Self-force in hyperbolic scattering

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Plan

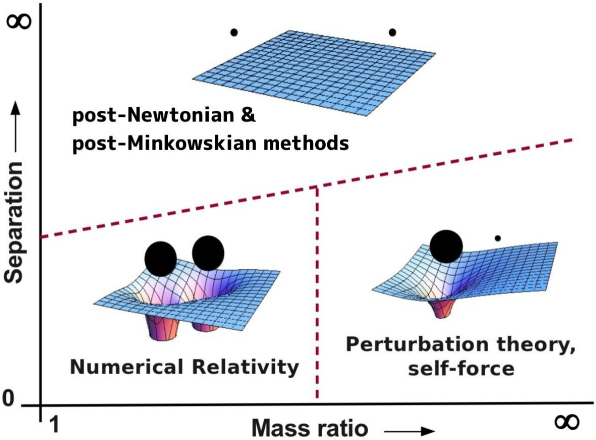
- **Part A: Introduction to gravitational self-force in classical GR**
  - Foundations
  - Sample of results for bound orbits

- **Part B: Self-force in black-hole scattering**
  - First results in a scalar-field toy model [LB & Long 2021, 2022]
  - Comparison with Amplitude PM calculations [LB, Bern et al., 2023]
  - Current and future work [Whittall & LB, 2023]

PART A: INTRODUCTION TO SELF-FORCE

Nonexpert review:
Barack & Pound, *Self-force and radiation reaction in general relativity*, 2019 Rep. Prog. Phys. **82** 016904 [arXiv:1805.10385]

Domains of the 2-body problem in classical GR

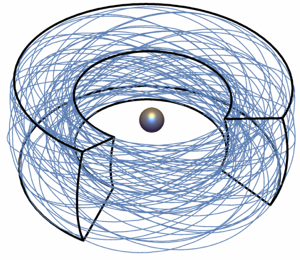


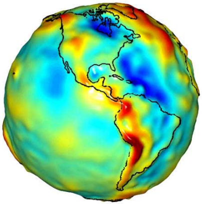
EMRIs as astrophysical probes of strong-field physics

- Small mass ratio, $q \sim \frac{10 M_\odot}{10^6 M_\odot} = 10^{-5} \ll 1$

  $\Rightarrow$ Slow evolution, over $T_{\rm RR} \sim T_{\rm orb}/q \gg T_{\rm orb}$

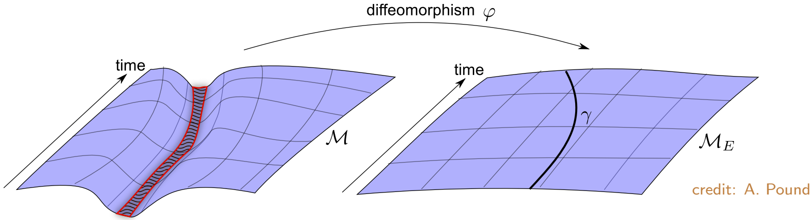
  $\Rightarrow$ Very many grav. wave cycles in LISA band: $N_{\rm orb} \sim 1/q \sim 10^5$
- Orbits very complicated.
  Geodesics generically tri-periodic & ergodic.





- Precision probe of strong-field geometry:
  - "black-hole geodesy"
  - tests of GR

Problem of motion



Body in exact spacetime

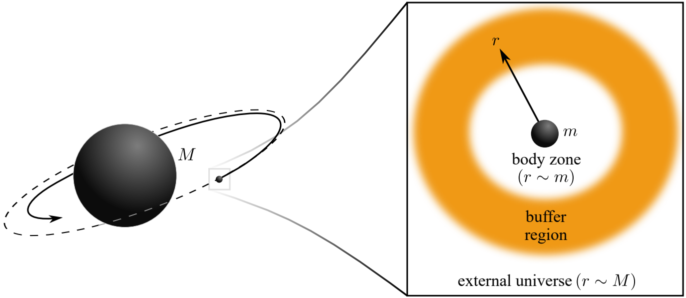
Effective representation of motion in background spacetime

Guiding principles:

- "Point particle" does not make sense as a fundamental object in GR, but "point particle equation of motion" can — in a certain effective way.
- No need for ad-hoc regularization; EoM rigorously derived via a limit process.

Matched Asymptotic Expansions

Mino, Sasaki & Tanaka (1997), building on Burke, d'Eath, Kates, Thorne & Hartle,. . .

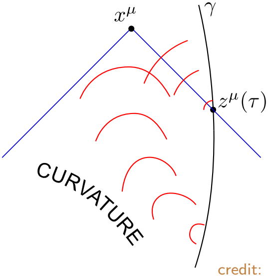


- Trajectory defined on background spacetime using a suitable far-zone limit; constrained by matching near & far expansions of the metric in the matching zone.
- No resort to "point particles": notion *derived* rather than assumed

