Non-invertible symmetries of Cardy-Rabinovici model and mixed gravitational anomaly

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Categorical aspects of symmetries at Nordita

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Overview

- Model: Cardy-Rabinovici model (a toy model for YM with θ angle)
- Method: non-invertible symmetry from "duality" & its anomaly
- Results:
    1. $SL(2,\mathbb{Z})$ transformations of the CR model can be understood as "**dualities**" between the CR model and its (appropriately) $\mathbb{Z}_N^{[1]}$-gauged model.
    2. From these "dualities," at self-dual parameters, we construct **non-invertible symmetries** and determine their fusion rules.
    3. We find **a "mixed gravitational anomaly"** of this symmetry for some cases, which rules out the trivially-gapped vacuum.

    (The conjectured phase diagram is consistent with this new constraint.)

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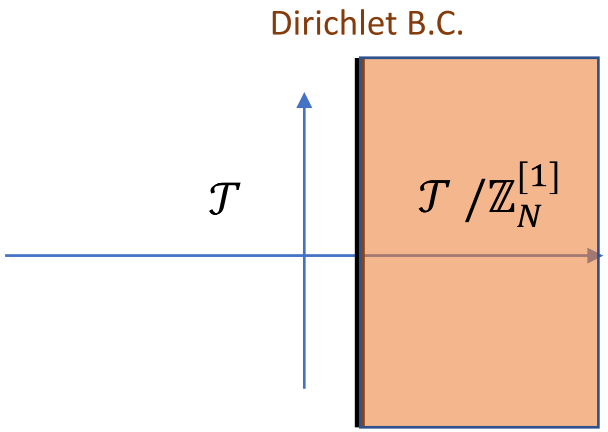
Non-invertible duality defect



"**Half-space gauging**": a popular way to construct non-invertible symmetries applicable to higher dimensions
[Koide, Nagoya, Yamaguchi '21; Choi, Córdova, Hsin, Lam, Shao '21; Kaidi, Ohmori, Zheng '21]

Idea: generalization of KW duality defect

2d example: Kramers-Wannier duality in Ising model.

$$\mathcal{T}\ /\ \mathbb{Z}_2 \simeq \mathcal{T} \quad \Longrightarrow$$ KW duality defect line = "half-space gauging"

Generalization to 4d: **self-duality by 1-form symmetry $\mathbb{Z}_N^{[1]}$ gauging leads to a similar defect**

$$\mathcal{T}\ /\mathbb{Z}_N^{[1]} \simeq \mathcal{T} \quad \Longrightarrow$$ **"half-space $\mathbb{Z}_N^{[1]}$ gauging"**
: 3-dim topological defect

application?

Note.) Gauging a p-form discrete symmetry causes a dual (d-p-2)-form symmetry
→ When d=4, only 1-form symmetry gauging can be self-dual.

