An Application of SymTFT: Group-Theoretical Duality Defects

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Nordita Workshop—Categorical Aspects of Symmetries
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Outline

1. Brief overview of duality defects.
2. What is, and why do we care about, group theoretical duality defects?
3. Why is SymTFT useful in probing group-theoretical-ness?
4. Criteria of group-theoretical-ness.
5. Example 1: $\mathbb{Z}_N^{(0)}$ duality defects in 2d.
6. Example 2: $\mathbb{Z}_N^{(1)}$ duality defects in 4d.
7. Relation to obstruction to duality preserving gapped phases.

A Brief Overview of Duality defects

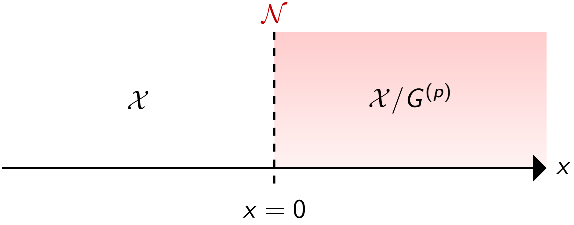
Duality Defects

- Given a quantum field theory $\mathcal{X}$ in d spacetime dimensions, with a finite, abelian, non-anomalous, p-form symmetry $G^{(p)}$. $\mathcal{X}$ has a duality symmetry if

$$\mathcal{X} = \mathcal{X}/G^{(p)}$$

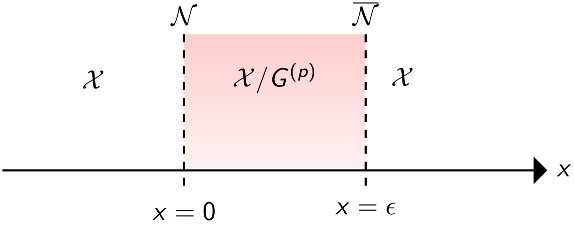
  possible only when $p + 1 = d/2$. In this talk, focus on $G^{(p)} = \mathbb{Z}_N^{(0)}$ in 2d, and $\mathbb{Z}_N^{(1)}$ in 4d.

- A duality symmetry comes with a duality defect $\mathcal{N}$: gauge $G^{(p)}$ on half of the spacetime, with Dirichlet boundary condition for $G^{(p)}$ Wilson operator at the defect locus.



Non-Invertible Fusion Rule

- The duality defects satisfy the non-invertible fusion rule



- Take $\epsilon \to 0$, $\mathcal{N} \times \overline{\mathcal{N}} = \mathcal{C}$
  $\mathcal{C}$ = condensation defect = sum over $G^{(p)}$ defects on $x = 0$ submanifold.
  Non-invertible duality defect. [Choi,Cordova,Hsin,Lam,Shao, 21',22']
  [Kaidi,Ohmori,YZ,21',22'],
  [Bhardwaj,Bottini,Schafer-Nameki,Tiwari,22']...

More Refined Data 1: Bicharacter

- There are different $G^{(p)}$ duality defects with the same fusion rule. [Tambara,Yamagami,98']...

- Take $G^{(p)} = \mathbb{Z}_N^{(0)}$ and $d = 2$: Gauging $\mathbb{Z}_N^{(0)}$ produces a $\mathbb{Z}_N^{(0)}$ quantum symmetry: generated by

$$\exp\left(\frac{2\pi i}{N}\int a^{(1)}\right)$$

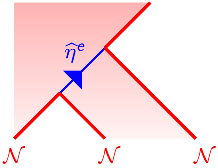
Alternatively, one can take the $\mathbb{Z}_N^{(0)}$ quantum symmetry to be generated by

$$\exp\left(\frac{2\pi i}{N} v \int a^{(1)}\right)$$

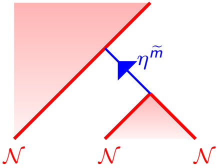
as long as $\gcd(v, N) = 1$, it still generates $\mathbb{Z}_N^{(0)}$. The choice of v was termed bicharacter.

