9 vs 10

Math	Physics
K-theoretic …	and KK modes
partition functions, invariants, moduli spaces	Precision 3d-3d

Based on:

- S.G., P.-S.Hsin, D.Pei (I, II)
- S.G., A.Sheshmani, S.-T.Yau

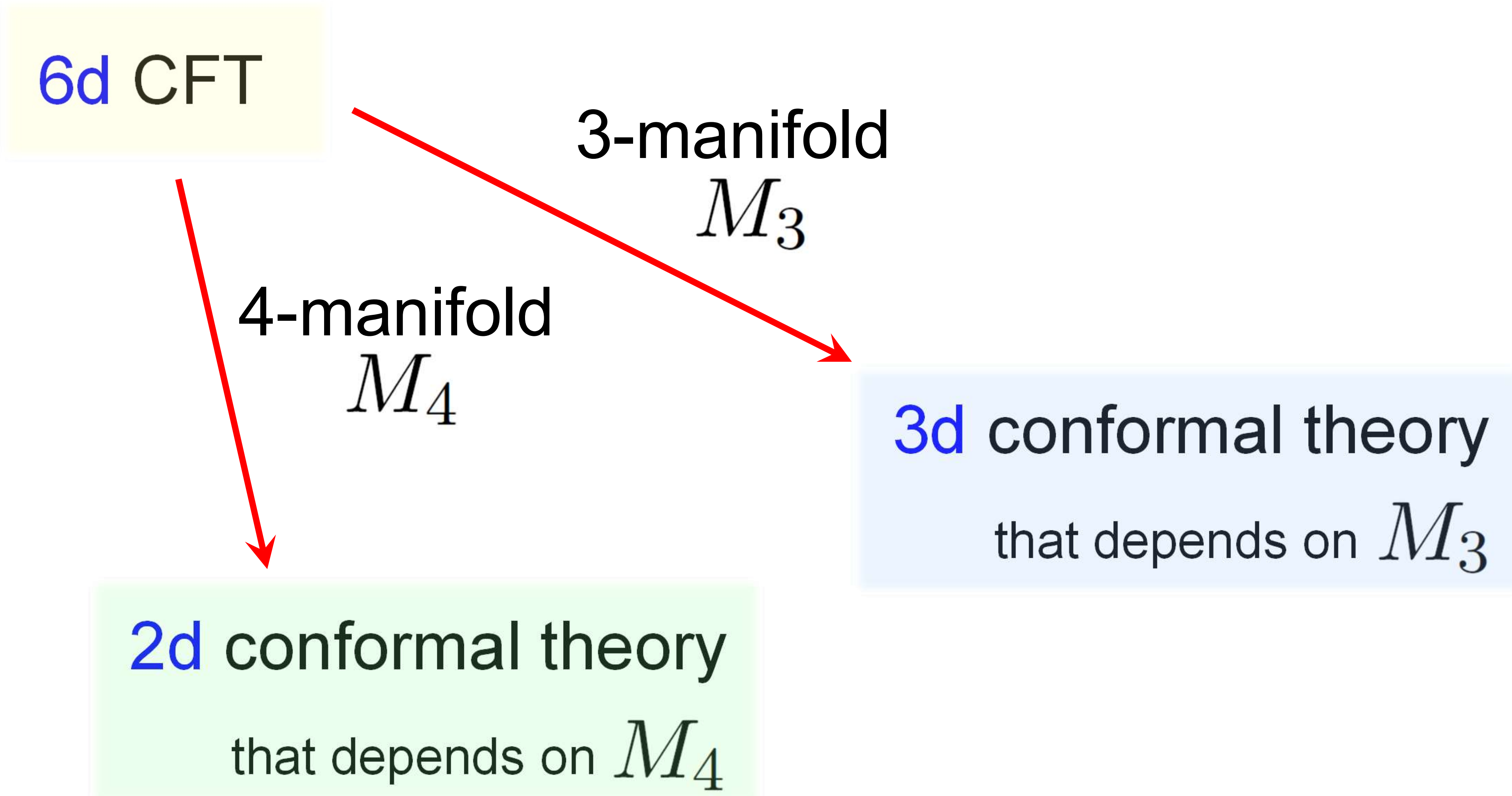
See also:

- S.Eckhard, H.Kim, S.Sch afer-Nameki, B.Willett
- C.Cordova, T.Dumitrescu, K.Intriligator
- M.Del Zotto, J.Heckman, D.Park, T.Rudelius

:

Definition: (6-n)-dimensional theory

$$T[M_n, \ldots] := \text{ 6d (0,2) theory on } M_n$$



Definition: (6-n)-dimensional theory

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G, polarization, ...

Today: G=U(1) single fivebrane
G=SU(2)

$$M_3 = S^1 \times S^2,\, T^3,$$
$$L(p,1) = S^3/\mathbb{Z}_p,\, \ldots$$

mapping tori

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G, polarization, ...

