Aspects of Self-Dualities and their Anomalies

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Categorical aspects of symmetries

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NP-QFT

Generalized symmetries

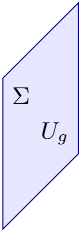
[Gaiotto, Kapustin, Seiberg, Willett 14]

New paradigm: symmetries in (Euclidean) QFT $=$ topological defect operators, of any dimension

Standard 0-form symmetry G: codimension-1 defects $U_g[\Sigma]$, $g \in G$ along submanifold Σ

that fuse according to G group structure:

$$U_g \times U_h = U_{gh}$$



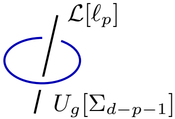
Charge conservation $=$ topological character of defects

⋆ $U(1)$ symmetry: $U_{\alpha \in [0,1)} = \exp\left[2\pi i \alpha \int_\Sigma *j\right]$ Here only *discrete* !

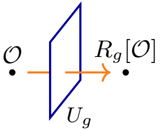
Categorical or Non-invertible Symmetries

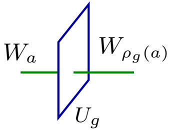
Various new structures:

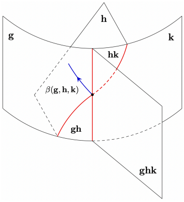
- Defects of higher codimension: p-form symmetries
  Charges carried by p-dimensional extended operators



- Symmetries that act on other symmetries (*e.g.*, n-groups):







from [FB, Cordova, Hsin 18]

- Fusion algebras instead of groups

$$U_a \times U_b = \sum_c N_{ab}^c \;\; U_c$$

- TQFT coefficients: $N_{ab}^c \;\rightarrow\; Z_{\text{TQFT}}[\Sigma_{d-p-1}]$ [Roumpedakis, Seifnashri, Shao 22]

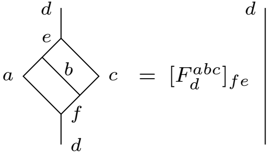
- Symmetries obtained by "condensing" other symmetries

In 2d the "most general" structure* is well understood

*: bosonic QFT, discrete & finite symmetries, no spacetime action (internal), ...

- (Unitary) Fusion category:

Objects:	top. line defects
Tensor product:	stacking of lines
Morphisms:	fusion algebra
	$U_a \times U_b = \sum_c N_{ab}^c \, U_c$
Associator or F-symbol:	



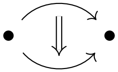
Includes standard 0-form symmetry G with 't Hooft anomaly

One reason for simplicity is that in d dimensions, only up to
$(d-2)$-form symmetries are non-trivial [Hellerman, Henriques, Pantev, Sharpe, Ando 06]

- Structure of (2d) fusion categories
  is equivalently described by certain (3d) modular tensor categories (MTCs)
  Drinfeld center

Related to the concept of Symmetry TFT

In d dimensions: look for $(d-1)$-category



- n-category:

Objects	0-form symmetry defects
1-morphisms between objects	junctions of 0-form defects, and 1-form defects
2-morphisms between 1-morphisms	junctions of junctions, ...
...	
n-morphisms	

In mathematics there are many different definitions.

It is not completely clear which one should appear in physics,
nor how to quantify all pieces of data, relations, constraints, ...

Topic of active research

Examples of non-invertible symmetries in $d > 2$ dimensions:

- Gauge a 0-form symmetry that acts on a higher-form symmetry.
  E.g.: 4d $SU(N)$ Yang-Mills with $\mathbb{Z}_N$ 1-form symmetry,
  gauge charge conjugation $C : U_a \to U_{-a}$

$$U_a \times U_b = U_{a+b} \qquad \to \qquad \begin{aligned} \widetilde{U}_{a\neq -a} &= U_a \oplus U_{-a} \\ \widetilde{U}_a \times \widetilde{U}_b &= \widetilde{U}_{a+b} + \widetilde{U}_{a-b} \end{aligned}$$

[Bhardwaj, Bottini, Schafer-Nameki, Tiwari 22; Antinucci, Galati, Rizi 22]

