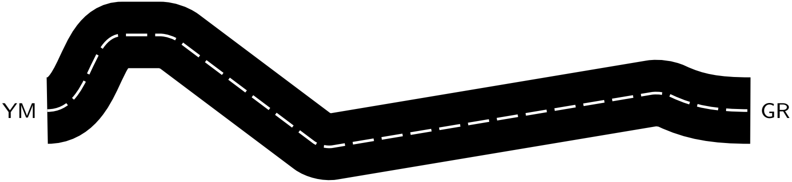
The Road to Seven Loops in $\mathcal{N} = 8$ SUGRA

Alex Edison

Amplifying Gravity at All Scales, July 11 2023



Northwestern

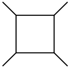
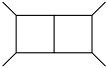
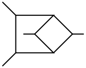
Why seven loops?

- SUSY arguments predict $L = 7$ counterterm in $D_c = 4$ (Bossard, Howe, Stelle; Green, Russo, Vanhove; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; many more)
- Similar counterterms proven absent for $\mathcal{N} = 4, 5$ at $L = \mathcal{N} - 1$ (Bossard, Howe, Stelle, Vanhove; Bern, Davies, Dennen; Bern, Davies, Dennen, Huang)
- Improved behavior observed in $D = 4$ kinematics (AE, Hermann, Parra-Martinez, Trnka)

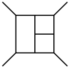
The journey so far:

History of direct calculations:

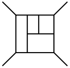
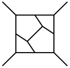
- 1&2 loops '80 -'90s (Green, Schwarz, Brink; Bern, Dixon, Dunbar, Perelstein, Rozowsky)

 $\sim s\,t\,u\,M^{\text{tree}}$  ,  $\sim s^2\left(s\,t\,u\,M^{\text{tree}}\right)$

- 3 loops '07-'10 (Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)

  Color-kinematics dual basis: 

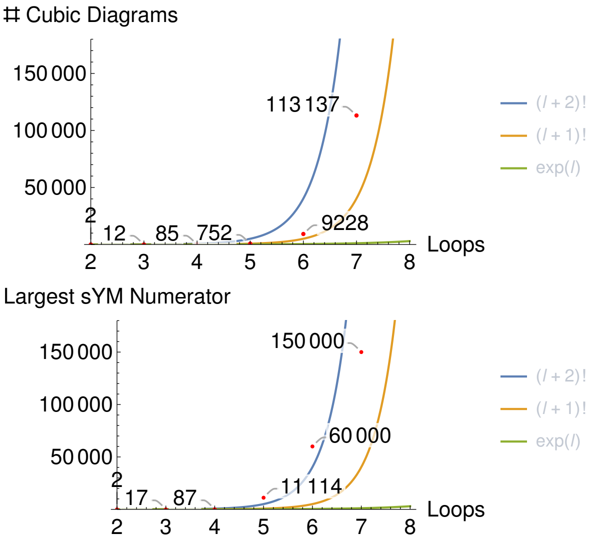
- 4 loops '09-'12 (Bern, Carrasco, Dixon, Johansson, Roiban)

  CK dual basis:  & 

- 5 loops 2018 (Bern, Carrasco, Chen, AE, Johansson, Parra-Martinez, Roiban, Zeng)

  CK dual basis has problems: generalized double-copy, thousands of diagrams

The road ahead

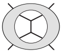
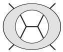
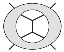


The road ahead

Can't be conquered just by "computing harder".
Need physics-driven insights!

Color-kinematics duality: (Bern, Carrasco, Johansson)

when kinematic numerators also obey Jacobi relations

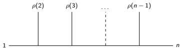
c [] $+c$ [] $+c$ [] $= 0 \Rightarrow n$ [] $+n$ [] $+n$ [] $= 0$

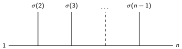
then $c \to n$ yields a valid gravity numerator

$$\mathcal{A} = \sum_g \frac{1}{S_g} \frac{c_g n_g}{D_g} \xrightarrow{c \to n} \mathcal{M} = \sum_g \frac{1}{S_g} \frac{n_g^2}{D_g}$$

Iterated application of kinematic Jacobi relations allow amplitude to be expressed in terms of small number of diagrams

- Tree-level: Construct "half-ladder" functions (...;Du,Teng;AE,Teng;Cheung, Mangan;...)