Equation of Motion with 1st-order self-force

Metric perturbation at x^μ is a sum of "**direct**" and "**tail**" contributions:

$$g^{\text{full}}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$



credit: A. Pound

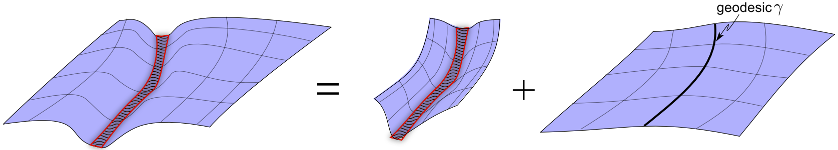
$$m\frac{D^2 z^\alpha}{d\tau^2} = m\nabla^{\alpha\beta\gamma} h^{\text{tail}}_{\beta\gamma}\Big|_{z(\tau)} =: F^\alpha_{\text{self}}$$

$$\nabla^{\alpha\beta\gamma} h_{\gamma\beta} := -\frac{1}{2}(g^{\alpha\beta}_{(0)} + u^\alpha u^\beta) u^\gamma u^\delta \left(2\nabla^{(0)}_\delta h_{\beta\gamma} - \nabla^{(0)}_\beta h_{\gamma\delta}\right)$$

"R field" reformulation [Detweiler & Whiting 2003]

- $h^{\text{tail}}_{\alpha\beta}$ is **not** a vacuum solution of the linearized Einstein equations
- But one can construct a vacuum solution $h^{R}_{\alpha\beta}$ [associated with a certain (a-causal) Green function in the Hadamard representation] such that $\nabla h^{\text{R}}\big|_z = \nabla h^{\text{tail}}\big|_z$.

$$\Rightarrow \quad F^{\alpha}_{\text{self}} = m\nabla^{\alpha\beta\gamma} h^{\text{R}}_{\beta\gamma} = \nabla^{\alpha\beta\gamma}\left(h^{\text{ret}}_{\beta\gamma} - h^{\text{S}}_{\beta\gamma}\right)$$



full metric $g^{\text{full}}_{\alpha\beta}$ = "self-field" $h^{\text{S}}_{\alpha\beta}$ + effective metric $g^{(0)}_{\alpha\beta} + h^{\text{R}}_{\alpha\beta}$

- **Interpretation:** orbit is a geodesic in the effective metric.
- Similar result for extended material objects [Harte 2010], 2nd-order self-force (Pound 2012), non-perturbative [Harte 2012]

Practical schemes in black-hole spacetimes:
I. Mode-sum method

- Subtraction of $h^S_{\alpha\beta}$ done mode-by-mode in a multipole expansion about large BH:

$$
\begin{aligned}
F_{\rm self}(z(\tau)) &= m\sum_{\ell=0}^{\infty}\left[(\nabla h^{\rm ret})^{\ell}-(\nabla h^{S})^{\ell}\right]\Big|_{z(\tau)} \\
&= \sum_{\ell=0}^{\infty}\left[m(\nabla h^{\rm ret})^{\ell}\Big|_{z(\tau)}-A(z)\ell-B(z)-C(z)/\ell\right]-D(z)
\end{aligned}
$$

- Regularization parameters derived analytically from local form of $h^S_{\alpha\beta}$; known for generic orbits in Kerr (LB & Ori 2000-03)

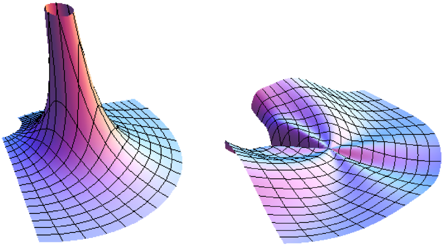
- Numerical input: Modes of $h_{\beta\gamma}$ obtained by solving metric perturbation equations with a particle (delta function) source and retarded boundary conditions.

Practical schemes in black-hole spacetimes:
II. Puncture (or "effective source") method

- Analytically construct Puncture field $h^P_{\alpha\beta} \approx h^S_{\alpha\beta}$ so that $\nabla h^P = \nabla h^{\rm S}$ at particle.
- Write linearized field equation $\delta G_{\mu\nu}(h) = T_{\mu\nu}$ in "punctured" form

$$\delta G_{\mu\nu}(h - h^P) = T_{\mu\nu} - \delta G_{\mu\nu}(h^P) =: S^{\rm eff}_{\mu\nu}$$

- Numerically solve for Residual field $h^{\rm Res} := h - h^P$. Then $F_{\rm self} = m\nabla h^{\rm Res}$

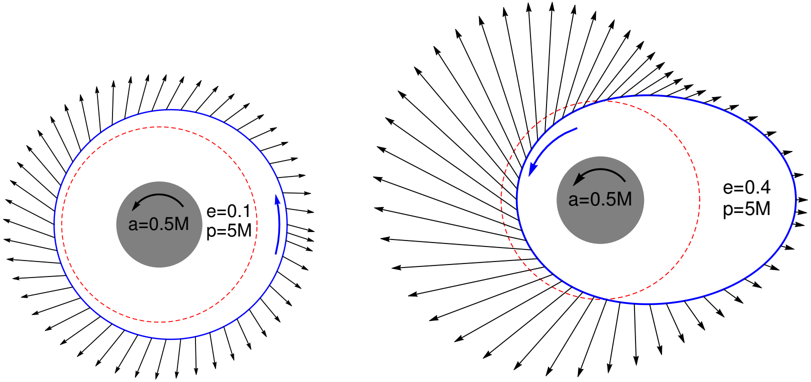


credit: J. Thornburg & B. Wardell

- Implementations (2007–) by
  - LB, Golbourn, Dolan, Thornburg,...
  - Detweiler, Vega, Diener, Wardell,...

