Higher-order calculations: Recent developments from Feynman integrals

Stefan Weinzierl

July 12, 2023

Section 1

Introduction

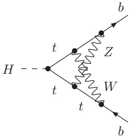
Scattering amplitudes

- We would like to make precise predictions for observables in scattering experiments from (quantum) field theory.
- Any such calculation will involve a scattering amplitude.
- Unfortunately we cannot calculate scattering amplitudes exactly.
- If we have a small parameter like a small coupling, we may use perturbation theory.
- We may organise the perturbative expansion of a scattering amplitude in terms of Feynman diagrams.

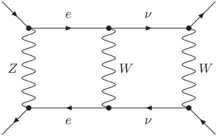
Scattering amplitude = sum of all Feynman diagrams

Applications

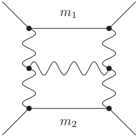
High-energy experiments: LHC



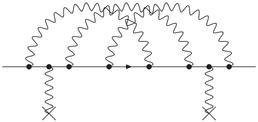
Low-energy experiments: Moller and P2



Gravitational waves:



Spectroscopy: Lamb shift



Standard techniques

- Dimensional regularisation ('t Hooft, Veltman '72, Bollini, Giambiagi '72, Ashmore '72):
  $D = 4 - 2\varepsilon$, used to regulate ultraviolet and infrared divergences.

- Integration-by-parts identities (Tkachov '81, Chetyrkin '81):
  leads to master integrals $I = (I_1, I_2, \dots, I_{N_F})$.

- Method of differential equations (Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99):

$$dI = A(x,\varepsilon)\, I$$

- Transformation to ε-factorised form (Henn '13):

$$dI = \varepsilon A(x)\, I$$

The method of differential equations

We want to calculate

$$I(\varepsilon, x)$$

as a Laurent series in ε.

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral (always possible).*
2. *Transform the differential equation into an* ε*-factorised form (**bottle neck**).*
3. *Solve the latter differential equation with appropriate boundary conditions (always possible).*

Example for an ε-factorised form

$$dI = \varepsilon A(x)\, I, \qquad A(x) = C_1 \omega_1 + C_2 \omega_2$$

with differential one-forms

$$\omega_1 = \frac{dx}{x}, \qquad \omega_2 = \frac{dx}{x-1},$$

and matrices

$$C_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Notation

$\mathbf{N_F} = N_{\text{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

$\mathbf{N_B} = N_{\text{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_{N_B})$.

$\mathbf{N_L} = N_{\text{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, ..., \omega_{N_L})$.

Vector bundles

- Fibre spanned by the master integrals $I = (I_1, ..., I_{N_F})$.

  (The master integrals $I_1(x), \dots, I_{N_F}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre.)

- Base space with coordinates $x = (x_1, ..., x_{N_B})$ corresponding to kinematic variables.

- Connection defined by the matrix A with differential one-forms $\omega = (\omega_1, ..., \omega_{N_L})$.

We would like to **transform** this vector bundle **to an ε-factorised form** through

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \;=\; \gamma^* \omega_j .$$

The **iterated integral** is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) \;=\; \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k) .$$

Chen '77

Multiple polylogarithms

Consider differential one-forms on $\mathbb{C} \cup \{\infty\}$ (the Riemann sphere) of the form

$$\omega^{\mathrm{mpl}}(z_j) = \frac{d\lambda}{\lambda - z_j}.$$

Definition (Multiple polylogarithms)

$$G(z_1, \dots, z_k; \lambda) = \int\limits_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int\limits_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \dots \int\limits_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}, \qquad z_k \neq 0$$

Caveats of iterated integrals

- In general, an individual iterated integral is **not** homotopy invariant.
  The linear combination making up a Feynman integral is, since the connection A is flat (integrable).
- If the differential one-forms ω_k transform nicely under a group of coordinate transformations, this does in general not imply that iterated integrals transform nicely as well.
  However, the vector space spanned by the master integrals does again.
  Suggests to use different bases of master integrals in different kinematic regions.