- Abelian symmetry with ABJ anomaly.
  $\mathbb{Q}/\mathbb{Z} \subset U(1)$ survives as non-invertible:

$$d * j = F \wedge F \qquad \Rightarrow \qquad U_{\theta\in\mathbb{Q}/\mathbb{Z}} = \exp\Big[2\pi i\theta \int_{\Sigma_3} *j + \mathcal{A}^{N,p}[F]\Big]$$

[Choi, Lam, Shao 22, Cordova, Ohmori 22]

- Self-duality symmetries.
  4d $SU(N)$ $\mathcal{N} = 4$ SYM at $\tau = i$ is almost self-dual, but $SU(N) \to PSU(N)$
  There exists a non-invertible version of self-duality

[Kaidi, Ohmori, Zheng 21; Choi, Cordova, Hsin, Lam, Shao 21 & 22]

Some questions

- **Q:** understand the structure of the symmetry.

This is some sort of fusion category, as opposed to a group, so requires more data.

Can we identify all such pieces of data in concrete examples?

Especially difficult in higher dimensions, where we deal with a $(d-1)$-category.

Full set of required data is not known.

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- **Q:** understand anomalies.

Anomalies are a very powerful tool: RG invariant, exactly-calculable observables.
Give constraints on IR dynamics (esp. at strong coupling).

"Standard" anomalies are quantified by an integer: an element of a cohomology group — *e.g.*, $\alpha \in H^{d+1}\big(G, U(1)\big)$ for internal 0-form symmetry G or more generally of a cobordism group, which are Abelian.

Standard anomalies are *additive*:
can be cancelled by adding matter or combining theories.

Such numbers not known for non-invertible symmetries. Might *not* be additive.

- Not even obvious how to *define* anomalies.

For standard continuous 0-form symmetry G (Lie group):

$$Z[A+d\lambda] = \exp\Big(\int_X \mathcal{A}(\lambda, A, dA)\Big)\, Z[A]$$

For invertible discrete symmetries:

- use discrete background fields
- background $\rightarrow$ network of defects
  gauge transformations that maintain the bundle $\rightarrow$ set of moves
  anomalies $\rightarrow$ phases picked up by those moves

For non-invertible symmetries:
not clear what a bundle, a background field, or an allowed move are.
Not clear how to associate a "number" to anomalies.

⋆ Possible definition of anomaly (yes/no):

- Anomaly if symmetry cannot be gauged.

Some questions

- **Q:** how do non-invertible symmetries appear in holography?

Is there a bulk gauge fields for non-invertible symmetries?

At least for discrete symmetries, the answer seems to be very much related to the concept of Symmetry TFT.

⋆ Example: 2d compact boson at rational R^2 [FB, Copetti, Di Pietro 22]

Holographically dual to $\dfrac{U(1)_k \times U(1)_{-k}}{\mathbb{Z}_k^{[1]}}$ 3d Chern-Simons theory

$U(1)_L \times U(1)_R$ 0-form symmetry is dual to a gauge field in the bulk.

Non-invertible symmetry of Verlinde lines: not dual to another gauge field, rather to non-gauge-invariant topological lines that can only lie on the boundary.

Symmetry TFT

⋆ Theory $\mathcal{T}_d$ with invertible (discrete) symmetry G and 't Hooft anomaly:

lives at the boundary of invertible TQFT$_{d+1}$ (or SPT phase) with symmetry G which captures its anomaly $\quad\rightarrow\quad$ anomaly TFT

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⋆ If part H of the symmetry is anomaly-free: gauge it to obtain a new theory $\mathcal{T}_d/H$ with new symmetry and new anomaly

Gaugin discrete symmetries does not introduce new dynamics
(it is RG invariant, reversible and topological):
it only reshuffles physical information between twisted and untwisted sectors

Upgrade SPT phase to a full-fledged (*i.e.*, non-invertible) TQFT such that:

- $\exists$ topological boundary conditions that reproduces original SPT;
- any other topological boundary condition corresponds to a possible gauging;
- anomalies appear as "lack of boundary conditions".

