad c = 0 \bmod 2$

  maps $SO(2N+1) \to SO(2N+1)$ and $USp(2N) \to USp(2N)$

  $\hat{S}$ maps $USp(2N) \leftrightarrow SO(2N+1)$

Previously $\mathcal{G}_N(\tau,\bar\tau)$

Integrated correlator $\quad \frac{1}{4}\mathbb{C}_{G_N}(\tau,\bar\tau) = \Delta_\tau\, \partial_m^2 \log Z_{G_N}(m,\tau,\bar\tau)|_{m\to 0}$

Results: Beautiful extension of the SU(N) case

- GNO duality explicit.
- Large-N limit gives results consistent with expected string theory results.
- Set of Laplace difference equations highly constrain results for all N.

Correlators with General Classical Gauge Group G_N

With :

$$\mathbb{C}_{G_N}(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty dt \left(B^1_{G_N}(t) e^{-t\pi \frac{|m+n\tau|^2}{\tau_2}} + B^2_{G_N}(t) e^{-t\pi \frac{|m+2n\tau|^2}{2\tau_2}} \right).$$

- $SU(N), \quad SO(2N)$ Invariance under $SL(2,\mathbb{Z})$ generated by $S: \tau \to -\frac{1}{\tau}, \quad T: \tau \to \tau+1$

$$B^2_{SU(N)}(s) = B^2_{SO(2N)}(s) = 0$$

- $SO(2N+1), \quad USp(2N),$ Invariance under $\Gamma_0(2)$ generated by $\hat{S}\,T\,\hat{S}$ and $T,$ where

$$\hat{S}: \tau \to -\frac{1}{2\tau}, \quad T: \tau \to \tau+1$$ not $\subset SL(2,\mathbb{Z})$

$$B^1_{SO(2N+1)}(t) = B^2_{USp(2N)}(t), \qquad B^1_{USp(2N)}(t) = B^2_{SO(2N+1)}(t)$$

- $\hat{S}$ Interchanges $B^1_{G_N}$ with $B^2_{G_N}$ GNO duality ⟶ $\mathbb{C}_{SO(2N+1)} \to \mathbb{C}_{USP(2N)}$

- **Formal expansion**

$$\mathbb{C}_{G_N}(\tau,\bar\tau) = -b_{G_N}(0) + \sum_{s=2}^{\infty} \left[b^1_{G_N}(s)\, E(s;\tau,\bar\tau) + b^2_{G_N}(s)\, E(s;2\tau,2\bar\tau) \right]$$

$$B^i_{G_N}(t) = \sum_{s=2}^{\infty} \frac{b^i_{G_N}(s)}{\Gamma(s)}\, t^{s-1} \quad i=1,2$$

noting that $E(s;\tau,\bar\tau) \underset{\hat{S}}{\to} E\left(s; -\frac{1}{2\tau}, -\frac{1}{2\bar\tau}\right) = E(s;2\tau,2\bar\tau)$

Yang-Mills Perturbation Expansion

Expansion Parameters

$$a_{SU(N)} = \frac{N g^2_{YM}}{4\pi^2}\,, \qquad a_{SO(n)} = \frac{(n-2) g^2_{YM}}{4\pi^2}\,, \qquad a_{USp(n)} = \frac{(n+2) g^2_{YM}}{8\pi^2}$$

$n = 2N \text{ or } 2N+1$

Proportional to **DUAL COXETER** NUMBERS

$$\mathbb{C}^{pert}_{G_N}(\tau_2) = -\,4c_{G_N}\Big[\frac{3\,\zeta(3)a_{G_N}}{2} - \frac{75\,\zeta(5)a^2_{G_N}}{8} + \frac{735\,\zeta(7)a^3_{G_N}}{16} - \frac{6615\,\zeta(9)\,(1+P_{G_N,1})\,a^4_{G_N}}{32}$$
$$+\frac{114345\,\zeta(11)\,(1+P_{G_N,2})\,a^5_{G_N}}{128} - \frac{3864861\,\zeta(13)\,(1+P_{G_N,3})\,a^6_{G_N}}{1024}$$
$$+\frac{32207175\,\zeta(15)\,(1+P_{G_N,4})\,a^7_{G_N}}{2048} + \mathcal{O}(a^8_{G_N})\Big]\,,$$