Self-force along fixed geodesic orbits

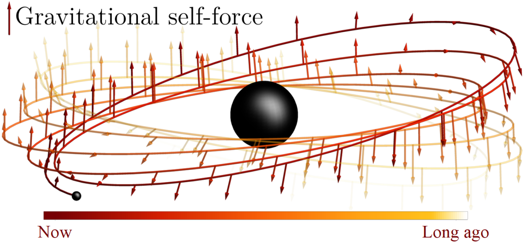
sample results for equatorial orbits in Kerr $(a = 0.5M)$



Based on data from M. van de Meent (2016)
using numerical implementation of Mano-Suzuki-Takasugi method
$+$ metric reconstruction $+$ mode-sum regularization.

Self-force along fixed geodesic orbits

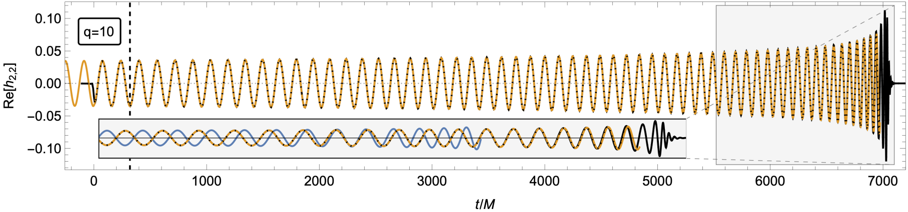
sample results for an inclined eccentric orbit in Kerr



$$a = 0.5M, \quad p = 10, \quad e = 0.1, \quad \cos\theta_{\min} = 0.3$$

[M. Van de Meent]

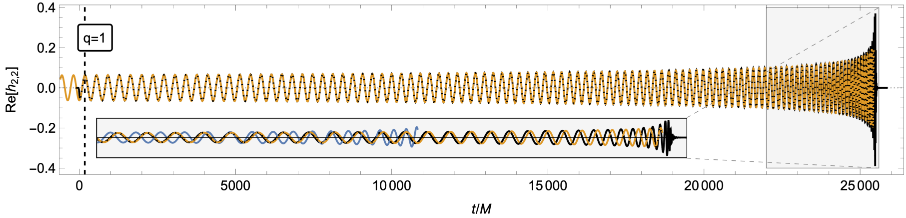
First 1-post-adiabatic waveforms from 2nd-order calculations



mass ratio 1:10

[Wardell et al. 2021]

First 1-post-adiabatic waveforms from 2nd-order calculations



mass ratio 1:1

[Wardell et al. 2021]

Conservative effects of the self-force

Now "turn off" dissipation:

$$m\frac{Dz^\alpha}{d\tau^2} = F^\alpha_{\text{cons}} := \frac{1}{2}\left[F^\alpha_{\text{self}}(h^{\text{ret}}) + F^\alpha_{\text{self}}(h^{\text{adv}})\right]$$

Motivation

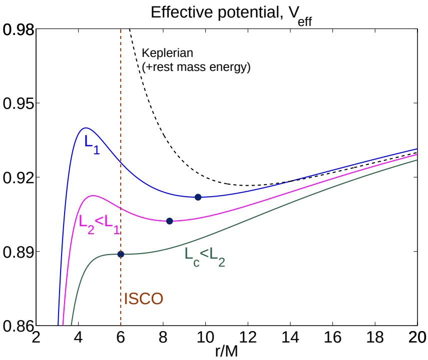
- Clean quantitative description of gauge-invariant finite-mass effects
- Comparison with post-Newtonian & post-Minskowskian calculations
- Strong-field calibration data for Effective One Body potentials
- Construct two-body Hamiltonian

$\mathrm{O(q)}$ shift in the ISCO frequency

Restoring force $\sim -mV'_{\rm eff} + F_{\rm self}$

inflection at $r_{\rm isco} = 6M + O(q)$.

$$\Rightarrow \Omega_{\rm isco} = 6^{-3/2}M^{-1} + \Delta\Omega_{\rm isco}$$



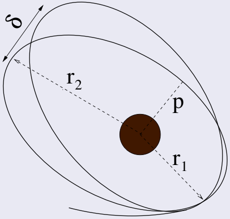
$$\left(\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}}\right)_{\rm SF} = 0.2513(6)\, q \qquad \text{[LB \& Sago 2009]}$$

$$= 0.25101546(5)\, q \qquad \text{[Akcay, LB, Damour \& Sago 2012]}$$

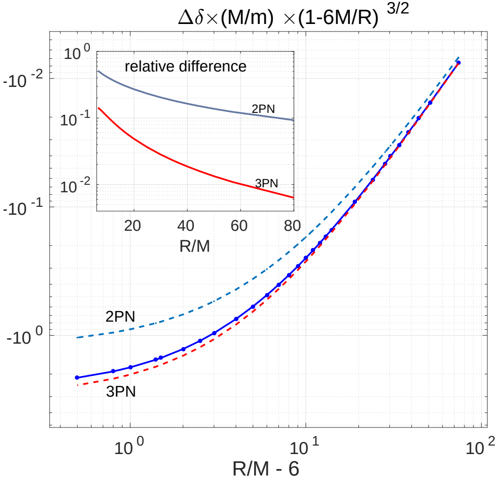
$$\left(\frac{\Delta\Omega_{\rm isco}}{\Omega_{\rm isco}}\right)_{\rm 3PN} = 0.434913\ldots\, q$$

$\mathrm{O(q)}$ correction to the periastron advance in slightly eccentric orbits [LB, Damour & Sago 2010]

$$\delta = 2\pi \left[\left(1 - \frac{6M}{R}\right)^{-1/2} - 1 \right] + \Delta\delta(R)$$



Use $R := \left(\frac{M+m}{\Omega^2}\right)^{1/3}$ as an "invariant" parametrization of the limiting circular orbit.