[Gaiotto, Kulp 20; Bhardwaj, Bottini, Schafer-Nameki, Tiwari 22; Freed, Moore, Teleman 22]

Plan of the talk

- Self-duality symmetries (focus on 4d)
- Self-duality symmetries in holography — for 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM
- RG flows (see L. Tizzano's talk)
- Anomalies of self-duality symmetries (in 2d and 4d) [to appear]

  (see A. Antinucci's poster)

  (see also P.-S. Hsin's talk)

Self-duality Symmetries of 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM

Simple example: 4d $\mathcal{N}=4$ super-Yang-Mills with gauge algebra $\mathfrak{su}(N)$

[Choi, Cordova, Hsin, Lam, Shao 22]

Global variants: $SU(N)$, $PSU(N)_p$, $\left[SU(N)/\mathbb{Z}_k\right]_p$ with $k|N$, $p \in \mathbb{Z}_k$

- $SL(2,\mathbb{Z})$ S-duality. S : $\tau \leftrightarrow -\frac{1}{\tau}$ where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

  $$SU(N) \leftrightarrow PSU(N)_0$$

Try to make S-duality into self-duality symmetry: $\tau = i$

But $SU(N)$ theory does not go to itself.

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Try to make S-duality into self-duality symmetry: $\tau = i$
But $SU(N)$ theory does not go to itself.

⋆ Can be corrected by topological operations:

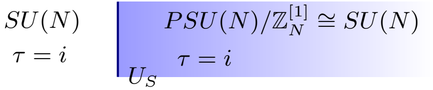
- $SU(N)$ has $\mathbb{Z}_N$ electric 1-form symmetry (shift A by $\mathbb{Z}_N$ center of the group)
  Measures N-ality of Wilson line operators
- $PSU(N)$ has $\mathbb{Z}_N$ magnetic 1-form symmetry (π_1 of gauge group)
  Measures N-ality of 't Hooft line operators (top. class of bundle)

Gauging those 1-form symmetries maps $SU(N) \leftrightarrow PSU(N)_0$

Self-duality Symmetries of 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM

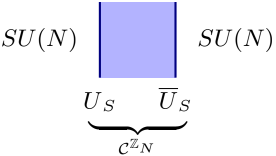
⋆ Combining S-duality with topological gauging maps the theory to itself $\rightarrow$ Symmetry

Doing it on a half space:
topological defect operator U_S



- Non-invertible 0-form self-duality symmetry:

$$U_S \times \overline{U}_S = \mathcal{C}^{\mathbb{Z}_N} \times \mathbb{1}$$
$$U_S \times U_S = \mathcal{C}^{\mathbb{Z}_N} \times U_C$$
$$U_S \times U_C = \overline{U}_S$$
$$\vdots$$



U_C: (invertible) charge conjugation
$\mathcal{C}^{\mathbb{Z}_N}$: 3d condensate of 1-form symmetry

[Choi, Cordova, Hsin, Lam, Shao 22]

- Similar story for non-invertible triality symmetry at $\tau = e^{2\pi i/3}$

  because $(CST)^3 = \mathbb{1}$ in $SL(2, \mathbb{Z})$

- Other examples , *e.g.* in $U(1)$ Maxwell theory

  [Kaidi, Ohmori, Zheng 21; Choi, Cordova, Hsin, Lam, Shao 21]

Self-duality in holography

AdS/CFT: 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM $\longleftrightarrow$ IIB string theory on $\text{AdS}_5 \times S^5$

- ⋆ How does self-duality symmetry and its non-invertibility appear from the holographic description?

$SL(2,\mathbb{Z})$ S-duality of τ $\longleftrightarrow$ $SL(2,\mathbb{Z})$ S-duality of axiodilaton $\tau = C_0 + i\, e^{-\phi}$

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- Global variants are described by boundary conditions for the topological sector:

[Aharony, Witten 98; Witten 98; Belov, Moore 04; Kravec, McGreevy, Swingle 14]

$$\int_{X_{10}} C_2 \wedge H_3 \wedge F_5 \qquad \longrightarrow \qquad \frac{N}{2\pi}\int_{X_5} B_2\, dC_2 \;\equiv\; \frac{N}{4\pi}\int \mathcal{B}^{\mathsf{T}} d\mathcal{B}$$