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G=SU(2)

fermions

6d: $B, \quad \phi_{1,\ldots,5}, \quad \psi = (\mathbf{4}, \mathbf{4})$

$SO(5)_R$

Question: $T[T^2] \stackrel{?}{=}$ 4d $\mathcal{N}=4$ super-Yang-Mills

Question: $T[T^2] \overset{?}{=}$ 4d $\mathcal{N} = 4$ super-Yang-Mills

Answer: No

6d: 2-form symmetry C ⇒ on T^2 :

$G = U(1)$: $C = U(1)$

$G = SU(2)$: $C = \mathbb{Z}_2$

2-form

1-form 1-form

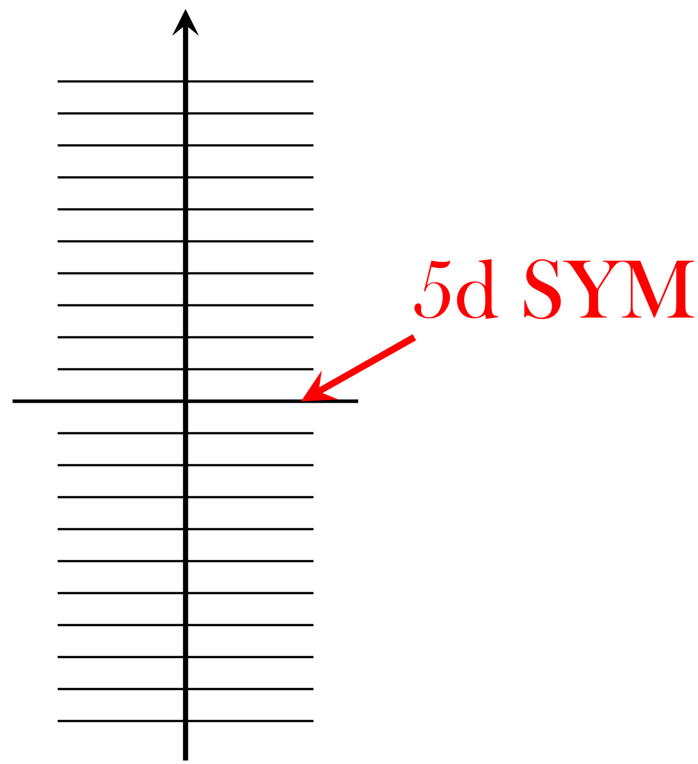
0-form

$T[M_n]$ for $M_n = T^n$

n=1: $T[S^1]$ 5d theory $G = U(1)$

6d		5d
B	$\longrightarrow$	$A, \ldots$
$\vec{\phi}$	$\longrightarrow$	$\vec{\phi}_{m \in \mathbb{Z}}$
ψ	$\longrightarrow$	$\psi_{m \in \mathbb{Z}}$

momentum on S^1



$$T[M_n] \text{ for } M_n = T^n$$

Question: $T[S^1] \overset{?}{=}$ 5d SYM

if yes, what in 5d **SYM** accounts for

$\vec{\phi}_m$ and ψ_m with $m \neq 0$

E.Witten
M.Douglas
N.Lambert, C.Papageorgakis, M.Schmidt-Sommerfeld
H.-C.Kim, S.Kim, E.Koh, K.Lee, S.Lee
Y.Tachikawa
A.Gustavsson
:

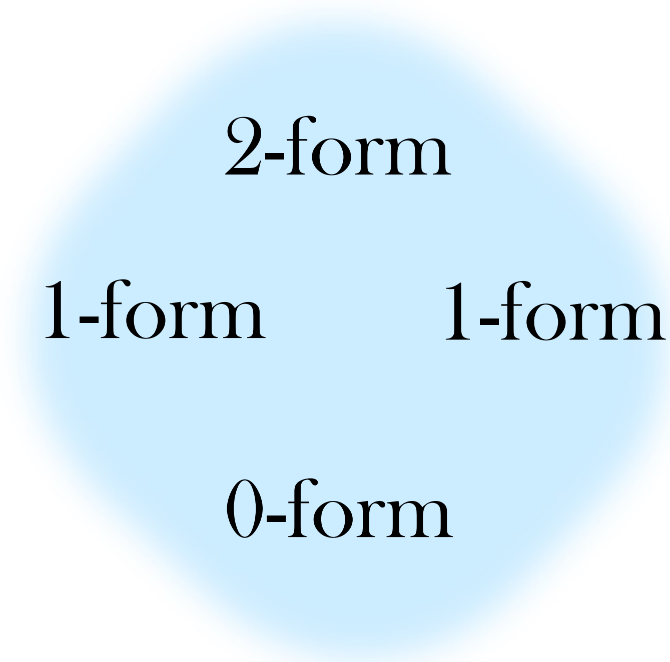
$$T[M_n] \text{ for } M_n = T^n$$

n=2: $T[T^2]$

$SO(5)$ R-symmetry

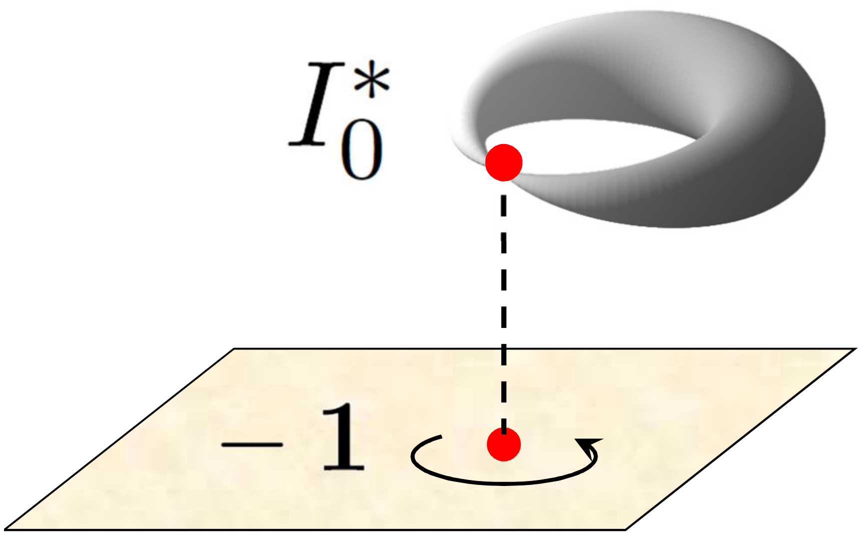
$\varphi \in G$

6d		4d
B	$\longrightarrow$	$A, \varphi, \ldots$
$\vec{\phi}$	$\longrightarrow$	$\vec{\phi}_{m,n}$
ψ	$\longrightarrow$	$\psi_{m,n}$