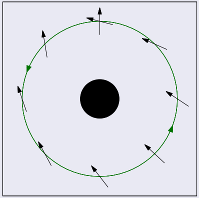
"Self-torque" and spin precession

[Dolan, Warburton, Harte, Le Tiec, Wardell & LB 2014]

In limit $s \ll m^2$, spin is parallel-transported along geodesic of $g + h^R$:

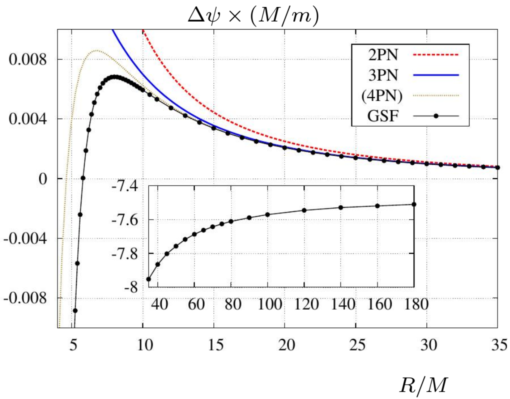
$$u^\beta \nabla^{(R)}_\beta u_\alpha = 0, \quad u^\beta \nabla^{(R)}_\beta s_\alpha = 0$$ [Harte 2012]

Circular orbit in Schwarzschild:
Spin undergoes simple precession:



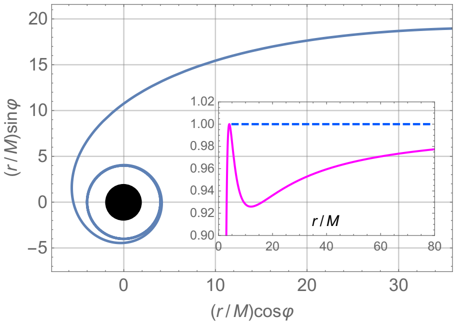
$$\psi(R) = 1 - \sqrt{1 - 3M/R} + \boldsymbol{\Delta\psi}$$

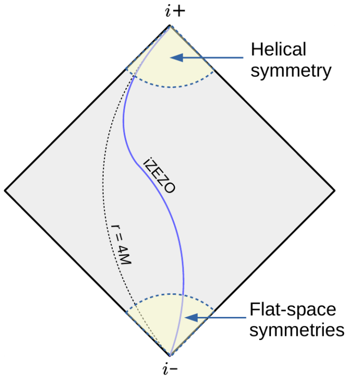
Precession angle per radian angular motion



SF effects on the ZEZO (Zero-binding-Energy Zoom-whirl Orbit)

[LB, Colleoni, Damour, Isoyama & Sago 2019]





Two asymptotic symmetries allow identification of the (unstable) circular orbit's energy and angular momentum as Bondi-type quantities (neglecting radiation).

SF effects on the ZEZO (Zero-binding-Energy Zoom-whirl Orbit)

[LB, Colleoni, Damour, Isoyama & Sago 2019]

Frequency of the asymptotic circular orbit:

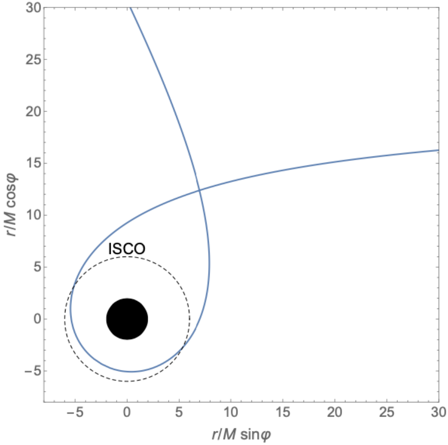
$$\begin{aligned} \Omega &= (8M)^{-1}\left[1+0.5536(2)q\right] && \text{our direct calculation} \\ &= (8M)^{-1}\left[1+0.55358516671q\right] && \text{IBCO redshift + 1st law} \\ &= (8M)^{-1}\left[1+0.32q\right] && \text{EOB (2009 model)} \\ &= (8M)^{-1}\left[1+0.553603030(1)q\right] && \text{EOB using 1st law} \end{aligned}$$

Angular momentum of the asymptotic circular orbit:

$$\begin{aligned} L &= 4M\mu\left[1-0.304(3)q\right] && \text{our direct calculation} \\ &= 4M\mu\left[1-0.3046742879q\right] && \text{IBCO redshift + 1st law} \\ &= 4M\mu\left[1-0.288(80)q\right] && \text{EOB (2009 model)} \\ &= 4M\mu\left[1-0.30467428782(6)q\right] && \text{EOB using 1st law} \end{aligned}$$