5d 2-form $\mathbb{Z}_N$ gauge theory (Chern-Simons-like) $\mathcal{B} = (B_2, C_2)$

electric top. b.c.	$B_2\big\|_{\partial X_5} = 0$	$SU(N)$
magnetic top. b.c.	$C_2\big\|_{\partial X_5} = 0$	$PSU(N)$
conformal b.c.	$B_2 - *C_2\big\|_{\partial X_5} = 0$	$U(N)$

Self-duality in holography

S-duality as $SL(2,\mathbb{Z})$ gauge theory in the bulk, spontaneously broken by $\langle\tau\rangle$

$SL(2,\mathbb{Z}_N)$ acts on 5d $\mathbb{Z}_N^{[1]}$ gauge theory (on spin manifolds) [Witten 98]

At special point part of it is unbroken: $\tau = i$ or $\tau = e^{2\pi i/3}$ (C always unbroken)

- Relevant topological theory is 5d $\mathbb{Z}_N^{[1]}$ gauge theory,

  further gauged by $\mathbb{Z}_4$ or $\mathbb{Z}_6$ $\quad\rightarrow\quad$ Symmetry TFT

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⋆ Consider $\mathfrak{su}(N)$ with N odd prime.

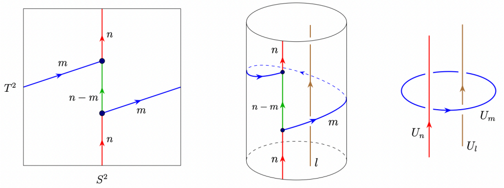
5d $\mathbb{Z}_N^{[1]}$ gauge theory has $\mathbb{Z}_N \times \mathbb{Z}_N$ 2-form symmetry

its surface symmetry defects U_n are generated by

$$\exp\Big(i\int B_2\Big) \qquad \text{and} \qquad \exp\Big(i\int C_2\Big)$$

- 0-form $SL(2,\mathbb{Z}_N)$ defects:

condensation defects of $\mathbb{Z}_N^{[2]} \times \mathbb{Z}_N^{[2]}$ (or subgroups) in 4d, with discrete torsion



E.g.: elements $M \in SL(2,\mathbb{Z}_N)$ s.t. $\operatorname{Tr} M \neq 2$: condense full $\mathbb{Z}_N \times \mathbb{Z}_N$ with

$$\mathcal{T} = \tfrac{\epsilon}{2}(\mathbb{1}+M)(\mathbb{1}-M)^{-1} \qquad\qquad \mathcal{T}: 2\times 2 \text{ symm. matrix}$$

For instance: $\quad S \to \mathcal{T} = \frac{1}{2}\mathbb{1}, \quad C \to \mathcal{T} = 0, \quad T$: gauge a $\mathbb{Z}_N$ subgroup

⋆ Lagrangian description:

$$S[\mathcal{T}] = \frac{N}{2\pi}\int_{\Sigma_4}\left[\mathcal{B}^\mathsf{T}(\Phi + d\Gamma) + \Phi^\mathsf{T} d\Psi + \frac{1}{2}\Phi^\mathsf{T}\mathcal{T}\Phi\right]$$

$\mathcal{T}$ is invertible $\quad\Leftrightarrow\quad \operatorname{Tr} M \neq -2.$ $\qquad$ In this case the 4d TQFT is invertible

(The case that $\mathcal{T}$ is not invertible is a bit special)

Symmetry defects admit a boundary $\rightarrow$ twist defects

- use gauge invariance to write a boundary action
- 4d invertible TQFT is $\mathcal{I}_{\mathcal{T}}(\mathcal{B}) = \frac{N}{2\pi}\int\left[\mathcal{B}^{\mathsf{T}} d\Gamma - \frac{1}{2}\mathcal{B}^{\mathsf{T}}\mathcal{T}^{-1}\mathcal{B}\right]$

determine [Hsin, Lam, Seiberg 18]

$$D[\mathcal{T}] = \mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) \qquad \text{minimal 3d TQFT (MTC)}$$

For C: $D[\mathcal{T}=0] = (\mathbb{Z}_N \times \mathbb{Z}_N)_0(\mathcal{B},\Phi)$

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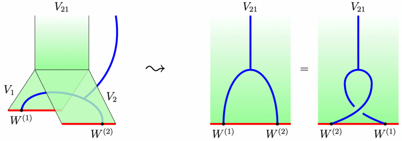
For C: $D[\mathcal{T}=0] = (\mathbb{Z}_N \times \mathbb{Z}_N)_0(\mathcal{B},\Phi)$

- Fusion of twist defects (from Lagrangian description or using MTC's): 5d bulk contributes!