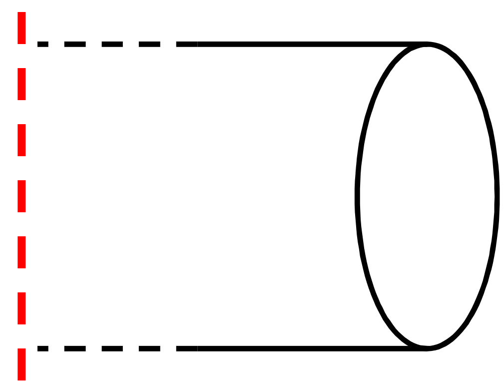


Coulomb branches

4d $\mathcal{N} = 4$ $G = SU(2)$



$T[T^2]$



$G = U(1)$: $\mathbb{C}^*$

$G = SU(2)$: $\mathbb{C}^*/\mathbb{Z}_2$

$\Downarrow$

$\Phi \to -\Phi$

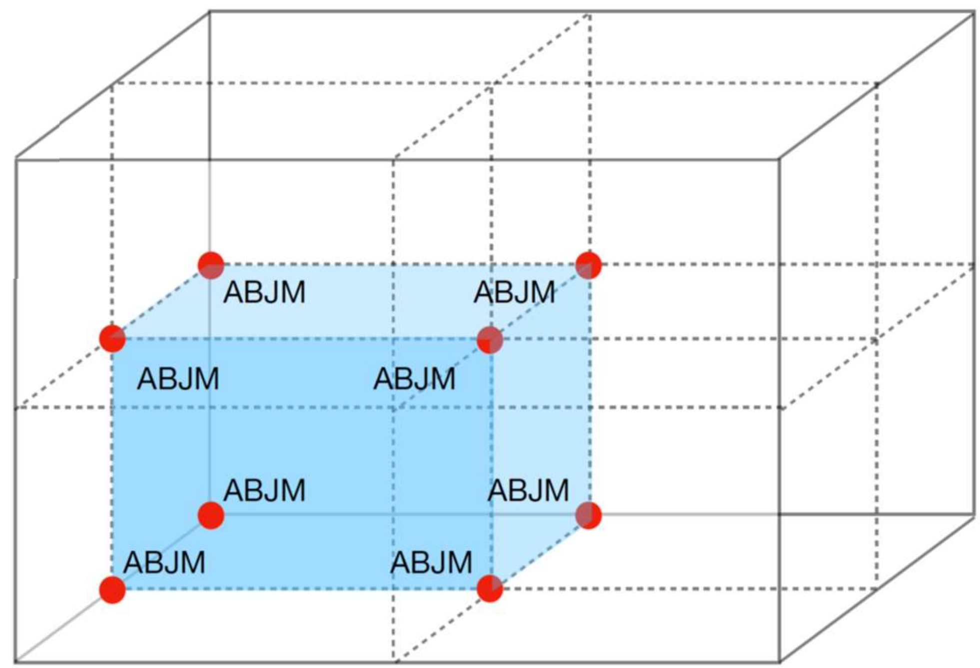
n=3: $T[T^3]$

$$\mathcal{M}_{U(1)} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}$$

$$\mathcal{M}_{SU(2)} = \frac{\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}}{\mathbb{Z}_2}$$

6d		3d
B	$\longrightarrow$	$A, \ \varphi_1, \ \varphi_2, \ \ldots$ dual to $\varphi_3 \in S^1$
$\vec{\phi}$	$\longrightarrow$	$\vec{\phi}_{m,n,k}$
ψ	$\longrightarrow$	$\psi_{m,n,k}$