PART B:
SELF-FORCE AND SCATTERING

Motivation



$v_\infty = 0.2, \quad b = 21M$

- Scattering angle $\chi(E_{\mathrm{COM}}, b)$ defined unambiguously (even with radiation); benchmark for strong-field dynamics
- Can probe down to light ring, $r = 3M$
- New way of calibrating EOB theory, using PM χ info [Damour 2016]
- Bound-to-unbound map [Kälin & Porto 2019+]

- $\chi_{1\mathrm{SF}}$ determines the **full** conservative dynamics to 4PM order (arbitrary q); $\chi_{2\mathrm{SF}}$ extends that to 6PM. [Damour 2019]
- 1GSF results give the full $O(q)$ piece of χ ("all PM orders")

Self-force correction to the scattering angle

$$\Delta\chi := \chi(E,b) - \chi_0(E,b) = O(q)$$

where $\chi_0 := \lim_{q\to 0}\chi$ is the geodesic limit at same E, b:

$$\Delta\chi_0 = 2k\sqrt{p/e}\,\mathrm{Elliptic}\left(\frac{\chi_\infty}{2}; -k^2\right)$$

with

$$k := 2\sqrt{\frac{e}{p-6-2e}}, \qquad \chi_\infty := \cos^{-1}(-1/e),$$

and with the transformation $(p,e) \leftrightarrow (E,b)$ obtainable in analytical form.

$$\Delta\chi = \int_{-\infty}^{\infty} A_\alpha(\tau; e, p) F^\alpha_{\mathrm{self}}(\tau)\, d\tau$$ [LB & Long 2022]

[*Note at $O(q)$ it suffices to evaluate F^α_{self} along the limiting geodesic.]

Conservative and dissipative effects

$$\Delta\chi_{\rm cons} = \int_{-\infty}^{\infty} A_\alpha^{\rm cons}(\tau; p, e) F^\alpha_{\rm cons}(\tau)\, d\tau$$

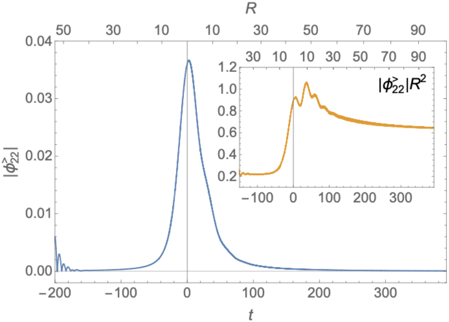
$$\begin{aligned}\Delta\chi_{\rm diss} &= \int_{-\infty}^{\infty} A_\alpha^{\rm diss}(\tau; p, e) F^\alpha_{\rm diss}(\tau)\, d\tau \\ &= \alpha_E(p, e)\Delta E_{\rm rad} + \alpha_J(p, e)\Delta J_{\rm rad}\end{aligned}$$

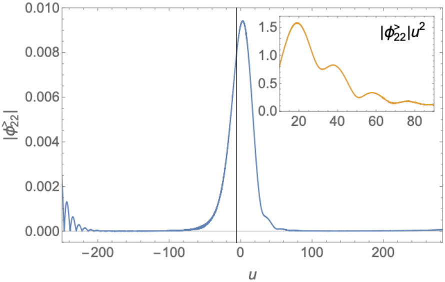
[LB & Long 2022]

Last result expresses $\Delta\chi_{\rm diss}$ in terms of total radiated energy and angular momentum, similar to relations in PM theory.

First attempt on gravitational scattering problem

Via metric reconstruction from a Hertz potential ϕ, obtained by numerically solving Teukolsky equation as a PDE in 1+1D [Long & LB 2021]





- $F_{\rm self} \sim \partial h_{\alpha\beta} \sim \partial^3\phi$.
  Hard to get good accuracy with current, rudimentary numerical method.
- On hold till better integration method developed (in progress).

Toy model: Scalar charge scattered off a Schwarzschild black hole

Charge sources a massless Klein-Gordon field Φ:

$$\nabla^\alpha \nabla_\alpha \Phi = -4\pi Q \int_{-\infty}^{\infty} \frac{\delta^4(x - z(\tau))}{\sqrt{-g(x)}} \, d\tau$$

We **ignore the gravitational self-force**, and consider only back-reaction from Φ:

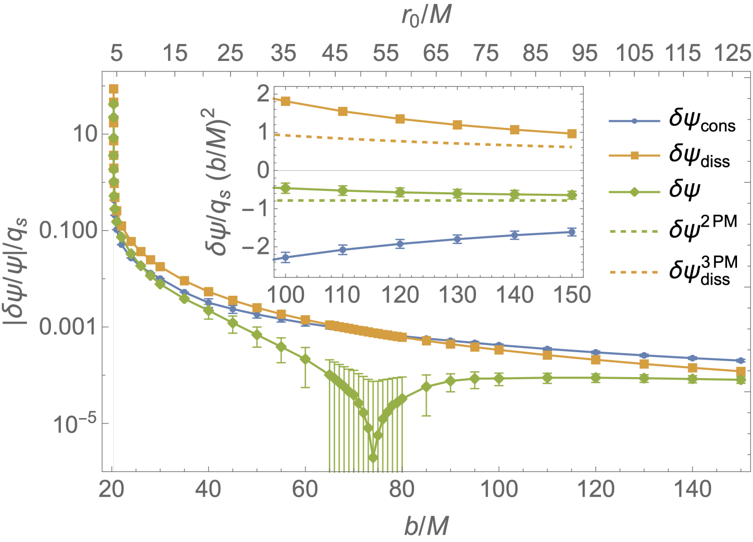
$$\frac{Dp^\alpha}{d\tau} = -Q\nabla^\alpha \Phi^R =: F^\alpha_{\rm self}$$