Section 2

Geometry

The base space

Question:

After a suitable coordinate transformation, can we relate the base space to a space known from mathematics?

The base space

- Assume we have $(n-3)$ variables $z_1, \dots, z_{n-3}$ and differential one-forms

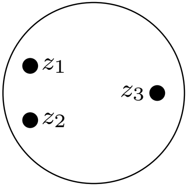
$$\omega_k \quad \in \quad \{ d\ln(z_1), d\ln(z_2), \dots, d\ln(z_1-1), \dots, d\ln(z_i-z_j), \dots \}$$

- The iterated integrals $I_\gamma(\omega_1, \dots, \omega_r; \lambda)$ are multiple polylogarithms.
- We require $z_i \notin \{0, 1, \infty\}$ and $z_i \neq z_j$:
  This defines the **moduli space** $\mathcal{M}_{0,n}$: The space of configurations of n points on a Riemann sphere modulo Möbius transformations.
- Usually the z_i are functions of the kinematic variables x and the arguments of the dlog-forms define the Landau singularities.

Multiple polylogarithms again

Take home message:

Feynman integrals, which evaluate to multiple polylogarithms are related to a Riemann sphere (a smooth complex algebraic curve of genus zero).



Section 3

Elliptic curves

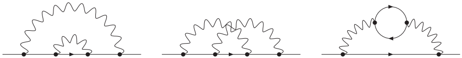
Beyond multiple polylogarithms

- Not every Feynman integral can be expressed in terms of multiple polylogarithms.
- Starting from two-loops, we encounter more complicated functions.
- The next-to-simplest Feynman integrals involve an **elliptic curve**.

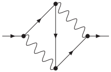
Elliptic curves

We do not have to go very far to encounter elliptic integrals in precision calculations: The simplest example is the two-loop electorn self-energy in QED:

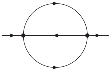
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the **kite graph**:



One sub-topology is the **sunrise graph** with three equal non-zero masses:



(Sabry, '62)

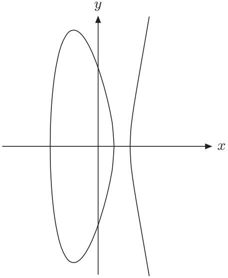
Elliptic curves

Where is the elliptic curve?

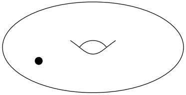
For the sunrise it's very simple: The second graph polynomial defines an elliptic curve in Feynman parameter space:

$$-p^2 a_1 a_2 a_3 + (a_1 + a_2 + a_3)(a_1 a_2 + a_2 a_3 + a_3 a_1) m^2 = 0.$$

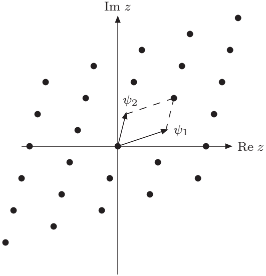
Three shades of an elliptic curve



Complex algebraic curve
$y^2 = 4x^3 - g_2 x - g_3$



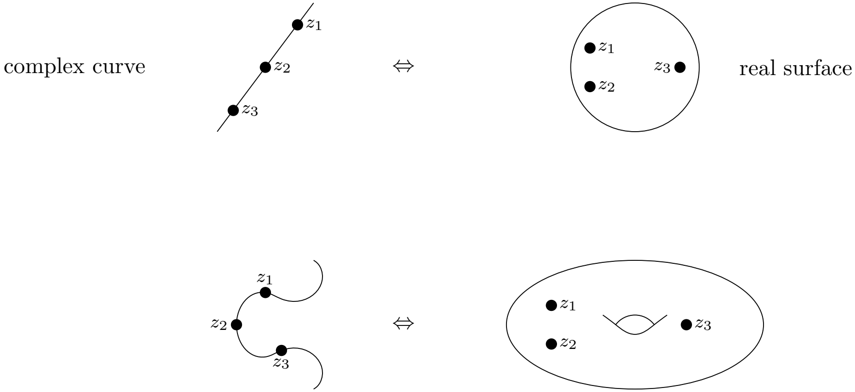
Real Riemann surface of genus one with one marked point



Complex plane modulo lattice: $\mathbb{C}/\Lambda$

Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.