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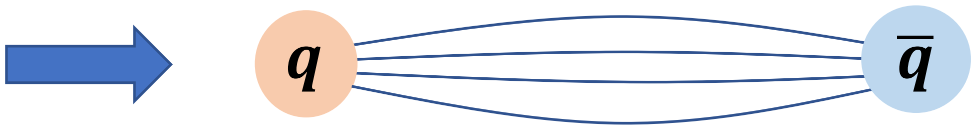
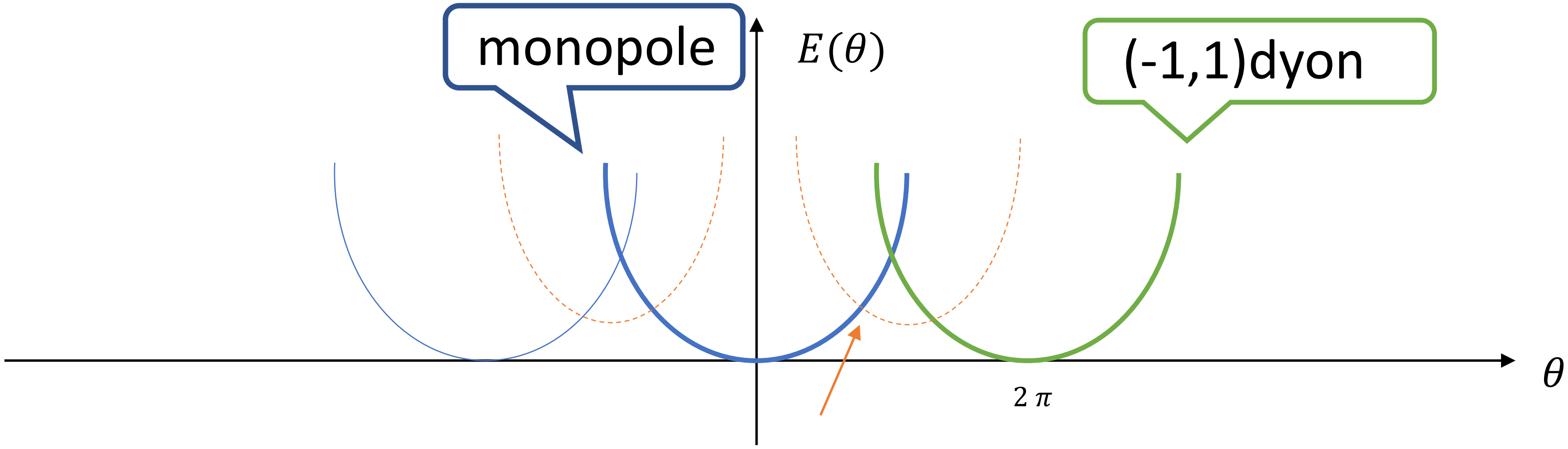
Motivation: quark confinement & θ angle

A popular understanding of quark confinement: dual superconductor picture

(monopole condensation)



Witten effect: monopole acquires electric charge $\theta/2\pi$ by increasing θ



cf.) Large-N → multi-branch quadratic function structure [Witten '80]

Cardy-Rabinovici model

A toy model mimicking such structure:

Cardy-Rabinovici model [Cardy and Rabinovici '82, Cardy '82]

- **4d U(1) gauge** + **charge-N Higgs** + **monopole**
- $\mathbb{Z}_N^{[1]}$ symmetry ($\sim$ $\mathbb{Z}_N^{[1]}$ center symmetry in $SU(N)$ YM)
- Formulated as a Villain-type lattice gauge theory. Symbolically,

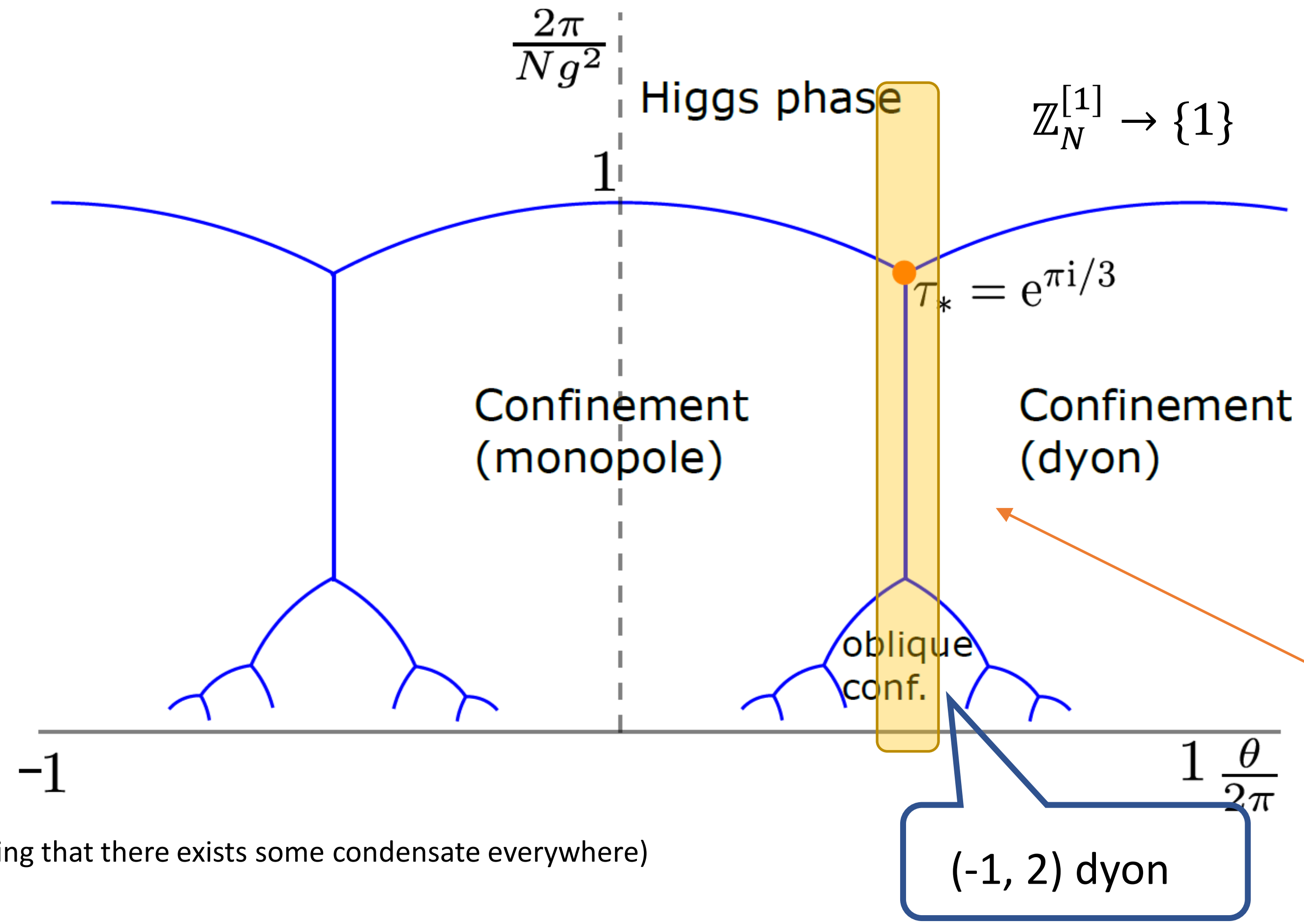
$$Z_{CR} = \int \mathcal{D}a \, e^{-S_{U(1)}[da]} \sum_{C,C':\mathrm{loops}} W^N(C)\, H(C')$$