$$\mathcal{M}_{\mathrm{ABJM}} = \mathbb{C}^4/\mathbb{Z}_2$$



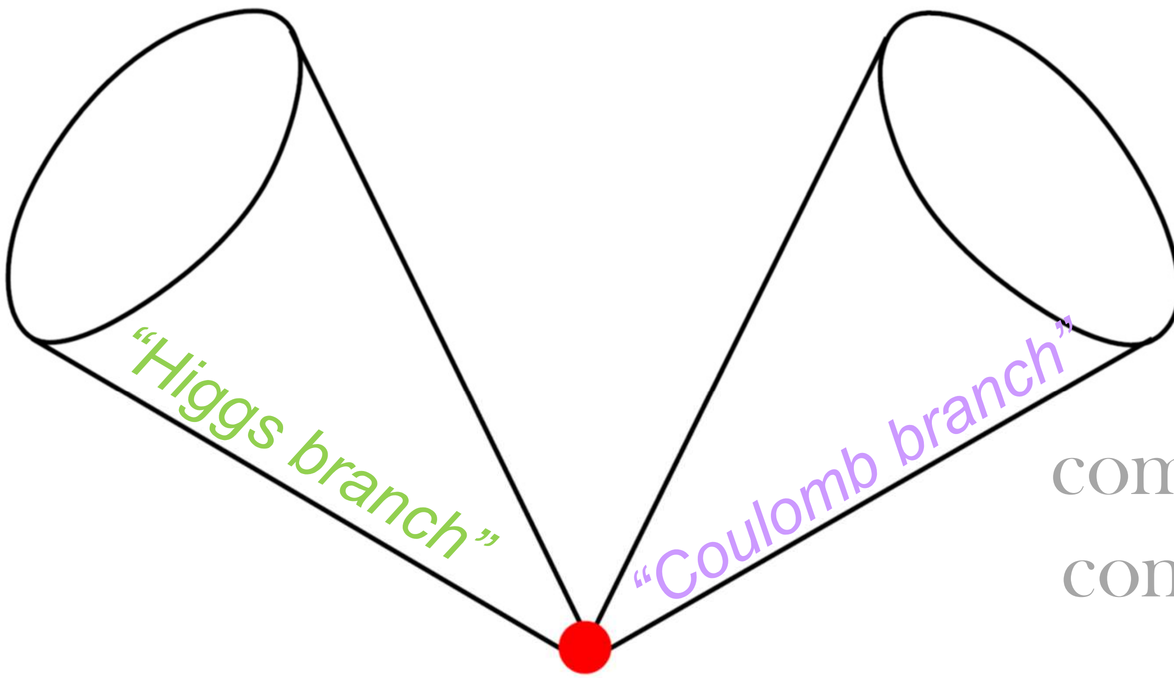
S.Chun, S.G., S.Park, N.Sopenko

$T[M_3]$ for $M_3 = S^3_0(K)$

9 such theories (polarizations)

Spin

Non-abelian complex flat connections (e.g. DGG)



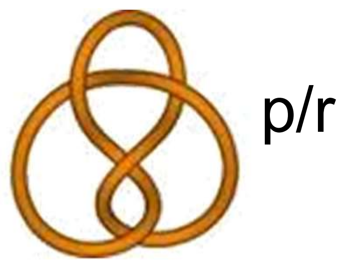
Abelian complex flat connections

3d $\mathcal{N}=2$ SCFT

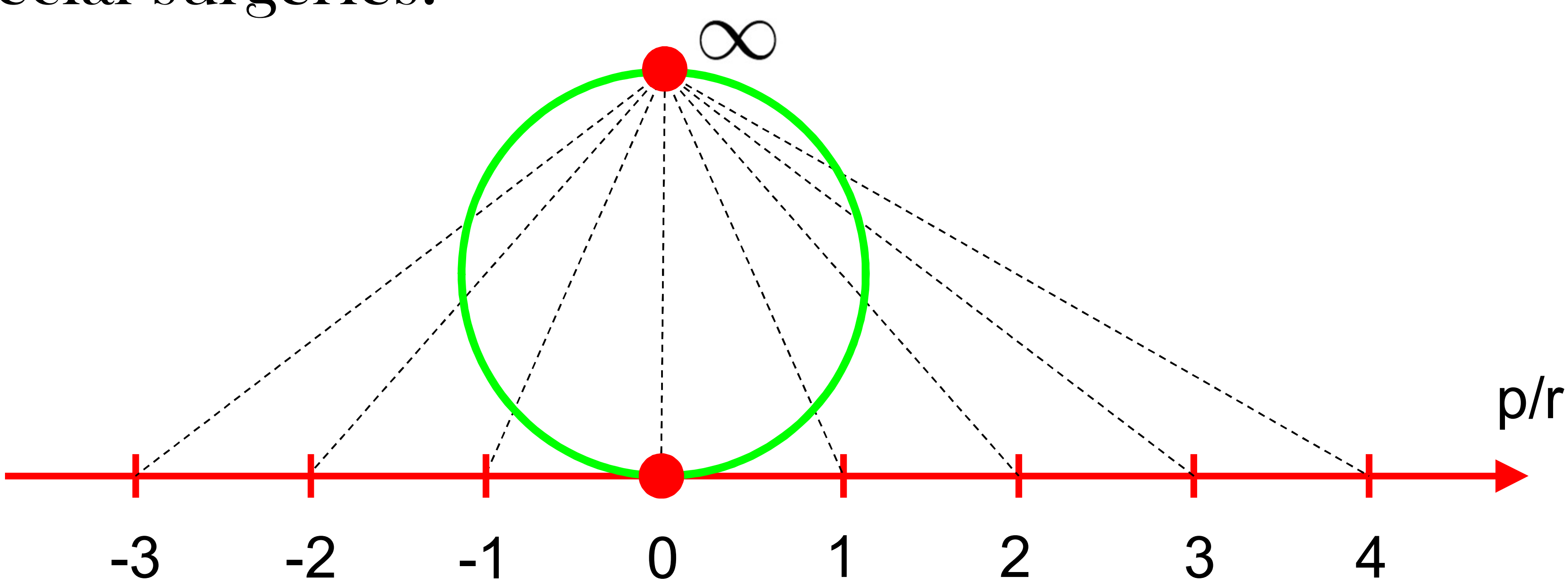
S.Chun, S.G., S.Park, N.Sopenko

Theorem [Lickorish, Wallace]:

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link in S^3.



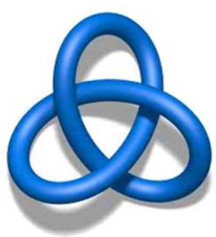
Special surgeries:



Property R

Theorem ("property R" conjecture): D.Gabai (1983)

If the 0-surgery on $K \subset S^3$ is homeomorphic to $S^1 \times S^2$, then K is the unknot.



The trefoil knot and the figure-8 knot are uniquely characterized by 0-surgery.