More Refined Data 2: Frobenius-Schur indicator

- Duality defects with the same fusion rule can also have different F-symbols.



$$= \quad \epsilon \sum_{\widetilde{m}=0}^{N-1} e^{-\frac{2\pi i}{N} e \widetilde{m}}$$



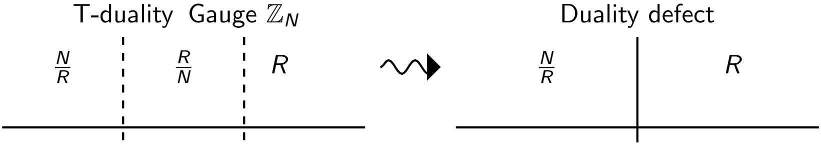
- The choice of $\epsilon = \pm 1$ was termed Frobenius-Schur indicator.

- In 2d, it is known that these refined data completely specify the Fusion Category $TY(\mathbb{Z}_N, v, \epsilon)$. [Tambara,Yamagami,98']

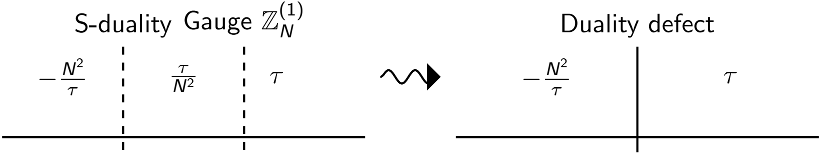
Candidate QFTs with Duality Symmetry

- What is the QFT $\mathcal{X}$ satisfying the self-duality condition $\mathcal{X} = \mathcal{X}/G^{(p)}$? Intuitively, such $\mathcal{X}$ is very limited.

- Example 1: 2d compact bosons at special $R = \sqrt{N}$.



- Example 2: 4d Maxwell theory with special coupling constant $\tau = iN$.



Candidate QFTs with Duality Symmetry

- It seems that the candidate $\mathcal{X}$ with self-duality is highly fine-tuned—showing $\mathcal{X} = \mathcal{X}/G^{(p)}$ requires either T or S-dualities, which hold for only very special theories, like compact bosons, Maxwell theory, $\mathcal{N} = 4$ super-Yang-Mills. These well-known examples are either free field theories, or CFTs.

- However, for some $G^{(p)}$, there exists an alternative, systematic, construction of the self-dual $\mathcal{X}$, which significantly expands the allowed candidates—Group Theoretical Duality Defects

Group Theoretical Duality Defects:

What? Why?

An Alternative Construction of Duality Defects

- Let's consider an example: $G^{(p)} = \mathbb{Z}_2^{(1)}$ in 4d, the QFT $\mathcal{X}$ satisfying $\mathcal{X} = \mathcal{X}/\mathbb{Z}_2^{(1)}$ admits an alternative construction. [Kaidi,Ohmori,YZ,21']

- Start with a 4d QFT $\mathcal{Y}$ with
  - global symmetry: $G^{(1)} \times H^{(0)} = \mathbb{Z}_2^{(1)} \times \mathbb{Z}_4^{(0)}$,
  - mixed anomaly: 5d inflow action $\frac{\pi}{2} A^{(1)} \mathcal{P}(B^{(2)})$.

- Many candidates for $\mathcal{Y}$:
  - $SU(2)$ Yang-Mills at $\theta = \pi$ ($\mathbb{Z}_4^{(0)} \to \mathbb{Z}_2^{T}$.)
  - $\mathcal{N} = 1$ $SU(2)$ SQCD
  - $SU(2)$ QCD with $N_f = 1$ adjoint fermion (w/ or w/o 4-fermion term)
  - any symmetric deformation of the above theories.

An Alternative Construction of Duality Defects

- Construct $\mathcal{X} := \text{Topo.Mani.}(\mathcal{Y})$, where Topo.Mani. $:= TS$
  - S: gauge $\mathbb{Z}_2^{(1)}$.
  - T: stack a 4d $\mathbb{Z}_2^{(1)}$ SPT.
- Claim: $\mathcal{X}$ is self-dual: $\mathcal{X} = S(\mathcal{X}) := \mathcal{X}/\mathbb{Z}_2^{(1)}$.
- Prove: S, T obey $SL(2, \mathbb{Z}_2)$ algebra, in particular $STS = TST$ and $T^2 = S^2 = 1$. Moreover, anomaly implies $T(\mathcal{Y}) = \mathcal{Y}$. Then

$$S(\mathcal{X}) = STS(\mathcal{Y}) = TST(\mathcal{Y}) = TS(\mathcal{Y}) =: \mathcal{X}$$

□

- As many $\mathcal{X}$ with duality defects as $\mathcal{Y}$'s with invertible symmetries:
  - $SO(3)$ Yang-Mills at $\theta = \pi$
  - $\mathcal{N} = 1$ $SO(3)$ SQCD.
  - $SO(3)$ QCD with $N_f = 1$ adjoint fermion (w/ or w/o 4-fermion terms)
  - any symmetric deformation of the above theories.