where $S_{U(1)}[da] = \frac{1}{2\,g^2} \int da \wedge * \, da + \frac{iN\theta}{8\,\pi^2} \int da \wedge da,$

$W(C)$: Wilson loop, $H(C)$: 't Hooft loop

Conjectured phase diagram

An energy vs. entropy argument for $W^{Ne}(C)\, H^{m}(C)$ [Cardy and Rabinovici '82, Cardy '82]



(assuming that there exists some condensate everywhere)

Complex coupling

$$\tau := \frac{\theta}{2\,\pi} + i\,\frac{2\pi}{Ng^2}$$

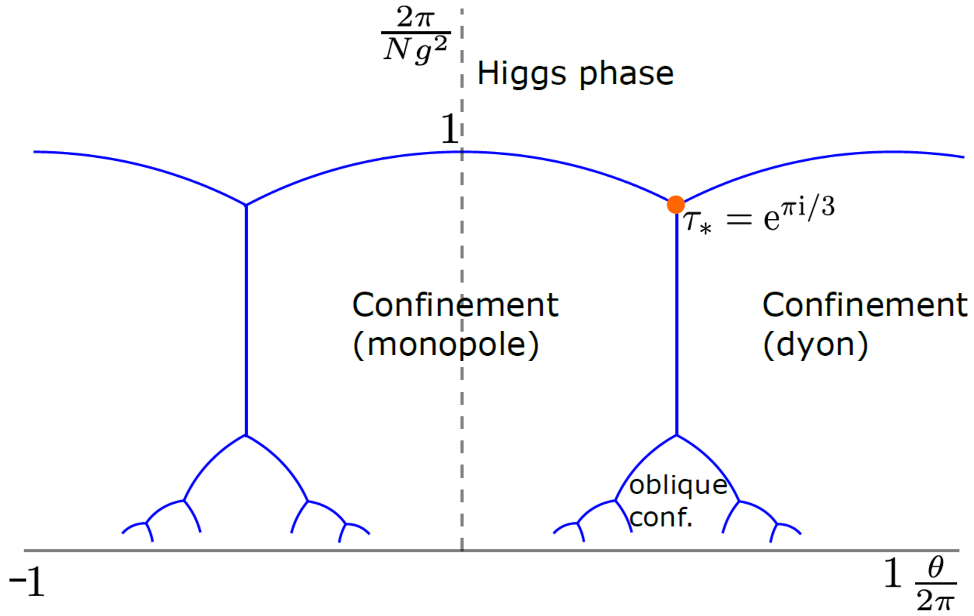
The same CP& $\mathbb{Z}_N^{[1]}$ mixed anomaly as $SU(N)$ YM [Honda and Tanizaki '19]