D.Gabai (1987)

$$M_3 = S^3_0(K)$$

Conjecture [Akbulut and Kirby '97]:

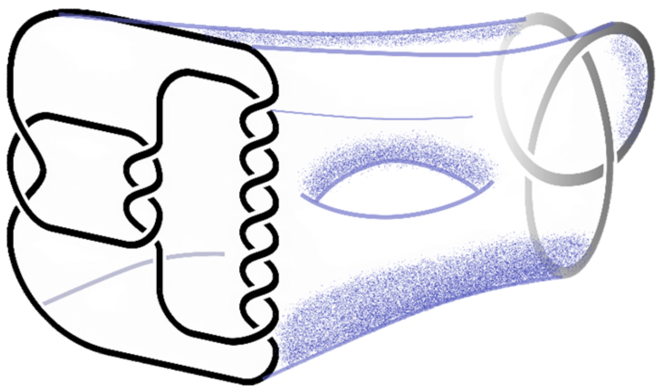
If 0-surgeries on two knots give the same 3-manifold,

$$S^3_0(K) \cong S^3_0(K')$$

then the knots are concordant.

FALSE

P.Kirk, C.Livingston (1999)



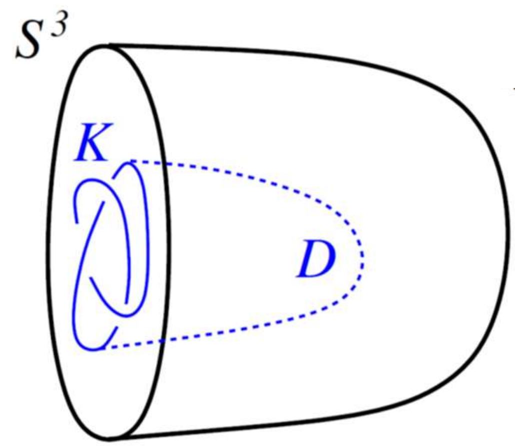
Conjecture:

If 0-surgeries on two knots give the same 3-manifold, then the knots with relevant orientations are concordant.

False if slice-ribbon conjecture is true. FALSE K.Yasui (2015)

Thm: For $M_3 = S^3_0(K)$ at least one of Rokhlin invariants vanishes. M.Hedden, M.H.Kim, T.Mark, K.Park (2018)

Cor: If M_3 is integral homology $S^1 \times S^2$ with two non-trivial Rokhlin invariants, then $M_3 \neq S^3_0(K)$.



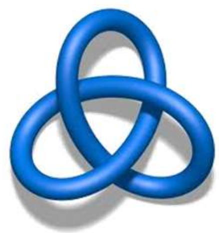
Thm: If K is slice, then L.Truong (2021)

$$b_2(M_4) \geq \frac{10}{8}|\sigma(M_4)| + 5$$

where $\partial M_4 = S^3_0(K)$, $b_2(M_4) \neq$ 1, 3, or 23, and M_4 is a two-handlebody (two-handles attached to a 4-ball).

Generalized Poincare conjecture:

Every homotopy 4-sphere is diffeomorphic to the standard 4-sphere.

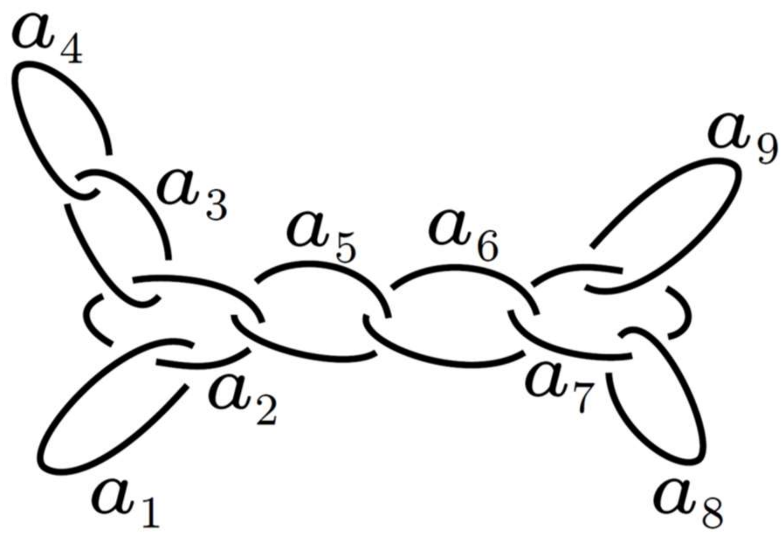


Theorem: If one finds a pair of knots which satisfy the following three properties:

- K and K′ have the same 0-surgery
- K is not slice
- K′ is slice

then the smooth 4-dimensional Poincare conjecture is false.

3d $\mathcal{N}=2$ theories labeled by



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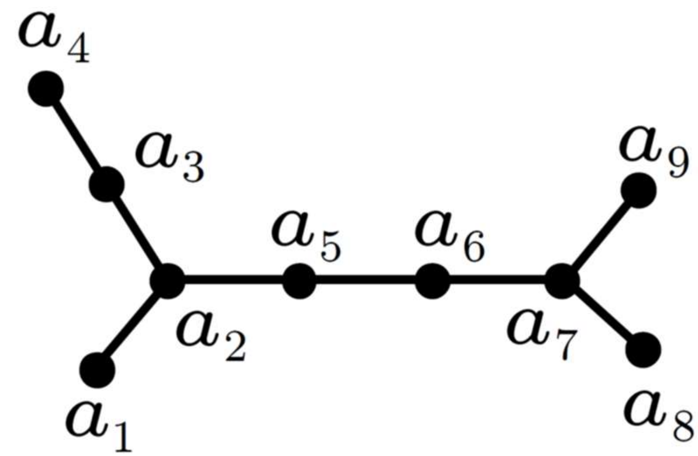