Assume $q_s := Q^2/(mM) \ll 1$. Then deviation from geodesic is small, and again

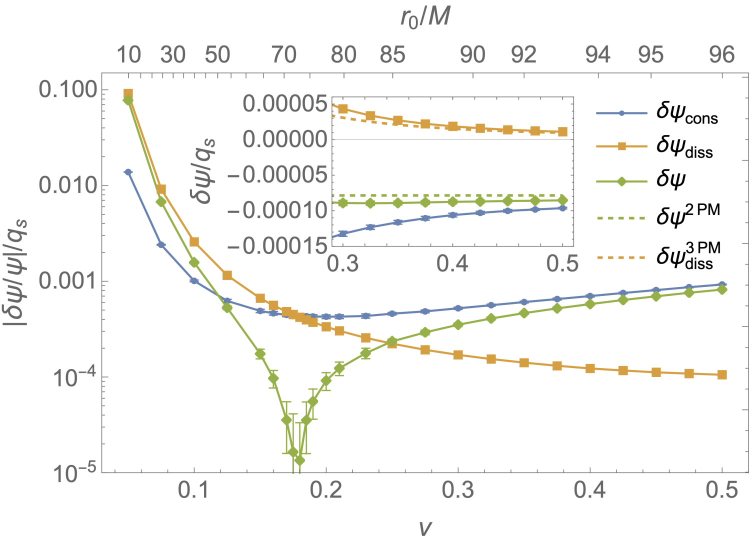
$$\Delta\chi = \int_{-\infty}^{\infty} A_\alpha(\tau; e, p) F^\alpha_{\rm self}(\tau) \, d\tau,$$

with $F^\alpha_{\rm self}$ again evaluated along the limiting geodesic.

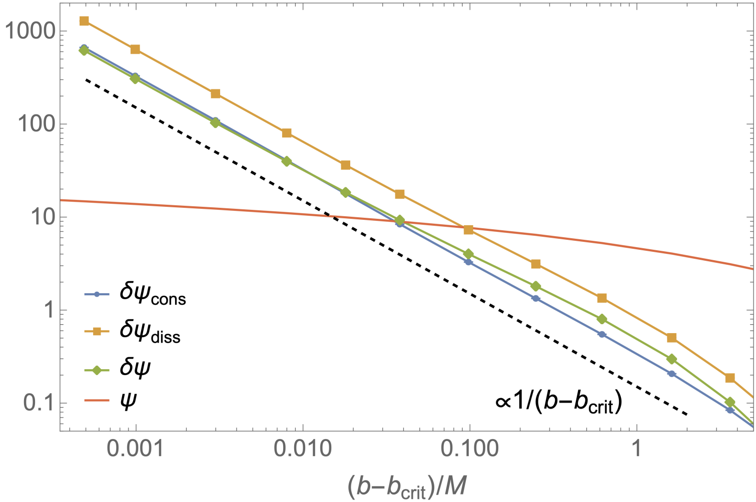
Scattering angle results $(v = 0.2)$ [LB & Long 2022]



Scattering angle results ($b = 100M$) [LB & Long 2022]



Scattering angle results (near plunge threshold) [LB & Long 2022]



Comparison with PM results from Amplitudes

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]

Expansion around flat space:

$$\delta\chi^{\text{PM}} = \sum_{i=0}^{\infty} \delta\chi_i \left(\frac{M}{b}\right)^i$$

2PM [Gralla & Lobo '22]:

$$\delta\chi_2^{\text{cons}} = -\frac{\pi}{4}\left(\frac{M}{b}\right)^2 \qquad \delta\chi_2^{\text{diss}} = 0$$

v : Velocity at infinity

b : Impact parameter

3PM:

$$\delta\chi_3^{\text{cons}} = -\frac{4\left(3-v^2\right)}{3v^2\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3 \qquad \delta\chi_3^{\text{diss}} = \frac{2\left(v^2+1\right)^2}{3v^3\sqrt{1-v^2}}\left(\frac{M}{b}\right)^3$$

LO

NLO

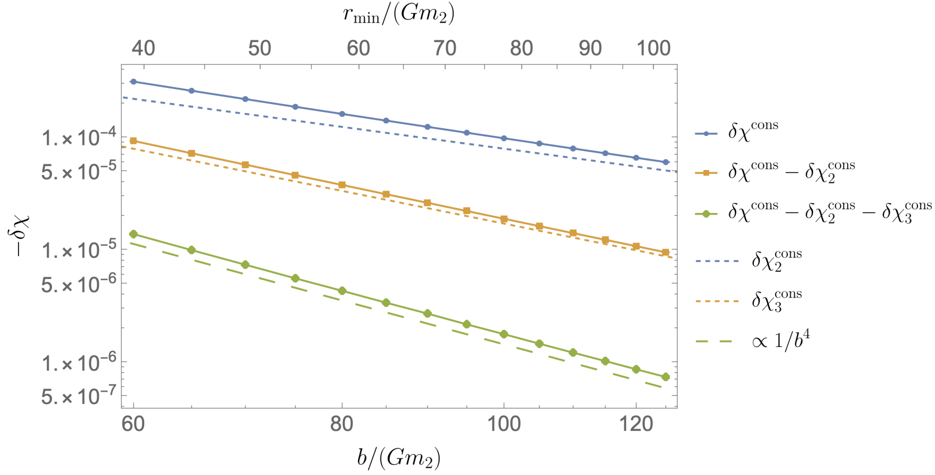
4PM dissipative:

$$\delta\chi_4^{\text{diss}} = \left(r_1 + r_2 \operatorname{arcsech}\left(\sqrt{1-v^2}\right) + r_3 \log\left[\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right]\right)\left(\frac{M}{b}\right)^4$$

r_i = rational coefficients

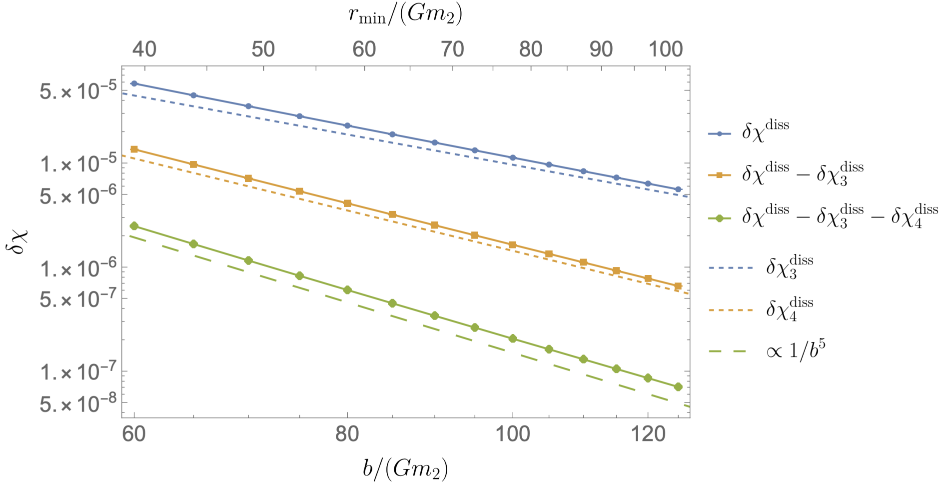
Comparison with PM: conservative sector

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]



Comparison with PM: dissipative sector

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]



Comparison with PM: numerical fitting

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]

PM expansion with free parameters:

$$\delta\varphi^{\rm PM}_{\rm cons} = \frac{a_2}{b^2} + \frac{a_3}{b^3} + \frac{a_4}{b^4} + \frac{a_5}{b^5} + \ldots$$

Up to 3PM can fit value or use **analytic value**.

a_2	a_3	a_4	a_5
-1.0886	–	–	–
-0.7535	-21.77	–	–
-0.7899	-16.17	-206.5	–
-0.7803	-18.49	-25.0	-4620
$\mathbf{-0.785398}$	-19.18	–	–
$\mathbf{-0.785398}$	-16.93	-176.2	–
$\mathbf{-0.785398}$	-17.20	-131.1	-1793
$\mathbf{-0.785398}$	$\mathbf{-16.9356}$	-175.9	–
$\mathbf{-0.785398}$	$\mathbf{-16.9356}$	-174.4	-107
$< 1\%$	$\sim 1\%$	~ -175	$< 0(?)$

4PM results from Amplitudes

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]

$$\delta\varphi_4^{\text{cons}} = \Bigg(r_1 + r_2 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right) + r_3 \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)^2 + r_4 \mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2$$

$$+ r_5 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)\mathsf{E}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right) + r_6 \mathsf{K}\left(-\frac{v^2+2\sqrt{1-v^2}-2}{v^2}\right)^2$$

$$+ r_7 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) + r_8 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \operatorname{arccosh}\left(\frac{1}{\sqrt{1-v^2}}\right)$$

$$+ r_9 \log\left(\frac{v}{2\sqrt{1-v^2}}\right) \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{10} \log\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right)$$

$$+ r_{11} \log^2\left(\frac{1}{2}\left(\frac{1}{\sqrt{1-v^2}}+1\right)\right) + r_{12}\,\boxed{\alpha} + r_{13}\frac{\boxed{\beta}}{v^2} + r_4 \log(b)\Bigg)\left(\frac{M}{b}\right)^4$$

Elliptic integrals

$r_i =$ rational coefficients

Free coefficients

Log term

Fitting for internal-structure coefficients c_1 and c_2

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]

Subtract known analytic parts of conservative 4PM:

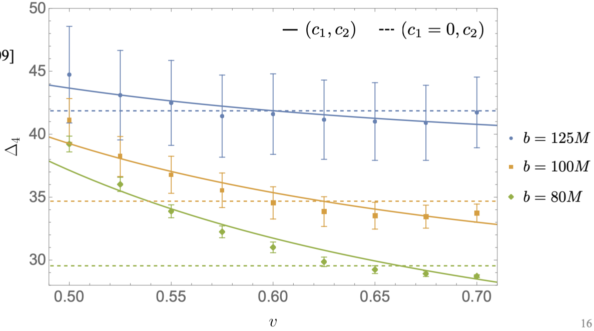
$$\Delta_4(v) := (\delta\chi_4^{\text{cons}} - \delta\chi_4^{\text{known}})b^4 = \boxed{\frac{3}{8}\pi M^4 \left[c_2 + c_1(5 - 4/v^2)\right]} + \mathcal{O}(1/b)$$

c_1 & c_2 are Love numbers.