Conjectured phase diagram

Complex coupling

$$\tau := \frac{\theta}{2\pi} + i\,\frac{2\pi}{Ng^2}$$

- **This phase diagram has $SL(2,\mathbb{Z})$ invariance: S ("electromagnetic" duality) and T ($\theta \to \theta + 2\pi$) transformations.**



$$S: \tau \mapsto -\frac{1}{\tau}, \quad \begin{pmatrix} e \\ m \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e \\ m \end{pmatrix} = \begin{pmatrix} -m \\ e \end{pmatrix}$$

$$T: \tau \mapsto \tau + 1, \quad \begin{pmatrix} e \\ m \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e \\ m \end{pmatrix} = \begin{pmatrix} e - m \\ m \end{pmatrix}$$

- However, the standard S transformation is not the duality of the CR model itself, because S-transformed model has electric charge-1 & magnetic charge-N matters.

→ **duality between the CR model and its $\mathbb{Z}_N^{[1]}$-gauged model**

Results

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Notations

- (The spacetime manifold is spin and torsion-free).
- The partition function with $\mathbb{Z}_N^{[1]}$ background B:

$$Z_{CR}^{\tau}[B] := \int \mathcal{D}a \, e^{-S_{U(1)}[da+B]} \sum_{C,C':\mathrm{loops}} W_{da+B}^{N}(C)\, H_{da+B}(C')$$

- The partition function of level-p $\mathbb{Z}_N^{[1]}$-gauged CR model with (dual) $\mathbb{Z}_N^{[1]}$ background B:

$$Z^{\tau}_{CR\,/\left(\mathbb{Z}_N^{[1]}\right)_p}[B] := \int \mathcal{D}b \, Z_{CR}^{\tau_*}[b]\, e^{\frac{iNp}{4\pi}\int b\wedge b} e^{\frac{iN}{2\pi}\int b\wedge B}$$