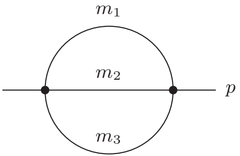
Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, ..., \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, ..., \mathbf{z_{n-1}})$

Iterated integrals on $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$

- Iterated integrals on $\mathcal{M}_{0,n}$ with at most simple poles are **multiple polylogarithms**.
  Most of the known Feynman integrals fall into this category.
- Iterated integrals on $\mathcal{M}_{1,n}$ are **iterated integrals of modular forms** and **elliptic multiple polylogarithms** (and mixtures thereof).
  The simplest example is the two-loop sunrise integral with non-zero masses.



Adams, S.W. '17, Broedel, Duhr, Dulat, Tancredi, '17,

Ch. Bogner, S. Müller-Stach, S.W., '19

Numerics

Physics is about numbers:

- Iterated integrals of modular forms and elliptic multiple polylogarithms can be evaluated numerically with arbitrary precision.
- Implemented in `GiNaC`.

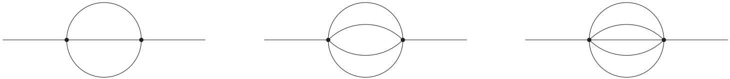
  Walden, S.W, '20

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.1)
  __,  _______  Copyright (C) 1999-2021 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type `warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> iterated_integral({Eisenstein_kernel(3,6,-3,1,1,2)},0.1);
0.23675657575197179243274817775862177623438999192840338805367
```

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of higher genus g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and **higher dimensional objects**: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

Section 4

Calabi-Yau manifolds

Calabi-Yau manifolds

Definition

A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.

Theorem (conjectured by Calabi, proven by Yau)

An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

Mirror symmetry

The **mirror map** relates a Calabi-Yau manifold A to another Calabi-Yau manifold B with Hodge numbers $h_B^{p,q} = h_A^{n-p,q}$.

Candelas, De La Ossa, Green, Parkes '91

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

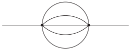
Calabi-Yau manifold A

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,1} & & 0 & \\ 1 & & h^{1,1} & & h^{1,1} & & 1 \\ & 0 & & h^{2,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

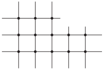
mirror image B

Fantastic Beasts and Where to Find Them

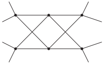
- Bananas



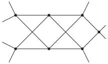
- Fishnets



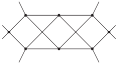
- Amoebas



- Tardigrades



- Paramecia



Aluffi, Marcolli, '09, Bloch, Kerr, Vanhove, '14
Bourjaily, McLeod, von Hippel, Wilhelm, '18
Duhr, Klemm, Loebbert, Nega, Porkert, '22

Bananas

- The l-loop banana integral with (equal) non-zero masses is related to a **Calabi-Yau $(l-1)$-fold**.
- An elliptic curve is a Calabi-Yau 1-fold, this is the geometry at two-loops.
- The system of differential equations for the equal mass l-loop banana integral can be transformed to an **ε-factorised form**.
  - Change of variables from $x = p^2/m^2$ to τ given by mirror map.
  - Transformation constructed from special local normal form of a Calabi-Yau operator.

    M. Bogner '13, D. van Straten '17
- Strong support for the conjecture that a transformation to an ε-factorised differential equation exists for all Feynman integrals.