Lines in $D[\mathcal{T}]$ are endpoints of surfaces in 5d, that braid.

$$\mathcal{A}^{N,-\mathcal{T}_2}(\mathcal{B}) \times_{\mathcal{B}} \mathcal{A}^{N,-\mathcal{T}_1}(\mathcal{B}) = \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} \times \mathcal{A}^{N,-\mathcal{T}_{12}}(\mathcal{B})$$
$$D[\mathcal{T}_2] \times D[\mathcal{T}_1] = \mathcal{A}^{N,-\mathcal{T}_1-\mathcal{T}_2} D[\mathcal{T}_{21}]$$

(assuming $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_1+\mathcal{T}_2$ are invertible)



- Project to gapped boundaries, labelled by $\mathscr{L} \subset \mathbb{Z}_N \times \mathbb{Z}_N$

$$U_n\Big|_X = 1 \quad \text{if } n \in \mathscr{L} \qquad \Leftrightarrow \qquad \text{certain Dirichlet b.c.'s for } \mathcal{B}$$

When $D[\mathcal{T}]$ lies on gapped boundary,
part of $\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B})$ decouples and can be thrown away:

$$\mathcal{A}^{N,-\mathcal{T}}(\mathcal{B}) = \underbrace{\mathcal{A}^{N,-t_\ell}(b_\ell)}_{\text{drop}} \times \underbrace{\mathcal{A}^{N,-t_\perp}(b_\perp)}_{D_{\mathscr{L}}[\mathcal{T}]}$$

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Decoupling is consistent with $\times_{\mathcal{B}}$. What is left gives the fusion rules:

$$D_{\mathscr{L}}[\mathcal{T}_2] \times D_{\mathscr{L}}[\mathcal{T}_1] = \mathcal{N}_{21}\, D_{\mathscr{L}}[\mathcal{T}_{21}]$$

We found explicit formulas for computing $\mathcal{N}_{21}$

⋆ For instance, on the electric boundary we find:

$$D_{\mathcal{L}}[\mathcal{T}] \times \overline{D}_{\mathcal{L}}[-\mathcal{T}] = \mathcal{C}^{\mathbb{Z}_N}$$
$$D_{\mathcal{L}}[\mathcal{T}_S] \times D_{\mathcal{L}}[\mathcal{T}_S] = \mathcal{C}^{\mathbb{Z}_N}\; D_{\mathcal{L}}[0]$$

- Finally, gauge the appropriate $\mathbb{Z}_n$ subgroup of $SL(2,\mathbb{Z})$:
  4d surfaces being gauged become transparent
  corresponding 3d boundaries get liberated

[use Bhardwaj, Bottini, Schafer-Nameki, Tiwari 22]

More precisely, gauge-invariant combinations are combination with Gukov-Witten operators:

$$\mathfrak{D}[\mathcal{T}] = \mathsf{GW}_{M(\mathcal{T})} \times D[\mathcal{T}]$$

Resulting fusion rules are not modified: are the ones presented before.

Comments:

- Twisted sectors of $SL(2,\mathbb{Z})$ in SUGRA are 7-branes
- Holographic construction extendable to class $\mathcal{S}$ [Antinucci, Copetti, Galati, Rizi 22]

[cfr. Bashmakov, Del Zotto, Hasan, Kaidi 22]

RG flows

[Argurio, Aguilera Damia, FB, Bevenuti, Copetti, Tizzano 22]

- Consider deformations that preserve the non-invertible symmetries

→ Constraints on dynamics: prevent generation of symmetry-breaking operators

→ Constraints on low-energy theory (in particular in presence of anomalies)