Central charge

$$c_{SU(N)} = \frac{N^2-1}{4}\,, \qquad c_{SO(n)} = \frac{n(n-1)}{8}\,, \qquad c_{USp(n)} = \frac{n(n+1)}{8}\,.$$

e.g. $SO(n)$

$$P_{SO(n),1} = -\frac{n^2-14n+32}{14(n-2)^3}\,, \qquad P_{SO(n),2} = -\frac{n^2-14n+32}{8(n-2)^3}$$

$$P_{SO(n),3} = -\frac{12n^4-221n^3+1158n^2-2432n+1856}{22(n-2)^5}$$

$$P_{SO(n),4} = -\frac{2\,(342n^5-7217n^4-48841n^3-153938n^2+239232n-149920)}{715(n-2)^6}$$

- **The "planar" pieces are identical for all gauge groups.**
- Non-planar terms first enter at four loops.
- The transformation $(N\,,g^2_{YM}) \leftrightarrow (-N, -g^2_{YM})$ **Symmetry of** $\mathbb{C}_{SU(N)}(\tau_2)$
  Interchanges $\mathbb{C}_{SO(2N)}(\tau_2)$ **and** $\mathbb{C}_{USp(2N)}(\tau_2)$

Laplace Difference Equations

$$\Delta_\tau \mathbb{C}_{SO(n)}(\tau,\bar\tau) - 2c_{SO(n)}\Big[\mathbb{C}_{SO(n+2)}(\tau,\bar\tau) - 2\,\mathbb{C}_{SO(n)}(\tau,\bar\tau) + \mathbb{C}_{SO(n-2)}(\tau,\bar\tau)\Big]$$

$n = N$ or $2N$

$$- n\,\mathbb{C}_{SU(n-1)}(\tau,\bar\tau) + (n-1)\,\mathbb{C}_{SU(n)}(\tau,\bar\tau) = 0\,.$$

$$\Delta_\tau \mathbb{C}_{USp(n)}(\tau,\bar\tau) - 2c_{USp(n)}\Big[\mathbb{C}_{USp(n+2)}(\tau,\bar\tau) - 2\,\mathbb{C}_{USp(n)}(\tau,\bar\tau) + \mathbb{C}_{USp(n-2)}(\tau,\bar\tau)\Big]$$
$$+ n\,\mathbb{C}_{SU(n+1)}(2\tau,2\bar\tau) - (n+1)\,\mathbb{C}_{SU(n)}(2\tau,2\bar\tau) = 0$$

- Identities $\mathbb{C}_{SO(3)}(\tau,\bar\tau) = \mathbb{C}_{SU(2)}(\tau,\bar\tau)\,, \quad \mathbb{C}_{SO(4)}(\tau,\bar\tau) = 2\,\mathbb{C}_{SU(2)}(\tau,\bar\tau)\,, \quad \mathbb{C}_{SO(6)}(\tau,\bar\tau) = \mathbb{C}_{SU(4)}(\tau,\bar\tau)$

- All integrated correlators can be related to $SU(N)$ correlators, and hence to the $SU(2)$ case.

e.g. $$\mathbb{C}_{SO(7)}(\tau,\bar\tau) = \left[\frac{8}{5}\mathbb{C}_{SU(2)}(\tau,\bar\tau) - \frac{12}{5}\mathbb{C}_{SU(3)}(\tau,\bar\tau) + \frac{3}{5}\mathbb{C}_{SU(4)}(\tau,\bar\tau) + \frac{4}{5}\mathbb{C}_{SU(5)}(\tau,\bar\tau)\right]$$
$$+ \left[\frac{3}{5}\mathbb{C}_{SU(2)}(2\tau,2\bar\tau) - \frac{12}{5}\mathbb{C}_{SU(3)}(2\tau,2\bar\tau) + \frac{8}{5}\mathbb{C}_{SU(4)}(2\tau,2\bar\tau)\right]\,,$$