vertex 

↔ 3d $\mathcal{N}=2$ vector multiplet with Chern-Simons coupling a

edge 

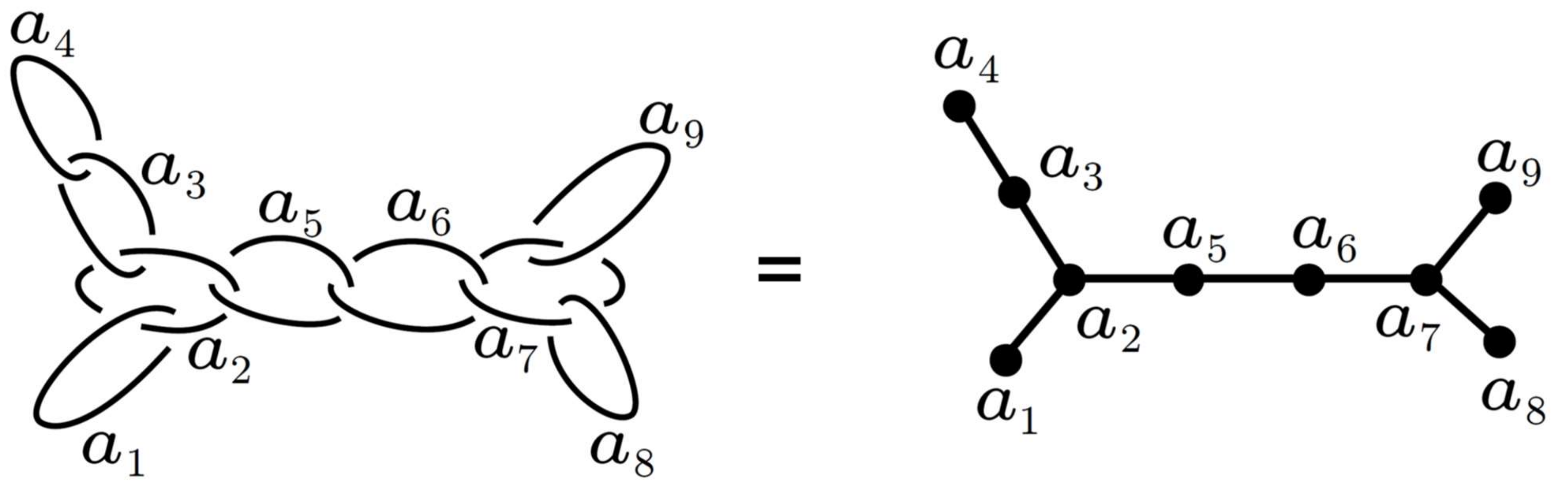
↔ Not known! non-Lagrangian ?

:

H.-J. Chung (2019, 2018, …)