Depending on IR phase, may or may not have spontaneous symmetry breaking

- $SL(2,\mathbb{Z})$ action on operators: [Intriligator 98; Kapustin, Witten 06]

Operators are assigned charge q under modular transformations if

$$M \cdot \mathcal{O}_q = \left(\frac{|c\,\tau_{\mathsf{YM}} + d|}{c\,\tau_{\mathsf{YM}} + d} \right)^{q/2} \mathcal{O}_q \qquad \text{for} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

(square root signals extension $SL(2,\mathbb{Z}) \to Mp(2,\mathbb{Z})$ because $S^4 = C^2 = (-1)^F$)

$\phi_{I=1\ldots6}$ have $q=0$; λ^A_α and Q^A_α have $q=1$; $\frac{1}{\sqrt{\mathbb{I}\mathrm{m}\,\tau}}(F - \tau\,\widetilde{F})$ has $q=2$

At $\mathbb{Z}_k$ self-dual point: $M \cdot Q^A_\alpha = e^{-\frac{i\pi}{k}} Q^A_\alpha$

For $\mathcal{N}=4$ SYM, a simple class of local deformations is by superpotential:

$$W \sim \mathrm{Tr}\,\Phi_1[\Phi_2,\Phi_3] \qquad\qquad \delta W = \sum_{i=1}^{3} m_i\,\mathrm{Tr}\,\Phi_i^2$$

(topological gauging involved in S does not act on local operators)

Superpotential should transform as $M\cdot W = e^{\frac{2\pi i}{k}}W$ to give invariant Lagrangian
$\rightarrow$ combine with suitable R-symmetry rotation

⋆ Examples:

$m_1, m_2, m_3 \neq 0$	$\mathcal{N}=1^*$ theory	gapped vacua or free photons
$m_1{=}m_2 \neq 0$, $m_3{=}0$	$\mathcal{N}=2^*$ theory	Coulomb branch
$m_1 \neq 0$, $m_2{=}m_3{=}0$	$\mathcal{N}=1$ Leigh-Strassler CFT	

(for details see L. Tizzano's talk)

$\mathfrak{su}(N)$ $\mathcal{N} = 1^*$ theory

The vacua are related to partitions of N. Can be gapped or have free photons. Gapped vacua are related to divisors of N ("rectangular" partitions) and counted by $\sigma_1 = \sum_{d|N} d$ [Donagi, Witten 95; Dorey 99; Dorey, Hollowood, Kumar 01]

- *E.g.*: $SU(2)$ 3 gapped vacua: 1 Higgsed H and 2 confined $C^{(0)}$, $C^{(1)}$

H :	$D_{(1,0)}$ = Wilson condenses	$\mathbb{Z}_2$ gauge theory (TQFT)
$C^{(0)}$:	$D_{(0,1)}$ = non-genuine 't Hooft cond.	SPT_0
$C^{(1)}$:	$D_{(1,1)}$ = non-genuine dyon cond.	SPT_1

S-duality: $H \overset{S}{\longleftrightarrow} C^{(0)}$ while $C^{(1)}$ is a singlet

(check using order parameter $\langle \mathrm{Tr}\, \Phi_i^2 \rangle = \partial_{m_i} W$)

$\mathfrak{su}(N)$ $\mathcal{N}=1^*$ theory

The vacua are related to partitions of N. Can be gapped or have free photons. Gapped vacua are related to divisors of N ("rectangular" partitions) and counted by $\sigma_1 = \sum_{d|N} d$ [Donagi, Witten 95; Dorey 99; Dorey, Hollowood, Kumar 01]

★ *E.g.*: $SU(2)$ 3 gapped vacua: 1 Higgsed H and 2 confined $C^{(0)}$, $C^{(1)}$

H :	$D_{(1,0)}$ = Wilson condenses	$\mathbb{Z}_2$ gauge theory (TQFT)
$C^{(0)}$:	$D_{(0,1)}$ = non-genuine 't Hooft cond.	SPT_0
$C^{(1)}$:	$D_{(1,1)}$ = non-genuine dyon cond.	SPT_1