Group Theoretical Duality Defects

- A duality defect in QFT $\mathcal{X}$ is group theoretical if there is another QFT $\mathcal{Y}$ with invertible symmetries, such that

$$\mathcal{X} \overset{Top.Mani.}{\longleftrightarrow} \mathcal{Y}$$

- Remarks:
  - Clarification on terminology: Group theoretical duality defect was also recently called non-intrinsically non-invertible duality defect. [Kaidi,Zafrir,YZ,22']
  - Group theoretical fusion category has been discussed extensively in math literatures [Drinfeld,Gelaki,Nikshych,Ostrik,07'], [Gelaki,Naidu,Nikshych,09'] Recently group theoretical 2-fusion category was studied in [Décoppet,Yu,23']

Why Shall We Care?

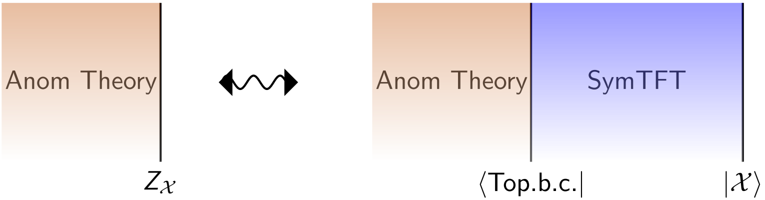
1. Ubiquitous: a large family of QFTs with duality defects.
2. Stable: Easy to add duality-preserving perturbation, friendly for studying duality-preserving RG flows.
3. Intimately related to the anomaly of duality symmetries.

It is difficult to exhaust all possible topological manipulations by brute-force ☹, so some organization principle is required—Symmetry TFT

Symmetry TFT of Duality Symmetry

Symmetry TFT

- SymTFT (together with its topological boundary condition) is a nice way to package the global symmetry and anomaly of the QFT. [Kong,Wen,Zheng,15',17'], [Ji,Wen,19'], [Gaiotto,Kulp,20'], [Kong,Lan,Wen,Zhang,Zheng,20'] [Apruzzi,Bonetti,García-Etxebarria,S.Hosseini,Schafer-Nameki,21'], [Freed,Moore,Teleman,22'], [Kaidi,Ohmori,YZ,22'], [Antinucci,Benini,Copetti,Galati,Rizi,22']... [Garcia Etxebarria's, Ji's and Kong's talks]



$\{$Symmetry, Anomaly theory$\}$ = $\{$SymTFT, Top.b.c.$\}$

Features of SymTFT

$$\{\text{Symmetry, Anomaly theory}\} = \{\text{SymTFT, Top.b.c.}\}$$

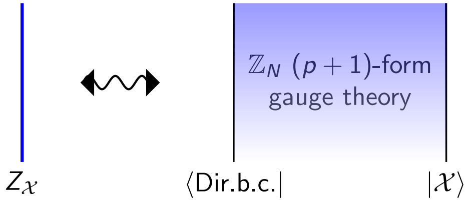
- "Background fields" of the symmetry only enters the top.b.c.
  $\rightarrow$ Topological manipulation only changes top.b.c.
  $\rightarrow$ SymTFT is an invariant under topological manipulation.

- When the symmetry is invertible, the SymTFT is Dijkgraaf-Witten (= gauged higher-group SPT).

A non-invertible symmetry is group theoretical if and only if its SymTFT is Dijkgraaf-Witten.

SymTFT of anomaly-free $\mathbb{Z}_N^{(p)}$ Symmetry

- Since the duality symmetry contains $\mathbb{Z}_N^{(p)}$ symmetry, it is useful to first discuss the SymTFT of anomaly free $\mathbb{Z}_N^{(p)}$ symmetry.