LARGE-N EXPANSIONS

Expansion parameters $\hat{N}_{SU(N)} = N$ $\quad \hat{N}_{SO(n)} = \frac{n}{2} - \frac{1}{4}$ $\quad \hat{N}_{USp(n)} = \frac{n}{2} + \frac{1}{4}$ $\quad n = 2N \text{ or } 2N+1$

- 't Hooft expansion $\quad \mathbb{C}_{G_N}(\lambda) \sim \sum_{g=0}^{\infty} (\hat{N}_{G_N})^{2-2g} f^{(g)}_{G_N}(\lambda_{G_N})$

where $\quad \lambda_{SU(N)} := g^2_{YM}\, N\,, \quad \lambda_{SO(n)} := g^2_{YM}\left(\frac{n}{2} - \frac{1}{4}\right), \quad \lambda_{USp(n)} := g^2_{YM}\left(\frac{n}{2} + \frac{1}{4}\right)$

= $g^2_{YM} \times$ **RAMOND-RAMOND FLUX**
in HOLOGRAPHICALLY DUAL STRING THEORY in $AdS_5 \times S^5/Z_2$ **(Orientifold)**

- Fixed- g^2_{YM}

$$2\,\mathbb{C}_{SO(n)}(\tau,\bar\tau) = \frac{(2\hat{N}_{SO(n)})^2}{4} - \frac{3(2\hat{N}_{SO(n)})^{\frac12}}{2^4} E(\tfrac32;\tau,\bar\tau) + \frac{45(2\hat{N}_{SO(n)})^{-\frac12}}{2^8} E(\tfrac52;\tau,\bar\tau)$$
$$+ (2\hat{N}_{SO(n)})^{-\frac32}\left[\frac{4725}{2^{15}} E(\tfrac72;\tau,\bar\tau) - \frac{111}{2^{13}} E(\tfrac32;\tau,\bar\tau)\right] + (2\hat{N}_{SO(n)})^{-\frac52}\left[\frac{99225}{2^{18}} E(\tfrac92;\tau,\bar\tau) - \frac{3825}{2^{16}} E(\tfrac52;\tau,\bar\tau)\right]$$
$$+ (2\hat{N}_{SO(n)})^{-\frac72}\left[\frac{245581875}{2^{27}} E(\tfrac{11}{2};\tau,\bar\tau) - \frac{10749375}{2^{25}} E(\tfrac72;\tau,\bar\tau) + \frac{40239}{2^{22}} E(\tfrac32;\tau,\bar\tau)\right]\Big] + O(\hat{N}_{SO(n)}^{-\frac92})\,.$$

- Furthermore $\quad \mathbb{C}_{USp(n)}(\tau,\bar\tau) \sim \mathbb{C}_{SO(n)}(2\tau, 2\bar\tau)$ $\quad$ with $\quad \hat{N}_{SO(n)} \to \hat{N}_{USp(n)}$

- Coefficients of highest-index Eisenstein series at each order same as in $\mathbb{C}_{SU(N)}(\tau,\bar\tau)$ expansion.

THE "SECOND CORRELATOR"

SOME PROPERTIES OF "GENERALISED" EISENSTEIN SERIES' $\mathcal{E}(s, s_1, s_2)$

$$\boxed{(\Delta_\tau - s(s-1))\mathcal{E}(s, s_1, s_2) = E(s_1)\, E(s_2)}$$

Either $s_1, s_2 \in \mathbb{N}$
Or $s_1, s_2 \in \mathbb{N} + \frac{1}{2}$

- The case $s = 4 \quad s_1 = s_2 = \frac{3}{2}$ is the prototype, which arose as the coefficient of $d^6 R^4$ in the ten-dimensional IIB superstring effective action.
- When $s_1, s_2 \in \mathbb{N}$, $\mathcal{E}(s, s_1, s_2)$ is a (linear combination of) MODULAR GRAPH FORMS which arise as coefficients in the low energy expansion of the genus-one amplitude in type II superstring theory.
- The function $\mathcal{E}(s, s_1, s_2)$ has four terms in its Laurent polynomial (zero Fourier mode)

$$\tau_2^{s_1+s_2}, \ \tau_2^{s_1+s_2+1}, \ \tau_2^{s_2-s_1+1}, \ \tau_2^{-s}$$

- Zero Fourier mode also has infinite sum of instanton/anti-instanton pairs $\sum_n a_n\, (q\,\bar{q})^n \qquad q = e^{2\pi i \tau}$