Section 5

The mirror map

The mirror map

- The point $x = \infty$ is a point of maximal unipotent monodromy, the Frobenius method gives solutions ordered by powers of logarithms.
- The holomorphic solution ψ_0 and the single-logarithmic solution ψ_1 are used to define a change of variables from x to τ (or q):

$$\tau = \frac{\psi_1}{\psi_0}, \qquad q = e^{2\pi i \tau}.$$

- In the context of Calabi-Yau manifolds the map from x to τ is called the **mirror map**.

Candelas, De La Ossa, Green, Parkes, '91

- In the special case of $l = 2$ the map corresponds to the transformation from x to the modular parameter τ of an elliptic curve.

Section 6

The special local normal form of a Calabi-Yau operator

Special local normal form

Consider a sequence which starts as

$$\begin{array}{rcl} l = 0: & & 1 \\ l = 1: & & \theta \\ l = 2: & & \theta \cdot \theta \\ l = 3: & & \theta \cdot \theta \cdot \theta \end{array}$$

We would like to understand the general term at l loops.

Special local normal form

We first compute the $(l = 4)$-term:

$$\begin{aligned} l &= 0: & & 1 \\ l &= 1: & & \theta \\ l &= 2: & & \theta \cdot \theta \\ l &= 3: & & \theta \cdot \theta \cdot \theta \\ l &= 4: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \end{aligned}$$

Special local normal form

The general term at l loops is given by

$$\theta \cdot \frac{1}{Y_{l-1}} \cdot \theta \cdot \frac{1}{Y_{l-2}} \cdot \theta \cdot \frac{1}{Y_{l-3}} \cdot \ldots \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_1} \cdot \theta$$

and we have

$$Y_1 = 1$$

and the duality

$$Y_j = Y_{l-j}.$$

Special local normal form

Up to seven loops we therefore have

$$
\begin{aligned}
l &= 0: & & 1 \\
l &= 1: & & \theta \\
l &= 2: & & \theta \cdot \theta \\
l &= 3: & & \theta \cdot \theta \cdot \theta \\
l &= 4: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 5: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 6: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta \\
l &= 7: & & \theta \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_3} \cdot \theta \cdot \frac{1}{Y_2} \cdot \theta \cdot \theta
\end{aligned}
$$

Special local normal form

- θ is the **Euler operator** $\theta = q\frac{d}{dq}$ in the variable q, the functions Y_j are called Y-invariants.
- $N = \theta^2 \frac{1}{Y_2} \theta \frac{1}{Y_3} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2$ is the **special local normal form** of a Calabi-Yau operator.
- Operators like N are related to **Picard-Fuchs operators** of Calabi-Yau Feynman integrals.
- From the factorisation of N we may construct the ε-factorised differential equation.

Section 7

The ansatz

The ansatz

- We set $D = 2 - 2\varepsilon$.
- Instead of $x = p^2/m^2$ we work with the variable τ (or q).
- We now **construct master integrals**

$$M = (M_0, M_1, \dots, M_l)^T,$$

which put the differential equation into an ε-factorised form.

- M_0 is proportional to the l-loop tadpole integral:

$$M_0 = \varepsilon^l I_{1\dots10}.$$

The ansatz

- $I_{1...11}$ has Picard-Fuchs operator $L^{(l)}$, the ε^0-part $L^{(l,0)}$ is of the form

$$L^{(l,0)} = \beta\theta^2 \frac{1}{Y_{l-2}} \theta \frac{1}{Y_{l-3}} \dots \frac{1}{Y_3} \theta \frac{1}{Y_2} \theta^2 \frac{1}{\psi_0}$$

- M_1 should start at order ε^l.
- $L^{(l,0)}$ **annihilates** $I_{1...11}$ modulo ε and modulo tadpoles.
- This suggests