- SymTFT $= \mathbb{Z}_N$ $(p+1)$-form gauge theory:

$$\frac{2\pi}{N}\widehat{b}^{(p+1)}\delta b^{(p+1)}$$

- Left Dirichlet boundary condition:
$\langle \mathrm{Dir}[B^{(p+1)}]| = \sum_{b^{(p+1)}} \delta\left(b^{(p+1)} - B^{(p+1)}\right) \langle b^{(p+1)}|$
- Right dynamical boundary condition: $|\mathcal{X}\rangle = \sum_{b^{(p+1)}} Z_{\mathcal{X}}[b^{(p+1)}]\,|b^{(p+1)}\rangle$

EM Exchange Symmetry

$$\frac{2\pi}{N}\widehat{b}^{(p+1)}\delta b^{(p+1)}$$

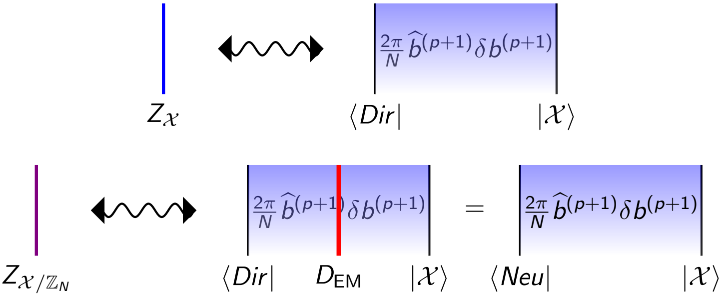
- The $\mathbb{Z}_N$ $(p+1)$-form gauge theory has an EM exchange symmetry

$$b^{(p+1)} \to u\widehat{b}^{(p+1)}, \qquad \widehat{b}^{(p+1)} \to (-1)^p v b^{(p+1)}$$

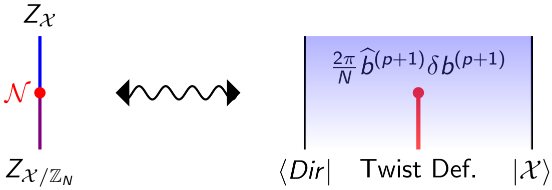
  where $uv = 1 \mod N$, i.e. $u = v^{-1}$ in $\mathbb{Z}_N$.
  The symmetry is $\mathbb{Z}_2^{\text{em}}$ for $p = 0$, and $\mathbb{Z}_4^{\text{em}}$ for $p = 1$.

- The EM exchange symmetry comes with a codim-1 topological defect D_{EM}, and can be explicitly constructed as a condensation defect of $(p+1)$-dim'al operators $e^{\frac{2\pi i}{N}\oint b^{(p+1)}}$ and $e^{\frac{2\pi i}{N}\oint \widehat{b}^{(p+1)}}$.
  [Seifnashri,Roumpedakis,Shao,22'], [Kaidi,Ohmori,YZ,22']

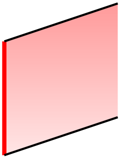
Gauging $\mathbb{Z}_N^{(p)}$ from Fusing with EM Exchange Symmetry Defect



Duality Interfaces from Fusing with Twist Defects



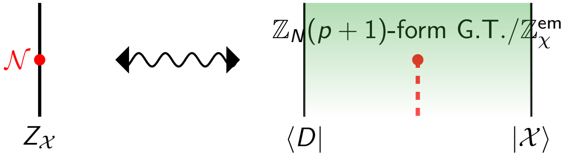
- The twist defect is the EM exchange symmetry defect on half space.



SymTFT of Duality Symmetry

- We further require the theory $\mathcal{X}$ to be invariant under gauging $\mathbb{Z}_N^{(p)}$, i.e. duality symmetry.

- The SymTFT of the duality symmetry should be such that the "tail" of the twist defect is transparent. $\Rightarrow$ The EM exchange symmetry should be gauged.



SymTFT of Duality Symmetry

	SymTFT	Duality Defect
Bicharacter	EM exchange sym $b^{(p+1)} \to u\widehat{b}^{(p+1)}$ $\widehat{b}^{(p+1)} \to (-1)^p v b^{(p+1)}$	choices of quantum symmetry defects $\exp(\frac{2\pi i}{N} v \oint b^{(p+1)})$
FS indicator	Discrete theta term for $\mathbb{Z}_\chi^{\text{em}}$ gauge field	F-move of duality defects

SymTFT of $\mathbb{Z}_N^{(p)}$ duality symmetry $_{v,\epsilon}$

$$||$$

$\mathbb{Z}_N(p+1)$-form gauge theory $/(\mathbb{Z}_\chi^{\text{em}})_{v,\epsilon}$

Criteria of Group Theoretical Duality Defects

Group Theoretical Condition

$\mathbb{Z}_N^{(p)}$ duality symmetry is group theoretical

$\mathbb{Z}_N(p+1)$-form gauge theory$/\mathbb{Z}_\chi^{\text{em}}$ is Dijkgraaf-Witten.