J.Eckhard, H.Kim, S.Schafer-Nameki, B.Willett

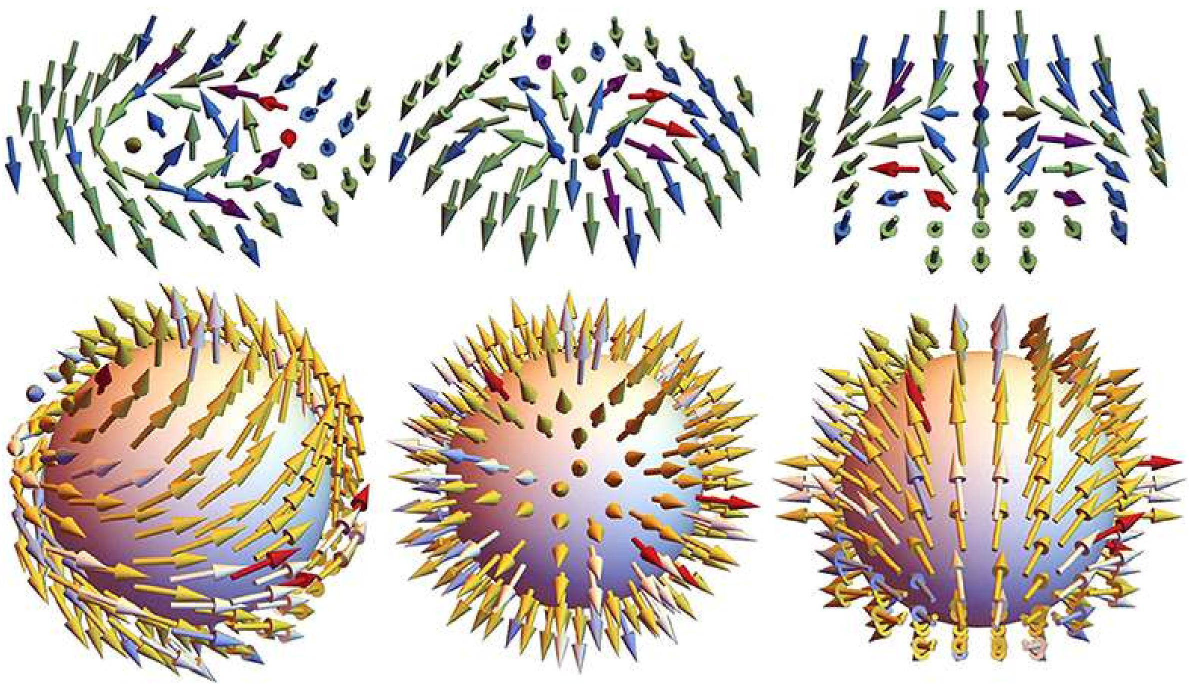
3d $\mathcal{N}=2$ theories labeled by



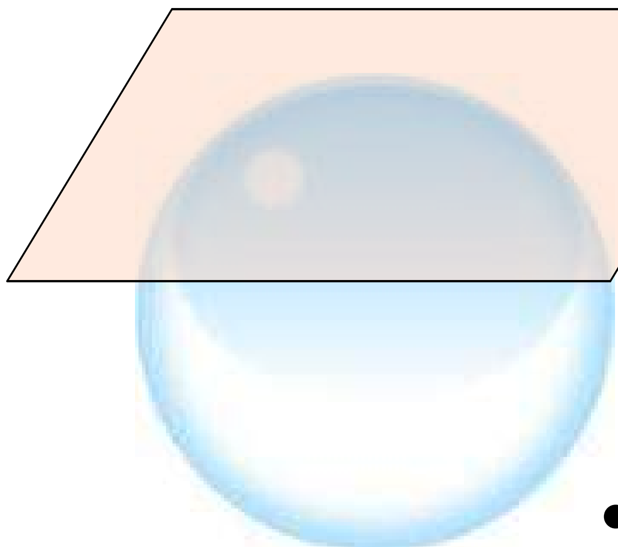
vertex a  ⟷ 3d $\mathcal{N}=2$ vector multiplet with Chern-Simons coupling a

edge ⟷ IR fixed point of a theory with non-linear matter *a la* Skyrme

S.Chun, S.G., S.Park, N.Sopenko



Tony Skyrme



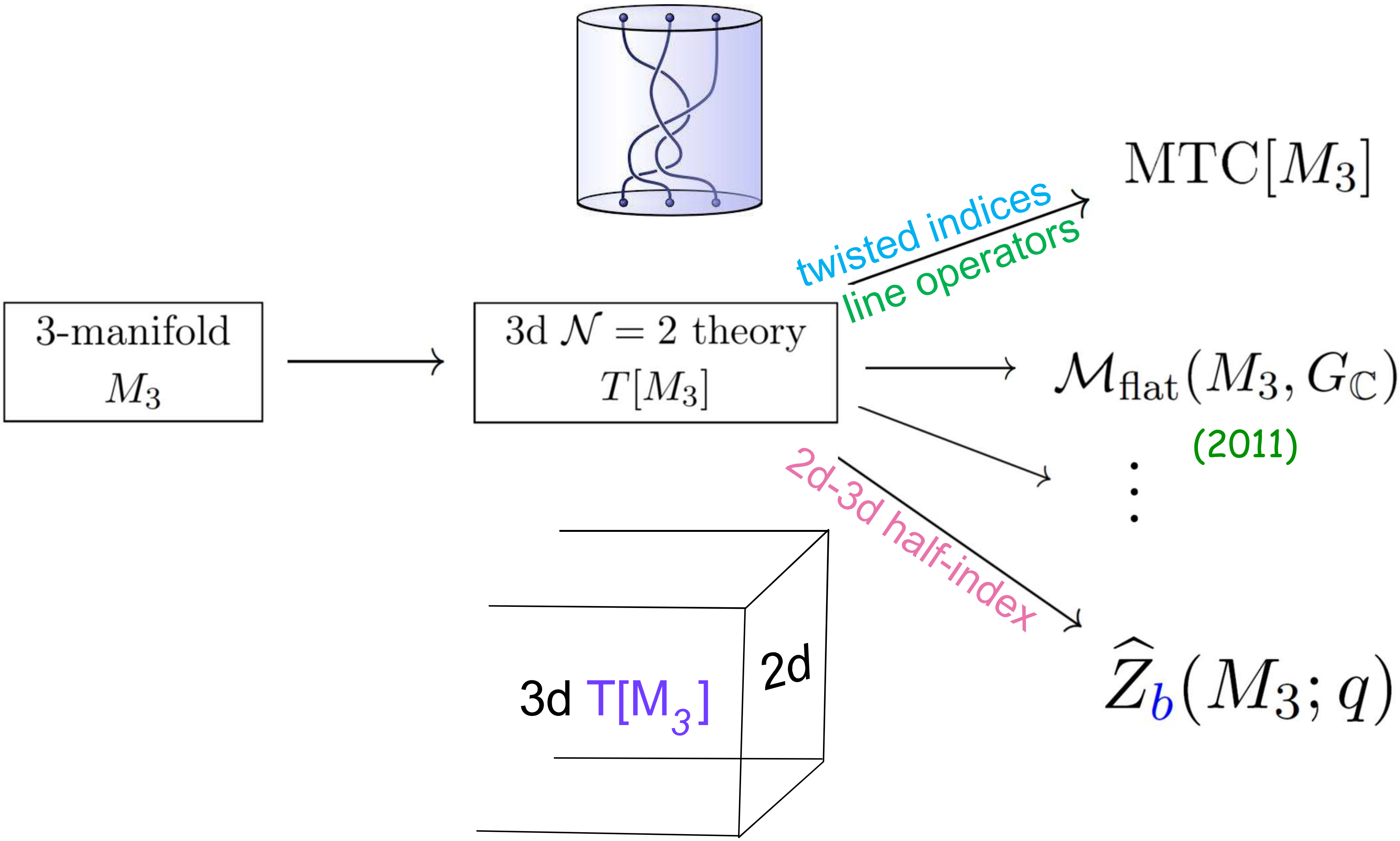
- Lie algebra

Both enjoy group action

- Group manifold

Tony Skyrme

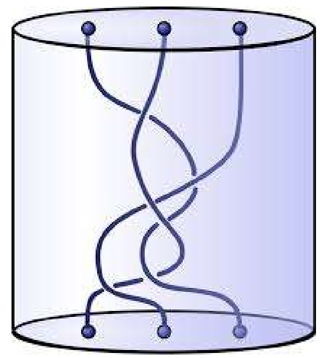
All related to complex Chern-Simons and modularity