The "Second Correlator"

$\mathcal{E}(s, s_1, s_2)$ ENTERS INTO THE EXPRESSION FOR THE SECOND CORRELATOR

$$\mathcal{G}_N^2(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)\big|_{m=0}$$

For $SU(N)$ gauge grroup

The large-N expansion up to order $1/N^3$

d^6R^4

$$\begin{aligned}\mathcal{G}_N^2(\tau,\bar\tau) =& \partial_m^4 \log Z\big|_{m=0} \sim 6N^2 + 6\sqrt{N} E(\tfrac32,\tau,\bar\tau) + C_0 - \frac{9}{2\sqrt{N}} E(\tfrac52,\tau,\bar\tau) + \frac{27}{2^3 N}\mathcal{E}(4,\tfrac32,\tfrac32,\tau,\bar\tau)\\
&+ \frac{1}{N^{\frac32}}\left[\frac{117}{2^8}E(\tfrac32,\tau,\bar\tau) - \frac{3375}{2^{10}}E(\tfrac72,\tau,\bar\tau)\right] + \frac{1}{N^2}\left[C_1 + \frac{14175}{704}\mathcal{E}(7,\tfrac52,\tfrac32,\tau,\bar\tau) - \frac{1215}{88}\mathcal{E}(5,\tfrac52,\tfrac32,\tau,\bar\tau)\right]\\
&+ \frac{1}{N^{\frac52}}\left[\frac{675}{2^{10}}E(\tfrac52,\tau,\bar\tau) - \frac{33075}{2^{12}}E(\tfrac92,\tau,\bar\tau)\right] + \frac{1}{N^3}\Big[\alpha_4\mathcal{E}(4,\tfrac32,\tfrac32,\tau,\bar\tau)\\
&+ \sum_{r=6,8,10}\left[\alpha_r\mathcal{E}(r,\tfrac32,\tfrac32,\tau,\bar\tau) + \beta_r\mathcal{E}(r,\tfrac52,\tfrac52,\tau,\bar\tau) + \gamma_r\mathcal{E}(r,\tfrac72,\tfrac32,\tau,\bar\tau)\right]\Big] + O(N^{-\frac72}),\end{aligned}$$

- This expansion includes integer powers of $1/N$, starting with the d^6R^4 coefficient.

IS THERE A LATTICE SUM REPRESENTATION OF THIS CORRELATOR?

Alday, Chester, Dorigoni, MBG, Wen - in preparation

Comments

- We have determined the functional form of the integrated correlators $\frac{1}{4}\mathbb{C}_{G_N}(\tau,\bar\tau) = \Delta_\tau\, \partial_m^2 \log Z_{G_N}(m,\tau,\bar\tau)|_{m\to 0}$ for all values of N and $\tau = \theta/2\pi + i\,4\pi/g^2_{YM}$

- Generalization to Maximal $U(1)$-violating **n**-point correlators. (Dorigoni, MBG, Wen arXiv:2202.05784)

e.g. $$\int \prod_{i=1}^{m+4} dx_i \langle O_2(x_1)O_2(x_2)O_2(x_3)O_2(x_4)\overbrace{O_\tau(x_5)\ldots O_\tau(x_{m+4})}^{m}\rangle$$ Modular weight $w=m$

- Also generalization to $\mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p$ $(p>2)$ (Paul. Perlmutter, Raj, arXiv:2209.06639; Brown, Wen, Xie arXiv: 2301.13195)

- Extension to the second correlator $\mathcal{G}_N^2(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)|_{m=0}$ (to be completed)

- Is there a generalization to higher derivatives w.r.t m - to determine the complete m-dependence??

These results add to our knowledge of superstring scattering amplitudes in $AdS_5\times S^5$ expanded around the large-radius (flat-space) and low energy limits.

Many possible extensions to other models such as ABJM and to exceptional gauge groups.