Group Theoretical Condition, Refined

- Luckily, for duality symmetry in 2d, whether the duality symmetry is group theoretical has been discussed by Mathematicians $\sim$15 years ago. [Drinfeld,Gelaki,Nikshych,Ostrik,0704.0195], [Gelaki,Naidu,Nikshych,0905.3117]

- Let me translate the main ideas in physics Language.

Criteria of $\mathbb{Z}_N^{(0)}$ Group Theoretical Duality Defects in 2d

$\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical

↓ ↑

$\mathbb{Z}_N$ gauge theory/$\mathbb{Z}_2^{\text{em}}$ is Dijkgraaf-Witten.

↓ ↑

$\mathbb{Z}_N$ gauge theory/$\mathbb{Z}_2^{\text{em}}$ contains Lagrangian subcategory.

↓ ↑

$\mathbb{Z}_N$ gauge theory contains $\mathbb{Z}_2^{\text{em}}$ stable Lagrangian subgroup.

Criteria of $\mathbb{Z}_N^{(p)}$ Group Theoretical Duality Defects in $2(p+1)$-dim

$\mathbb{Z}_N^{(p)}$ duality symmetry is group theoretical

$\Downarrow \quad \Uparrow$

$\mathbb{Z}_N$ $(p+1)$-form gauge theory$/\mathbb{Z}_\chi^{\text{em}}$ is Dijkgraaf-Witten.

$\Downarrow \quad \Uparrow$

$\mathbb{Z}_N$ $(p+1)$-form gauge theory$/\mathbb{Z}_\chi^{\text{em}}$ contains Lagrangian subcategory.

⇣ $\quad \Uparrow$

$\mathbb{Z}_N$ $(p+1)$-form gauge theory contains $\mathbb{Z}_\chi^{\text{em}}$ stable Lagrangian subgroup.

Example 1: $\mathbb{Z}_N^{(0)}$ duality defects in 2d

Group Theoretical Duality Defects in 2d

$\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical

$\downarrow \quad \uparrow$

3d $\mathbb{Z}_N$ gauge theory contains $\mathbb{Z}_2^{\mathrm{em}}$ stable Lagrangian subgroup.

Our goals are:

1. Determine when 3d $\mathbb{Z}_N$ gauge theory contains $\mathbb{Z}_2^{\mathrm{em}}$ stable Lagrangian subgroup.
2. Establish $\uparrow$ arrow.
3. Establish $\downarrow$ arrow.

Lagrangian subgroups of 3d $\mathbb{Z}_N$ gauge theory

- In 3d $\mathbb{Z}_N$ gauge theory, anyons are labeled by (e, m). The topological spin of the anyon (e, m) is $\theta_{(e,m)} = e^{\frac{2\pi i}{N} em}$. The mutual braiding of two pairs of anyons are determined by the self-spins $\mathcal{B}_{(e,m),(e',m')} = \theta_{(e+e',m+m')}/\theta_{(e,m)}\theta_{(e',m')}$.

- Lagrangian subgroup $\mathcal{A}$ of a 3d Abelian TQFT consists of anyons/topological lines satisfying:
  1. all anyons in $\mathcal{A}$ are bosonic.
  2. any pair of two anyons have trivial mutual braiding.
  3. any anyon not within $\mathcal{A}$ must braid non-trivially at least with one anyon in $\mathcal{A}$.

- Using definition and some elementary-school number theory, one can show

$$\mathcal{A} = \{x(p, 0) + y(0, N/p) \quad | \quad x \in \mathbb{Z}_{N/p}, y \in \mathbb{Z}_p\}$$

  where p is a factor of N.

$\mathbb{Z}_2^{\mathbf{em}}$ Stable Lagrangian Subgroup

- The $\mathbb{Z}_2^{\text{em}}$ symmetry premutes the two generating anyons in $\mathcal{A}$ as

$$(p,0) \to (0,up), \qquad (0,N/p) \to (vN/p,0)$$

- $\mathcal{A}$ is $\mathbb{Z}_2^{\text{em}}$ stable means the resulting anyons also belong to $\mathcal{A}$,

$$(0,up) = (xp, yN/p), \qquad (vN/p,0) = (zp, wN/p)$$

- From $up = yN/p$ mod N, multiply both sides by p, one finds $p^2 = 0$ mod N, so $p^2/N \in \mathbb{Z}$.
- From $vN/p = zp$ mod N, multiply both sides by N/p, one finds $(N/p)^2 = 0$ mod N, so $(N/p)^2/N \in \mathbb{Z} \Leftrightarrow N/p^2 \in \mathbb{Z}$.