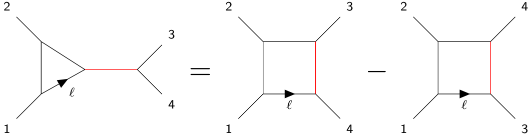
$\mathcal{A}^{\text{tree}}_{\text{sYM}} = \sum_{\sigma,\rho\in S_{n-2}} c\Big($ 

$\Big) m(1,\rho,n|1,\sigma,n) N \Big($ 

$\Big)$

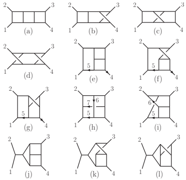
$$\mathcal{M}^{\text{tree}}_{\text{GR}} = \sum_{\sigma,\rho\in S_{n-2}} \tilde{N}(1,\rho,n) m(1,\rho,n|1,\sigma,n) N(1,\sigma,n)$$

- Loop-level: All diagram numerators can be written in terms of a small basis, *e.g.*

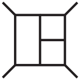


Racing through three loops

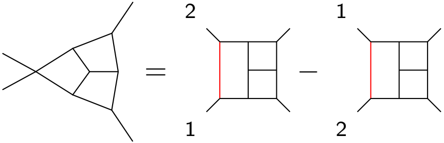
CK relations: all twelve diagrams can be written in terms of "tennis court". Start from no-triangle ansatz



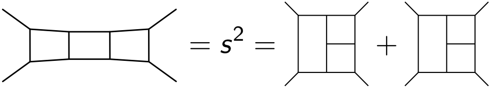
$\sim$



$$= s(\tau_{45} + \tau_{15}) + (\alpha s + \beta t)(s + \tau_{15} - \tau_{25})$$



$$\to \beta = \frac{1}{3}$$

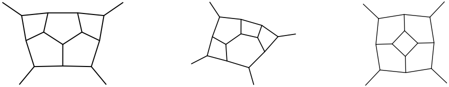


$$\Rightarrow \alpha = -\frac{1}{3}$$

All other cuts satisfied for free

The roadblock at 5 loops

Minimal CK dual basis of 3 diagrams:



Able to match all max cuts. Inconsistent with next-to-max cuts.

Can we still extract SUGRA from sYM?

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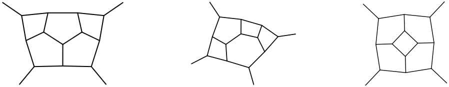

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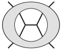
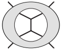
n_{sYM} [] $+\, n_{\text{sYM}}$ [] $+\, n_{\text{sYM}}$ [] $= J$

$\Rightarrow n_{\text{SUGRA}}$ () $\sim \sum_{g\in\text{cubic}} \frac{J_{\{g,1\}}J_{\{g,2\}}}{d_g}$

Turns 752 diagrams in sYM into thousands of SUGRA contacts

Scenic overlooks: learning more from lower loops

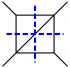
Difficult to look inside 5 loops. Need smaller problems to explore challenges.

- 1-loop 6-point $\mathcal{N}=4$: many less-than-perfect approaches

  (Bjerrum-Bohr,Dennen,Monteiro,O'Connel; Mafra, Schlotterer; He, Schlotterer, Zhang; Bridges, Mafra)

  Solution: Physics-tamed initial ansatz , careful contact analysis

  (AE, He, Johansson, Schlotterer, Teng, Zhang)

- 2-loop 4-point pure Yang–Mills: local CK basis cannot satisfy all cuts (*e.g.* ). Similar problem to 5 loops!