with the following normalization,

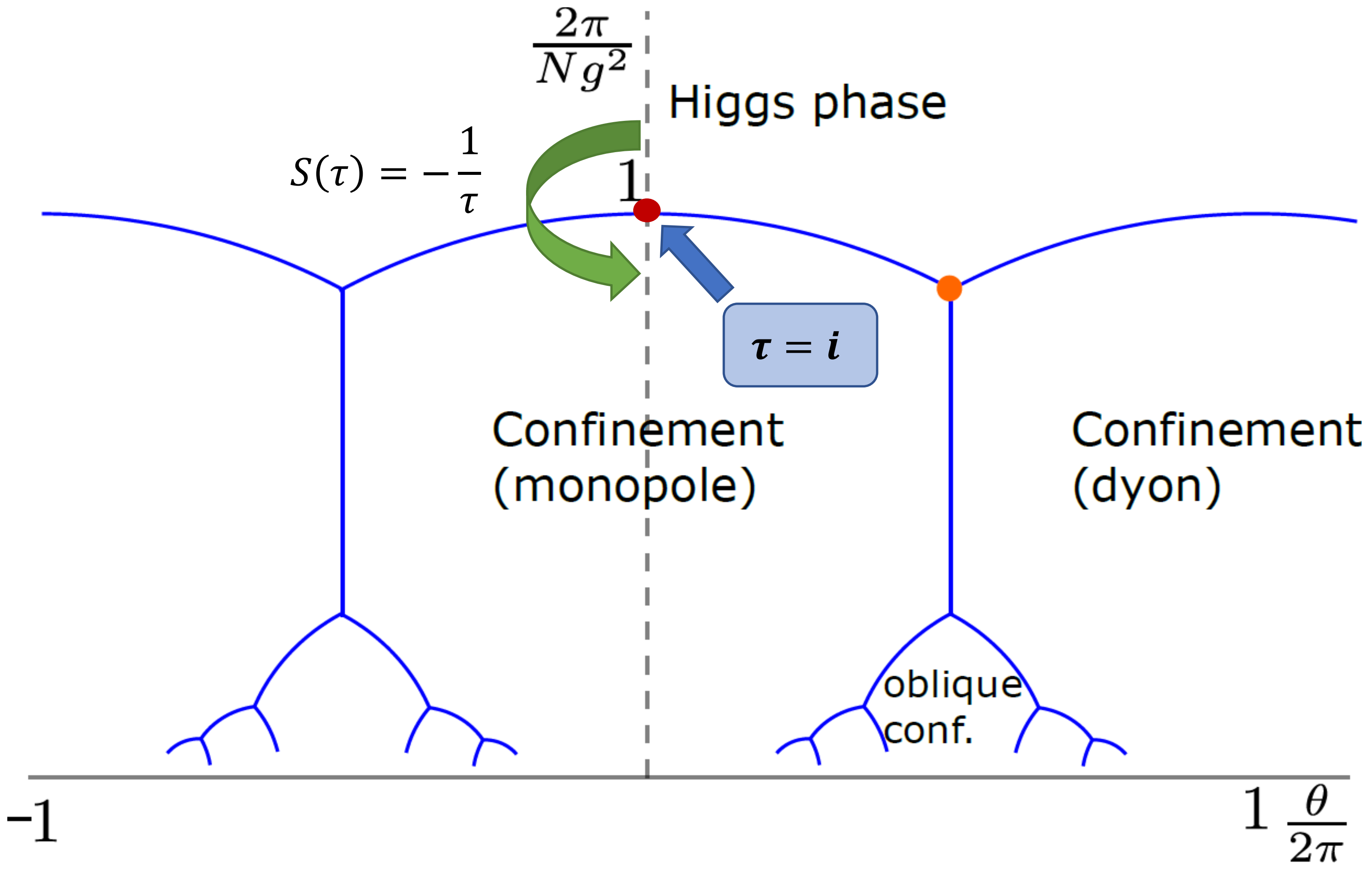
$$\int \mathcal{D}b \ldots := \frac{|H^0(X;\mathbb{Z}_N)|}{|H^1(X;\mathbb{Z}_N)|} \sum_{b\in H^2(X;\mathbb{Z}_N)} \ldots$$

Warm-up: S-defect

For Maxwell theory, constructed in [Choi, Córdova, Hsin, Lam, and Shao '21]

Complex coupling

$$\tau := \frac{\theta}{2\pi} + i\,\frac{2\pi}{Ng^2}$$



The S “self-duality” at $\tau = i$ can be realized as

$\mathbb{Z}_N^{[1]}$-gauging → scale of U(1) by N

$da + b \to da'/N$

[U(1) gauge + (1,0) matter + (0,N) matter] system at coupling $\tau = i/N^2$

$$\boldsymbol{CR} \,/\, \left(\mathbb{Z}_{\boldsymbol{N}}^{[\mathbf{1}]}\right)_{\mathbf{0}} \simeq \boldsymbol{CR}$$

[U(1) gauge + (N,0) matter + (0,1) matter] system with coupling $\tau = i$

electromagnetic
S transform

CP symmetry

We can construct non-invertible defects by half-space gauging

$$\mathcal{D}(M) \times \mathcal{D}(M) = \mathcal{C}(M)\, \frac{1}{N} \sum_{\Sigma \in H_2(M, \mathbb{Z}_N)} \eta\,(\Sigma)$$

$$\eta\,(\Sigma) \times \mathcal{D}(M) = \mathcal{D}(M) \times \eta\,(\Sigma) = \mathcal{D}(M)$$