$\mathbb{Z}_N$ gauge theory contains an $\mathbb{Z}_2^{\text{em}}$ stable Lagrangian subgroup

$\Updownarrow$

$N = p^2$ is a prefect square.

$\mathbb{Z}_N$ gauge theory $/\mathbb{Z}_2^{\text{em}}$ = Dijkgraaf-Witten

- Recall the action of the $\mathbb{Z}_N$ gauge theory

$$\frac{2\pi}{N}\widehat{b}\delta b$$

- When $N = p^2$, the $\mathbb{Z}_2^{\text{em}}$ stable Lagrangian subalgebra is generated by $e^{\frac{2\pi i}{N}\oint pb}$ and $e^{\frac{2\pi i}{N}\oint p\widehat{b}}$, and the $\mathbb{Z}_2^{\text{em}}$ symmetry exchanges them. It motivates us to introduce the $\mathbb{Z}_p$ fields, via $b = p\widehat{a} + c$, $\widehat{b} = p\widehat{c} + a$, so that the $\mathbb{Z}_2^{\text{em}}$ symmetry exchanges a and c, (and similarly $\widehat{a}$ and $\widehat{c}$)

$$\begin{pmatrix} a \\ c \end{pmatrix} \to \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}, \qquad \begin{pmatrix} \widehat{a} \\ \widehat{c} \end{pmatrix} \to \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \begin{pmatrix} \widehat{a} \\ \widehat{c} \end{pmatrix}$$

- In terms of $\mathbb{Z}_p$ fields, the Lagrangian can be rewritten as

$$\frac{2\pi}{p}\widehat{a}\delta a + \frac{2\pi}{p}\widehat{c}\delta c + \frac{2\pi}{N}a\delta c$$

  This is a $\mathbb{Z}_p \times \mathbb{Z}_p$ Dijkgraaf-Witten theory, and the $\mathbb{Z}_2^{\text{em}}$ symmetry becomes a "flavor rotation" symmetry!

$\mathbb{Z}_N$ gauge theory $/\mathbb{Z}_2^{\text{em}}$ = Dijkgraaf-Witten

- Now, it is straightforward to gauge the "flavor rotation" symmetry (via twisted cocycle approach), and show the resulting theory is Dijkgraaf-Witten.

- After gauging "flavor rotation" symmetry, the Lagrangian becomes

  $$\frac{2\pi}{p}\widehat{\mathbf{a}}_{ij}^T K^{x_{ij}}(K^{x_{jk}}\mathbf{a}_{kl}-\mathbf{a}_{jl}+\mathbf{a}_{jk})+\frac{\pi}{N}\mathbf{a}_{ij}^T\sigma^1 K^{x_{ij}}(K^{x_{jk}}\mathbf{a}_{kl}-\mathbf{a}_{jl}+\mathbf{a}_{jk})+\epsilon\pi x_{ij}x_{jk}x_{kl}$$

  where $K=\begin{pmatrix}0 & v\\ u & 0\end{pmatrix}$.

- x_{ij} is the flat Z_2^{em} dynamical gauge field. The last term is the discrete theta term, related to the FS indicator.

- This is a DW gauge theory with gauge group $(\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_2$, for all choices of the bicharacter v (or u) and the FS indicator ϵ, as expected.

- It is possible to find explicit topological manipulation that maps the duality defect to invertible defect.

Summary so far

$\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical

↑

N is a perfect square

- We have extablished the upward arrow: $\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical if N is a perfect square.
  [Drinfeld,Gelaki,Nikshych,Ostrik,0704.0195], [Gelaki,Naidu,Nikshych,0905.3117], [Sun,YZ,23']

- How about the downward arrow?

Gauging $\mathbb{Z}_2^{\mathbf{em}}$ of $\mathbb{Z}_N$ gauge theory

- One data that is easy to obtain from directly gauging $\mathbb{Z}_2^{\text{em}}$ of $\mathbb{Z}_N$ gauge theory is the spectrum of line operators. See e.g. [Teo,Hughes,Fradkin,15'], [Barkeshli,Bonderson,Cheng,Wang, 14'], [Bhardwaj,Bottini,Schafer-Nameki,Tiwari,22'], [Kaidi,Ohmori,YZ,22'], [Antinucci,Benini,Copetti,Galati,Rizi,22'] for discussions toward physicists.