$$M_1 = \frac{\varepsilon^l}{\psi_0} I_{1...11}.$$

The ansatz

- We construct a derivative basis. The **factorisation** of $L^{(l,0)}$ in the variable q suggests for the master integrals $M_2 - M_l$

$$M_j = \frac{\mathbf{1}}{\mathbf{Y_{j-1}}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} + \text{junk} \right],$$

- **Griffiths transversality**:

$$M_j = \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} \mathbf{F_{(j-1)k} M_k} \right],$$

with a priori unkown but ε-independent functions $F_{ij}(\tau)$.

Summary of the ansatz

$$
\begin{aligned}
M_0 &= \varepsilon^l I_{1\dots 10} \\
M_1 &= \frac{\varepsilon^l}{\psi_0} I_{1\dots 11} \\
M_j &= \frac{1}{Y_{j-1}} \left[\frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} M_{j-1} - \sum_{k=1}^{j-1} F_{(j-1)k} M_k \right] \qquad \text{for } j \geq 2
\end{aligned}
$$

The differential equation

The ansatz leads to the differential equation

$$\frac{1}{2\pi i}\frac{d}{d\tau}M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M.$$

- The first l rows are in an ε-factorised form.
- Determine the functions F_{ij} such that the $(l+1)$-th row is in ε-factorised form.

The differential equation

The condition that in the $(l+1)$-th row only terms of order ε^1 are present leads to

- differential equations
- **algebraic equations** from self-duality

$$\frac{1}{2\pi i}\frac{d}{d\tau}M = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & & 0 & 0 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & F_{(l-2)4} & \dots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & F_{(l-1)4} & \dots & F_{(l-1)(l-1)} & 1 \\ * & * & * & * & * & \dots & * & * \end{pmatrix} M$$

The differential equation

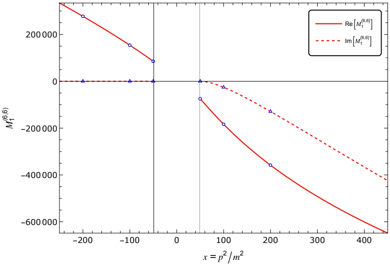
- The equations for F_{ij}'s have a natural triangular structure and can be solved systematically.
- We arrive at the **differential equation in ε-factorised form**:

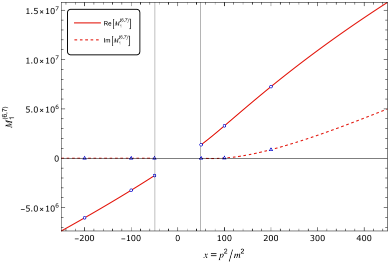
$$dM = \varepsilon A M$$

Section 8

Results and potential applications

Results: Six loops





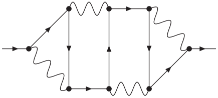
Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

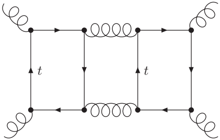
Pögel, Wang, S.W. '22

Examples

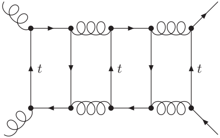
- Electron self-energy in QED
  (related to a Calabi-Yau 3-fold).



- Dijet production at N^3LO
  (related to a Calabi-Yau 2-fold).



- Top pair production at N^4LO
  (related to a Calabi-Yau 3-fold)

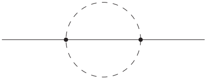


Conclusions

- Feynman integrals are needed for precision calculations in perturbative quantum field theory.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- It is helpful to relate a Feynman integral to a geometric object (spheres, elliptic curves, Calabi-Yau *n*-folds, ...).
  Algebraic geometry gives us information on the original Feynman integral.

Outlook

- The geometry we associate with a Feynman integral might not be unique.
- For example, we may use a more complicated geometry instead of a simpler one:



  - The sunrise integral with one non-zero mass and two massless internal lines evaluates to multiple polylogarithms. This corresponds to **genus 0**.
  - We may express the integral in terms of iterated integrals of modular forms. This corresponds to **genus 1**.
- A straightforward determination of the geometry might not lead to the simplest one.