  Partial solution: CK *on cuts*. BUT need to specify all diagrams separately, thousands of terms each (Bern, Dennen, Nohle)

- 1-loop 4-point in OSS at α'^7: Bubble diagram cannot be "pure contact": n () $\not\propto s^2$ (AE,Tegevi)

  $\Rightarrow$ Interplay between CK and UV structure

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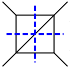
- 2-loop 4-point pure Yang–Mills: local CK basis cannot satisfy all cuts (*e.g.* ). Similar problem to 5 loops!

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Taming the ansatz

Often resort to sorting the needle from a haystack: write down generic ansatz, impose conditions by solving large linear systems.

How much of the physics can we bake in?

- Symmetry → invariant theory, *e.g.*

$$n_{\{k_i,\ell\}}\Bigg(\quad\Bigg) = \mathbb{Q}[p_1,..,p_6|s_1,s_2,s_3]$$

- Gauge invariance → tensor bases:
  - Helicity-centric, nonlocal: $\hat{\kappa}_{(ab)(cd)}$ (Johansson, Kälin, Mogull; Kälin, Mogull, Ochirov)
  - D-dim local + perms: $s\,t\,A^{\text{tree}}, s\,t\,A^{F^3}, \ldots$ (Bern, AE, Kosower, Parra-Martinez)
- CK duality:
  - Composition rules (Carrasco, Rodina, Yin, Zekioglu)
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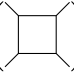

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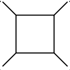
$$n_{\{k_i,\ell\}}\Big($$  $$\Big) = \mathbb{Q}[p_1, .., p_6 | s_1, s_2, s_3]$$

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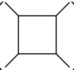
$n_{\{k_i,\ell\}}$ () $= \mathbb{Q}[p_1, .., p_6 | s_1, s_2, s_3]$

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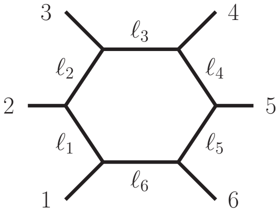
Example: 6 points with 5 parameters

(AE,He,Johansson, Schlotterer, Teng, Zhang)

Use covariant structures that appear in forward limit/max cut: (AE, He, Schlotterer, Teng)

$$N_{\text{hex}} \supset \{(\varepsilon_1 \cdot \ell_1)(\varepsilon_2 \cdot \ell_2) t_8(f_3, f_4, f_5, f_6) + \text{perms}, \\ (\varepsilon_1 \cdot \ell_1) t_8(f_2, f_{[3,4]}, f_5, f_6) + \text{perms}, \\ t_8(f_1, f_{[2,3]}, f_{[4,5]}, f_6) + \text{perms}, \\ t_8(f_1, f_{[2,[3,4]]}, f_5, f_6) + \text{perms}, \\ t_{12}(f_1, f_2, f_3, f_4, f_5, f_6)\}$$



$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$$
$$f_{[i,j]} = f_i f_j - f_j f_i$$

$$t_8(f_w, f_x, f_y, f_z) = \text{tr}(f_w f_x f_y f_z) - \frac{1}{4}\text{tr}(f_w f_x)\text{tr}(f_y f_z) + \text{cyc}(x, y, z)$$

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$$
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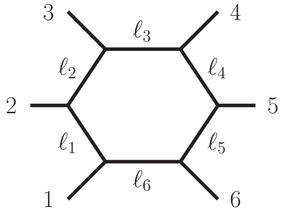
& t_{12}(f_1, f_2, f_3, f_4, f_5, f_6) \}
\end{aligned}
$$