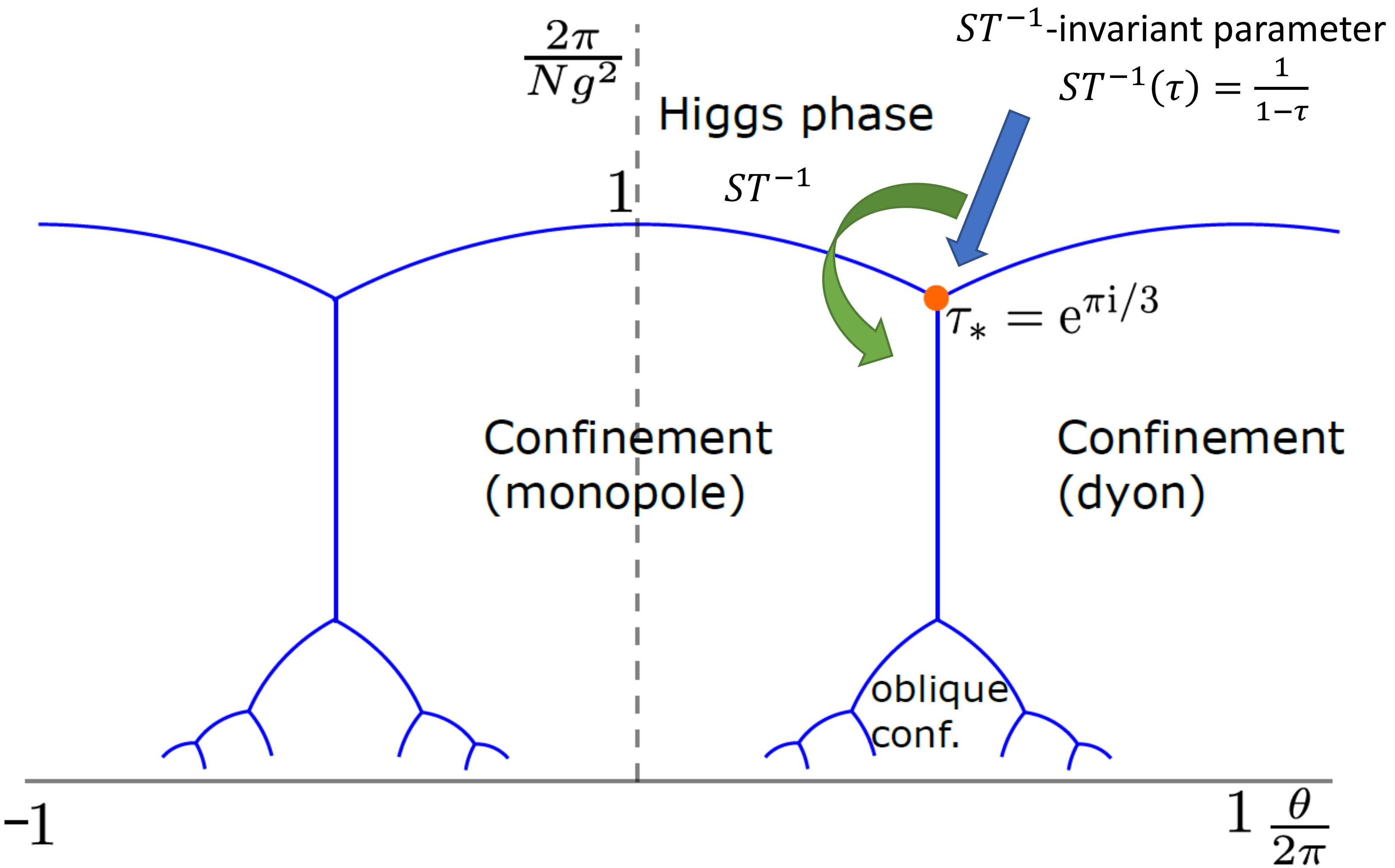
The trivially-gapped phase is ruled out for $N > 2$.

A simple guess from the conjectured phase diagram $Z_{mono}[B] + Z_{Higgs}[B]$ is consistent with these constraints.

Nontrivial example: ST^{-1} defect

Complex coupling

$$\tau := \frac{\theta}{2\pi} + i\,\frac{2\pi}{Ng^2}$$



The ST^{-1} "self-duality" at $\tau = \tau_* = e^{i\pi/3}$ can be realized as

$da + b \to da'/N$

discrete θ for b $\to$ 2π shift of θ

⇓

[U(1) gauge + (1,0) matter + (0,N) matter] system with coupling $\frac{\tau_* - 1}{N^2}$

$$\boldsymbol{CR} \,/\, \left(\mathbb{Z}_N^{[1]}\right)_{-1} \simeq \boldsymbol{CR}$$