S-duality: $H \xleftrightarrow{S} C^{(0)}$ while $C^{(1)}$ is a singlet

(check using order parameter $\langle \mathrm{Tr}\, \Phi_i^2 \rangle = \partial_{m_i} W$)

- Spontaneous symmetry breaking of (discrete) non-invertible symmetry
  $\rightarrow$ degenerate vacua with inequivalent physical properties

Non-invertible symmetry relates untwisted and twisted sectors

$SU(2)$

Vacuum	H	$C^{(0)}$	$C^{(1)}$
Cond. line	$D_{(1,0)}$	$D^U_{(0,1)}$	$D^U_{(1,1)}$
TQFT	$\mathbb{Z}_2$	SPT_0	SPT_1
SSB	✓	✓	×

$PSU(2)_1$

Vacuum	H	$C^{(0)}$	$C^{(1)}$
Cond. line	$D^U_{(1,0)}$	$D^U_{(0,1)}$	$D_{(1,1)}$
TQFT	SPT_0	SPT_1	$\mathbb{Z}_2$
SSB	✓	✓	×

In $PSU(2)_1$ S-duality is an invertible symmetry.

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$PSU(2)_1$

Vacuum	H	$C^{(0)}$	$C^{(1)}$
Cond. line	$D^U_{(1,0)}$	$D^U_{(0,1)}$	$D_{(1,1)}$
TQFT	SPT_0	SPT_1	$\mathbb{Z}_2$
SSB	✓	✓	×

In $PSU(2)_1$ S-duality is an invertible symmetry.

⋆ Further information from cubic anomalies:
S has no cubic anomaly
TS can have cubic anomaly:
only in the absence of anomaly there can be a trivially-gapped IR phase (not guaranteed)

See Tizzano's talk.

• Other examples: IR CFTs with non-invertible symmetry. Inherited duality.
Gauge coupling does not run. Action of S-duality matches with SUGRA.

Anomalies of self-duality symmetries

- A symmetry is anomalous if it cannot be gauged

This definition requires precise definition of n-category and of gauging

⋆ Strategy: Start in 2d, where the problem is solved.

Rephrase in terms of 3d Symmetry TFT. The result has a natural generalization.
Leads to a Proposal for 4d, that can be considered in examples.

Anomalies of self-duality symmetries

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Leads to a Proposal for 4d, that can be considered in examples.

- 2 dimensions. Eg: Ising CFT

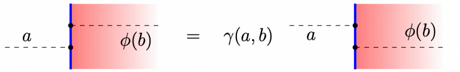
$$\eta \times \eta = \mathbb{1} \,, \qquad \eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N} \,, \qquad \mathcal{N} \times \mathcal{N} = 1 \oplus \eta$$

η: generator of $\mathbb{Z}_2$ symmetry $\qquad$ $\mathcal{N}$: generator of Kramers-Wannier self-duality

⋆ Tambara-Yamagami fusion category $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$: [Tambara, Yamagami 98]

$$a \times b = (a+b) \,, \qquad \mathcal{N} \times a = a \times \mathcal{N} = \mathcal{N} \,, \qquad \mathcal{N} \times \mathcal{N} = \bigoplus_{a \in \mathbb{A}} a$$

$\mathbb{A}$: non-anomalous Abelian symmetry group (self-duality under gauging it)
γ: symmetric non-degenerate bi-character on $\mathbb{A}$ that fixes F-symbols
$\epsilon = \pm 1$: Frobenius-Schur indicator of $\mathcal{N}$ (F-symbol of $\mathcal{N}$)



- Gaugingː exhibit symmetric Frobenius algebra $\mathcal{A}$ [Fuchs, Runkel, Schweigert 02]

$$\mathcal{A} = \left(\bigoplus_{b\in\mathbb{B}} b\right) \oplus n\mathcal{N} \qquad\qquad m : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$$

Gauging depends on subgroup $\mathbb{B}\subset\mathbb{A}$ and discrete torsion $[\nu]\in H^2\big(\mathbb{B},U(1)\big)$

Conditions for gauging: [Tambara 00; Meir, Musicantov 12; Thorngren, Wang 19]
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- Symmetry TFT: [Gelaki, Naidu, Nikshych 09]