Expect $c_1 = 0$ [Binnington & Poisson '09]

b/M	c_1	c_2
80	0.94	−21.2
100	0.68	−25.9
125	0.31(±0.38)	−33.6
80	**0**	−25.1
100	**0**	−29.4
125	**0**	−35.5

Fixed (Fitting error)



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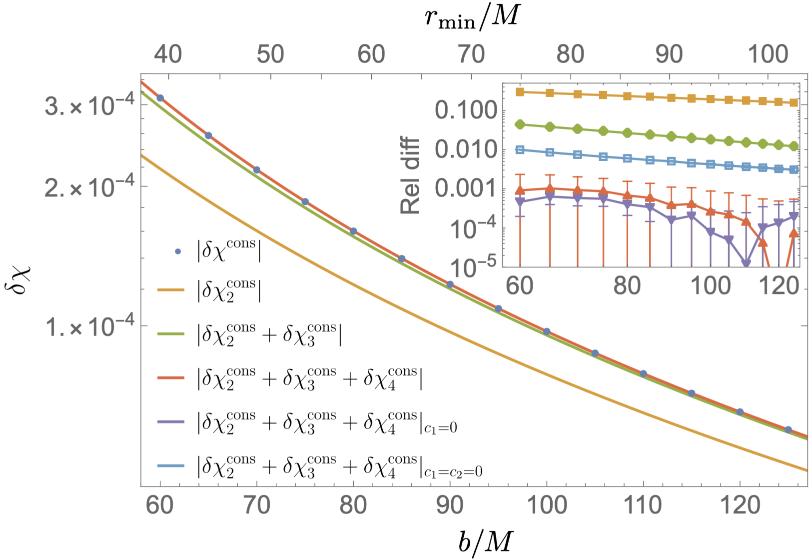
Comparison through 4PM: conservative sector

[LB, Bern, Herrmann, Long, Parra-Martinez, Roiban, Ruf, Shen, Solon, Teng, Zeng 2023]

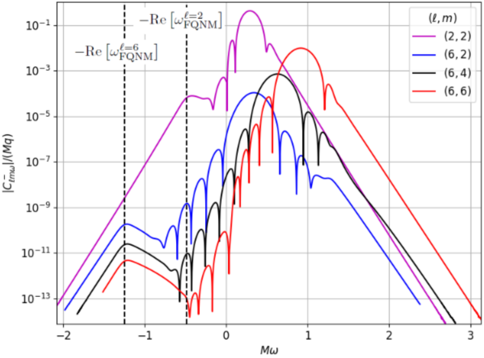


Frequency-domain approach for improved precision

[Whittall and LB 2023]

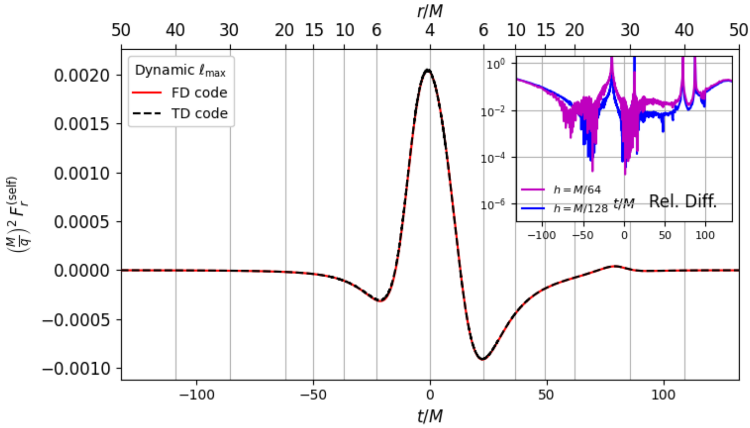
$$\Phi = \sum_{\ell,m} \int d\omega \, \phi_{\ell m\omega}(r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}$$

- Reconstruct Φ and F_{self} from numerical solutions of ODEs for $\phi_{\ell m\omega}(r)$
- **Much** more precise than TD method in strong field



Frequency-domain approach for improved precision

[Whittall and LB 2023]



- FD code loses accuracy fast at large r, where small frequencies dominate.
- Work in progress to analytically derive F_{self} at large r in $1/r$ expansion.

Prospects

- **Improved TD method**: spectral, hyperbolidal slicing, compactification. Two ongoing efforts, one using mode-sum regularization (with R. Mecado), another using puncture (with P. Diener).
- **Analytical self-force at large $\mathbf{r}$** is under way, to capitalize on improved accuracy enabled by FD method.
- **Road to gravitational scattering:**
  - Direct Lorenz-gauge calculation [Akcay, Warburton, LB 2013]
  - Radiation-gauge reconstruction [Pound, Merlin, LB 2013]
  - Lorenz-gauge reconstruction [Dolan, Durkan, Kavanagh, Wardell 2023]
- **2nd-order self-force?** No disparate timescales perhaps makes scattering problem easier than bound-orbit case.