- Before gauging $\mathbb{Z}_2^{\text{em}}$, the operators include
  - N^2 lines labeled by (e, m)
  - $\mathbb{Z}_2^{\text{em}}$ defect: D_{em}

- Gauging $\mathbb{Z}_2^{\text{em}}$ amounts to three step process (at the level of operator spectrum)
  - keep the lines invariant under $\mathbb{Z}_2^{\text{em}}$
  - promote the twist defect to genunie line by forgetting the "tail".
  - attach a quantum line dual to $\mathbb{Z}_2^{\text{em}}$ to the above lines.

Gauging $\mathbb{Z}_2^{\text{em}}$ of $\mathbb{Z}_N$ gauge theory

- Applying the three step process to $\mathbb{Z}_N$ gauge theory, we obtain the following spectrum of operators:
  - $2N$ invertible lines
  - $N(N-1)/2$ non-invertible lines of quantum dim 2.
  - $2N$ non-invertible lines of quantum dim $\sqrt{N}$.

- However, it is well-known that in a (bosonic) DW theory, all lines should have integer dimensions. $\Rightarrow$ N must be a perfect square! [Kaidi,Ohmori,YZ,22']

$\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical

$\downarrow$

N is a perfect square

Summary of $\mathbb{Z}_N^{(0)}$ Duality Defect in 2d

$\mathbb{Z}_N^{(0)}$ duality symmetry is group theoretical

$\Downarrow \quad \Uparrow$

N is a perfect square

Example 2: $\mathbb{Z}_N^{(1)}$ duality defects in 4d

Group Theoretical Duality Defects in 4d

- The main idea is the same, but there are interesting new technical features. We will be brief.

$\mathbb{Z}_N^{(1)}$ duality symmetry is group theoretical

5d $\mathbb{Z}_N$ 2-form gauge theory contains $\mathbb{Z}_4^{\text{em}}$ stable Lagrangian subgroup.

$\mathbb{Z}_4^{\mathrm{em}}$ stable Lagrangian subgroup

- The Lagrangian subgroup is given by surface operators

$$\mathcal{A} = \{x(p,0) + y(\ell, N/p) \quad | \quad x \in \mathbb{Z}_{N/p}, y \in \mathbb{Z}_p\}$$

- Stability of $\mathcal{A}$ under $\mathbb{Z}_4^{\mathrm{em}}$ if and only if $N = L^2 M$, where -1 is the quadratic residue of M, and L is an arbitrary integer. [Sun,YZ,23']

- The condition again does not depend on the choice of bicharacter v and FS indicator ϵ.

5d $\mathbb{Z}_N$ 2-form gauge theory contains $\mathbb{Z}_4^{\mathrm{em}}$ stable Lagrangian subgroup.

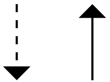
$$\Updownarrow$$

$N = L^2 M$, where -1 is the quadratic residue of M

Group Theoretical Duality Defects in 4d

- Applying similar discussion, when $N = L^2 M$ and -1 being the quadratic residue of M, one can explicitly show that the $\mathbb{Z}_4^{\text{em}}$ gauged theory is a Dijkgraaf-Witten.

$\mathbb{Z}_N^{(1)}$ duality symmetry is group theoretical



$N = L^2 M$ and -1 being the quadratic residue of M

- However, the discussion using quantum dimension does not generalize to 4d, hence the downward arrow remains a conjecture.

Repeated Occurrence

- The sequence $N = L^2 M$, or its subsequence, have appeared in various contexts.
  - The special case of prime N was found in [Bashmakov, Del-Zotto, Hasan, Kaidi,22'].
  - The same full sequence $N = L^2 M$ was found independently to be group theoretical when studying the Hanany-Witten effect in string theory. [Apruzzi,Bonetti,S.W.Gould,Schafer-Nameki 23']
  - A special subsequence for $L = 1$ was found to be a necessary condition for anomaly free. [Choi,Cordova,Hsin,Lam,Shao,21',22']
  - The same full sequence $N = L^2 M$ was found recently in studying the obstruction to duality preserving gapped TQFTs [Apte,Cordova,Lam,22'].