3d gauge theory for $\mathbb{A}$ dubbed DW($\mathbb{A}$). Has 1-form symmetry $\mathbb{A}\times\mathbb{A}^\vee$.
γ induces isomorphism $\phi : \mathbb{A}\to\mathbb{A}^\vee$
and automorphism Φ of $\mathbb{A}\times\mathbb{A}^\vee : (a,\alpha)\mapsto\big(\phi^{-1}(\alpha),\phi(a)\big)$
Gauge $\mathbb{Z}_2[\Phi]$ with discrete torsion $\epsilon\in H^3\big(\mathbb{Z}_2,U(1)\big)\cong\mathbb{Z}_2$
This is Drinfeld center of TY category.

⋆ Topological boundary conditions $\leftrightarrow$ discrete gaugings in boundary theory
$\leftrightarrow$ gaugings of the bulk theory that trivialize it completely

Our result: [Antinucci, FB, Copetti, Galati, Rizi (to appear)]

⋆ Topological b.c.'s where $\mathcal{N}$ is trivial at the boundary require:

- A duality-invariant Lagrangian algebra $\mathscr{L}_D$ in DW($\mathbb{A}$):

$$\Phi(\mathscr{L}_D) = \mathscr{L}_D$$

  Lag. algebras classified by $\mathbb{B}$ and $[\nu] \leftrightarrow \chi_\nu$ alternating bicharacter on $\mathbb{B}$

  Conditions for duality invariance are *equivalent* to 1^{st} obstruction

- Gauging $\mathscr{L}_D$ in DW($\mathbb{A}$) gives an SPT for $\mathbb{Z}_2[\Phi]$ dubbed Y.

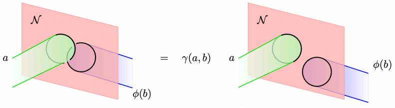
  (We find a simpler formula for Y, equivalent to the one in the literature)

  Gauging of $\mathbb{Z}_2$ is possible only if $Y\epsilon = 1$ — equivalent to 2^{nd} obstruction

Self-duality symmetries in 4d

4d theory with 1-form symmetry $\mathbb{A}$, (on spin manifolds)
self-dual under gauging of $\mathbb{A}$ (possibly with torsion)

Symmetric non-degenerate bicharacter γ induced by braiding on symmetry defect:



Cubic anomaly $\epsilon \in \Omega_5^{\text{spin}}(G)$ for self-duality

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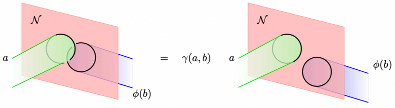

- 5d Symmetry TFT:

2-form gauge theory for $\mathbb{A}$, with 2-form symmetry $\mathbb{A} \times \mathbb{A}^\vee$
Natural $SL(2,\mathbb{Z})$ action: $S : (a, \alpha) \mapsto (-\phi^{-1}(\alpha),\, \phi(a))$
$T : (a, \alpha) \mapsto \big(a + \phi^{-1}(\alpha),\, \alpha\big)$
Gauge $\mathbb{Z}_4[S]$ or $\mathbb{Z}_3[CST]$ with discrete torsion ϵ

Obstructions: Impose that $\exists$ duality-invariant Lagrangian algebra $\mathscr{L}_D$ in DW($\mathbb{A}$)
and that the resulting SPT for G cancels ϵ

- Lagrangian algebras are classified by $\mathbb{B} \subset \mathbb{A}$
  and $[\nu] \in H^4(B^2\mathbb{B}, U(1)) \quad \leftrightarrow \quad \chi_\nu$ symmetric bicharacter on $\mathbb{B}$
  Duality invariance boils down to certain algebraic conditions (1^{st} obstruction)
- After gauging $\mathscr{L}_D$ in DW($\mathbb{A}$), we find a simple formula for SPT$_G$ (2^{nd} obstr.)

Conclusions

We discussed various aspects of self-duality symmetries:

- their structure
- how it can be obtained from holography
- dynamics they can lead to along RG flows
- part of their anomalies

Self-duality symmetries are just one example

- All should fit into the correct definition of a $(d-1)$-category

Thank you!