But allow contact freedom

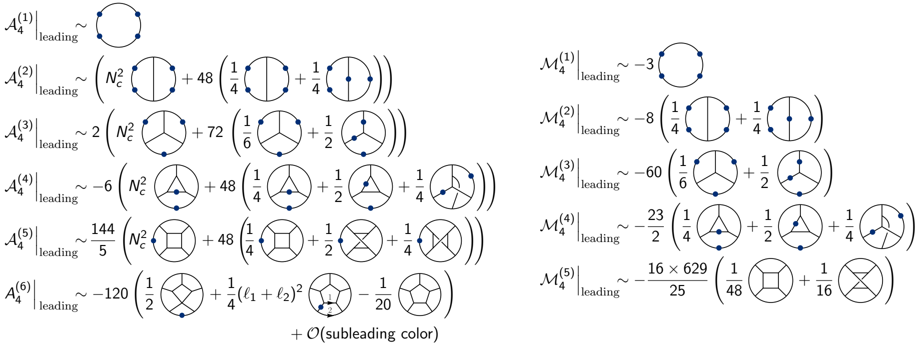
$$
\begin{aligned}
N_{\text{hex}} \supset \{ & \varepsilon_1 \cdot \varepsilon_2 t_8(f_3, f_4, f_5, f_6)\{\ell_6^2 + \ell_2^2, \ell_1^2\}, \\
& \varepsilon_1 \cdot \varepsilon_3 t_8(f_2, f_4, f_5, f_6)\{\ell_6^2 + \ell_3^2, \ell_1^2 + \ell_2^2\}, \\
& \varepsilon_1 \cdot \varepsilon_4 t_8(f_2, f_3, f_5, f_6)\{\ell_6^2 + \ell_4^2\}\} \\
& + \text{cyc}
\end{aligned}
$$

$$
f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu
$$

$$
f_{[i,j]} = f_i f_j - f_j f_i
$$

$$
\begin{aligned}
t_8(f_w, f_x, f_y, f_z) = & \, \text{tr}(f_w f_x f_y f_z) \\
& - \frac{1}{4} \text{tr}(f_w f_x) \text{tr}(f_y f_z) \\
& + \text{cyc}(x, y, z)
\end{aligned}
$$

Lots of data about UV



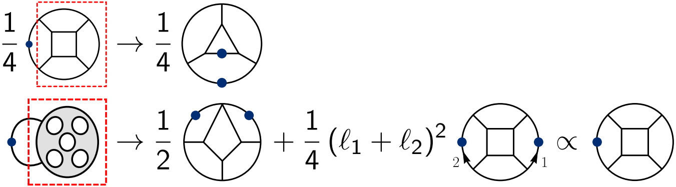
 = dots ↔ doubled propagators $= \dfrac{1}{(\ell^2)^2}$

(Green, Schwarz, Brink; Bern, Rozowsky, Yan; Bern, Dixon, Dunbar, Perelstein, Rozowsky; Carrasco, Dixon, Johansson, Roiban; Bern, Carraco, Dixon, Douglas, von Hippel, Johansson; Bern, Carrasco, Chen, AE, Johansson, Parra-Martinez, Roiban, Zeng)

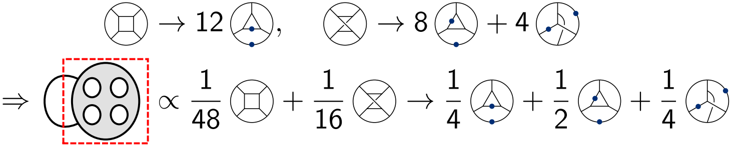
Inter-loop consistency of sYM and SUGRA UV:

(Bern, Carrasco, Chen, AE, Johansson, Parra-Martinez, Roiban, Zeng)

- Planar sYM



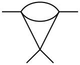
- SUGRA (similar to non-planar sYM below $L = 5$)



Can we target them directly?

Ongoing explorations

- 1-loop 4-point HD scalars:
  Possible to identify "redundant" CK terms? Finitely generated? Combinatoric/geometric interpretations?