Relation to obstruction to duality preserving SPT and TQFTs

Four Types of Obstructions

Given a $\mathbb{Z}_N^{(1)}$ duality defect, it is useful to distinguish four types of obstructions to

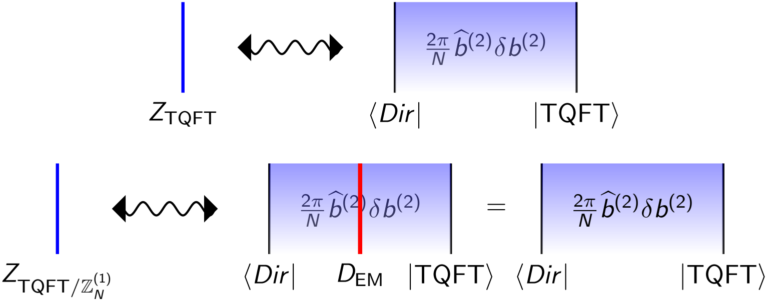
(*a*) existence of SPT equipped with the duality defect with an unspecified bicharacter and FS indicator. $\Leftrightarrow$ SPT $=$ SPT$/\mathbb{Z}_N^{(1)}$. [Choi,Cordova,Hsin,Lam,Shao,21']

(*b*) existence of SPT equipped with the duality defect with specified bicharacter and FS indicator. $\Leftrightarrow$ Anomaly of duality symmetry labeled by N, v, ϵ.

(a') existence of TQFT (with unique GS on S^3) equipped with the duality defect with an unspecified bicharacter and FS indicator. $\Leftrightarrow$ TQFT $=$ TQFT$/\mathbb{Z}_N^{(1)}$. [Apte,Cordova,Lam,22']

(b') existence of TQFT (with unique GS on S^3) equipped with the duality defect with specified bicharacter and FS indicator. $\Leftrightarrow$ Obstruction gapped phase preserving duality symmetry labeled by N, v, ϵ.

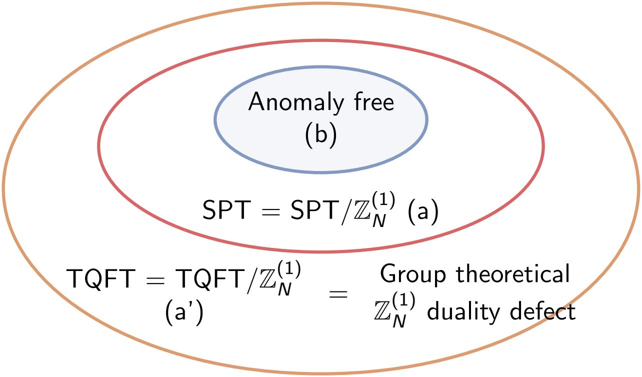
SymTFT Interpretation of Obstruction (a')



if $D_{\text{EM}}|\text{TQFT}\rangle = |\text{TQFT}\rangle$, i.e. $|\text{TQFT}\rangle$ is $\mathbb{Z}_4^{\text{em}}$ stable. This is precisely the condition required by the group-theoretical duality defect, and also reproduces [Apte,Cordova,Lam,22']. See Po-Shen Hsin's and Francesco Benini's talks next Tuesday on SymTFT interpretation of (b) and (b').

Anomaly free ⇒ Group Theoretical

Duality Symmetry is Anomaly Free $= (b) \subset (a) \subset (a') =$ Group Theoretical



Summary

- QFTs with group theoretical duality defect are ubiquitous, including many gauge theories. Friendly to study duality-preserving deformation/RG flows.

- The SymTFT is very useful to determine when a duality defect is group theoretical. The problem can be translated to the existence of EM stable Lagrangian subgroup of finite gauge theories.
  - In 2d QFT, the $\mathbb{Z}_N^{(0)}$ duality defect is group theoretical if and only if N is a prefect square, irrelevant to the choice of bicharacter and FS indicator.
  - In 4d QFT, the $\mathbb{Z}_N^{(1)}$ duality defect is group theoretical if and only if $N = L^2 M$ where -1 is a quadratic residue of M, irrelevant to the choice of bicharacter and FS indicator.

- The group theoretical condition coincides with the obstruction to symmetric gapped phases (with unspecified FS indicator), and this can be easily interpreted using SymTFT. Being group theoretical is a necessary condition for being anomaly free.

Thank you!