[U(1) gauge + (N,0) matter + (0,1) matter] system with coupling $-\frac{1}{\tau_* - 1} = \tau_*$

$$\left(Z^{\tau_*}_{CR\,/\left(\mathbb{Z}_N^{[1]}\right)_{-1}}[B] = N^{\frac{\chi}{2}} e^{-\frac{i\pi}{3}\sigma} Z^{\tau_*}_{CR}[B] \right)$$

cf.) [Witten '95] for Maxwell theory

mixed gravitational anomaly

We can construct non-invertible defects by half-space gauging

$$\mathcal{D}(M) \times \mathcal{D}(M) \times \mathcal{D}(M) \propto \mathrm{C}(M) \sum_{\Sigma \in H_2(M, \mathbb{Z}_N)} \eta(\Sigma)$$

$$\eta(\Sigma) \times \mathcal{D}(M) = \mathcal{D}(M) \times \eta(\Sigma) = \mathcal{D}(M)$$

cf.) [Choi, Córdova, Hsin, Lam, Shao '22]

The trivially-gapped phase is ruled out.

∵ Any SPT partition function cannot satisfy this relation on, e.g., K3 surface.

cf.) [Apte, Córdova, Lam '22]

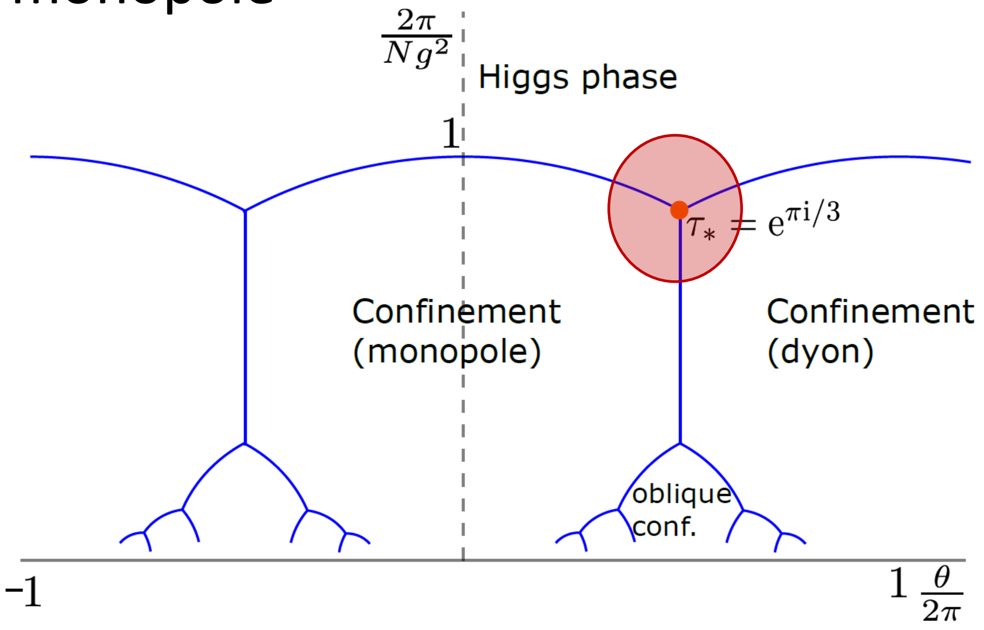
Anomaly and conjectured phase diagram

A natural guess for low-energy theories of Higgs, monopole-condensed, and dyon-condensed phases:

$$Z_{Higgs}[B] = \int \mathcal{D}a\mathcal{D}b\ e^{\frac{iN}{2\pi}\int b\wedge(da+B)}$$

$$Z_{mono}[B] = 1,$$

$$Z_{dyon}[B] = e^{\frac{iN}{4\pi}\int B\wedge B}$$



A linear combination of them:

$$Z[B] = Z_{mono}[B] + e^{\frac{\pi i}{3}\sigma} Z_{dyon}[B] + N^{-\frac{\chi}{2}} e^{\frac{2\pi i}{3}\sigma} Z_{Higgs}[B]$$

matches the anomaly: $Z^{\tau_*}_{CR/\left(\mathbb{Z}_N^{[1]}\right)_{-1}}[B] = N^{\frac{\chi}{2}} e^{-\frac{i\pi}{3}\sigma} Z^{\tau_*}_{CR}[B]$!

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