- 2-loop 4-point pions: (AE,Mangan,Pavao WIP)
  Very few non-trivial cuts(, , ) leads to lots of flexibility in representation. Are there identifiable building blocks? Can it be extended from something simple like ZM theory?

- 2-loop 4-point YM revisited
  Minimal non-locality? Interesting tensor structures?

Long-term goals

- $\mathcal{N} = 5$ SUGRA
  - Explicit 0 found at $L = 4$ (Bern, Davies, Dennen; Bern, Davies, Dennen, Huang)
  - Behavior at $L = 5$ indication of $\mathcal{N} = 8$ at $L = 8$
  - Need good control over tensor structures, generalized double copy
  - Lots of space for testing at lower loops
- 6 loops $\mathcal{N} = 8$
  - Gather data for UV bootstrap
  - Have 6 loop sYM, but in a very gross form (Carrasco, AE, Johansson)

N^kM	0	1	2	3	4	$\sum$
cuts	5548	41649	156853	363963	576582	1,144,595
non-zero contacts	4420	16776	37653	56717	36087	151653

  - Generalized double copy between contacts?

Thank you for your attention!

Questions? See:

- AE (This week)
- Bern (Weeks 3-4)
- Johansson(Weeks 3-4)
- Roiban (Week 4)

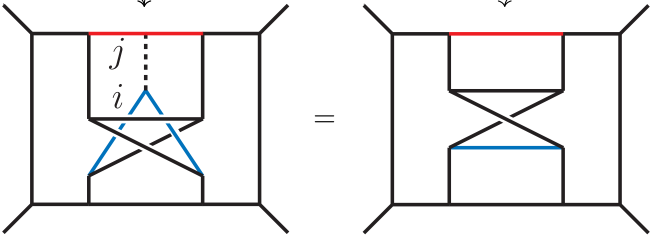
Cut construction via HID

$$\mathcal{P}^{(k)}_{\gamma,\text{ans}} = \mathcal{C}^{(k)}_{\gamma} - \mathcal{R}^{(k)}_{\gamma,\text{MMC}}$$

Trying to determine | Need to eval | Known higher cuts: $\sum \frac{n}{p^2}$

H Identity: Evaluate $\mathcal{P}$ via constraining limits, on which $\mathcal{C}^{(k)}_{\gamma} \to \mathcal{C}^{(k)}_{\gamma_{L-1}}$

$$\lim_{\ell_m \to 0} \mathcal{P}^{(k)}_{\gamma,\text{ans}} = -\left(\lim_{\ell_m \to 0} \mathcal{R}^{(k)}_{\gamma,\text{MMC}} \right) + (2 p_i \cdot p_j) \mathcal{C}^{(k)}_{\gamma \backslash \ell_m}$$



Cut construction techinicals

$\lim_{\ell_m \to 0}$ realized as projection on invariants: π_{ℓ_m}
We want to undo the projection

$$\mathcal{P}_{\gamma^{(k)}} = \pi_{\ell_m}\mathcal{P}_{\gamma^{(k)}} + \ker \pi_{\ell_m} \quad \in \quad \underbrace{\left\langle \pi_{\ell_m}\mathcal{P}_{\gamma^{(k)}}\ , \ker \pi_{\ell_m} \right\rangle}_{\text{Polynomial Ideal}}$$

But this must simultaneously hold on every ℓ_m!

$$\Rightarrow \mathcal{P}_{\gamma^{(k)}} \in \mathcal{I}_{\gamma^{(k)}} \equiv \underbrace{\bigcap_{\ell_m \in \gamma^{(k)}} \left\langle \pi_{\ell_m}\mathcal{P}_{\gamma^{(k)}}\ , \ker \pi_{\ell_m} \right\rangle}_{\text{algorithmic calculation}}$$

(Almost) always find that lowest degree generator of $\mathcal{I}_{\gamma^{(k)}}$ is unique and the correct degree to be $\mathcal{P}$. Often much faster than solving ansatz.