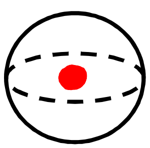
S.G., P.Putrov, C.Vafa (2016)

Examples:

$$\mathrm{MTC}\left[\begin{matrix}\text{Lens}\\ \text{space}\end{matrix}\right] = \mathrm{Ver}\,(G_k)$$

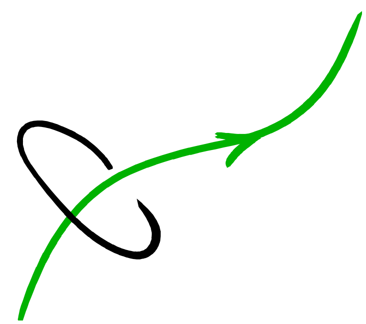


state-operator in 3d:



$$\mathcal{H}(S^2) = \text{ local operators}$$

$$\mathcal{H}(T^2) = \text{ line operators} = K^0(\mathcal{C})$$



6d perspective: $T^2 \times \mathbb{R} \times M_3$

$$\mathcal{H}_{T[M_3]}(T^2) = K^0\,(\mathrm{MTC}[M_3])$$

$$\mathrm{MCG}(M_3) \times SL(2,\mathbb{Z})$$

$$\mathcal{H}_{\mathrm{VW}}(M_3)$$

Homology *a la* Floer

approximate relations:

differ by KK modes on S^1

$$\mathcal{H}_{\text{VW}}(M_3) \simeq \mathcal{H}_{T[M_3]}(T^2) = H_{T^A[M_3]}(S^1)$$

differ by
KK modes on T^2

$$\cong QH^*\left(\mathcal{M}_{\text{flat}}(M_3, G_{\mathbb{C}})\right)$$

$$\mathcal{H}_{\text{VW}}(M_3) \;=\; ?$$

Special family:

$M_3 = S^1 \times \Sigma_g$

$g = 0:$ Gluck twist

$g = 1:$ knot surgeries, log-transforms, ...

On $M_4 = \mathbb{R} \times M_3$ same as **KW** equations

"u-plane"

VW:	A	$B \in \Omega^{2,+}(M_4, \mathfrak{g})$ $C \in \Omega^0(M_4, \mathfrak{g})$	$\phi,\ \overline{\phi}\ \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})$
KW:	A	$\phi \in \Omega^1(M_4, \mathfrak{g})$	$\sigma,\ \overline{\sigma}\ \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})$

play an important role in the gauge theory approach to the geometric Langlands program

THE SELF-DUALITY EQUATIONS ON A RIEMANN SURFACE

N. J. HITCHIN

[Received 15 September 1986]

Introduction

In this paper we shall study a special class of solutions of the self-dual Yang–Mills equations. The original self-duality equations which arose in mathematical physics were defined on Euclidean 4-space. The physically relevant solutions were the ones with finite action—the so-called 'instantons'. The same equations may be dimensionally reduced to Euclidean 3-space by imposing invariance under translation in one direction. These equations also have physical relevance—the

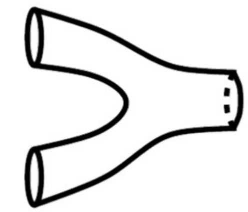
Theorem: in Vafa-Witten theory on $M_3 = S^1 \times \Sigma$

$$\mathcal{M}_{\mathrm{VW}}(G, M_3) \cong \mathcal{M}_H^{\mathcal{E}}(G, \Sigma)$$

moduli space of $\mathcal{E}$-valued Higgs bundles with $R_i = (2, 0, 0)$.

S.G., A.Sheshmani, S.-T.Yau

➡ explicit expression for $\mathcal{H}_{\mathrm{VW}}(\Sigma_g \times S^1)$
and its equivariant character
cf. equivariant Verlinde formula



<u>Claim:</u> when $\pi_1(G) = 1$ the Gluck involution acts trivially on $\mathcal{H}_{\mathrm{VW}}(S^2 \times S^1)$ and $Z_{\mathrm{VW}}(M_4, G)$ can not detect the Gluck twist.

<u>Remark:</u> the moduli spaces are different for

$$M_4 = \mathbb{R} \times S^1 \times \Sigma_g \cong \mathbb{C}^* \times \Sigma_g$$ ("K-theoretic")

and

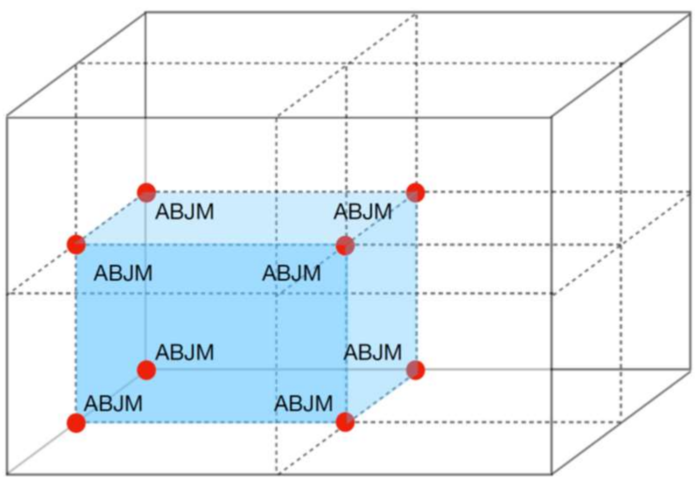
$$M_4 = \mathbb{R} \times \mathbb{R} \times \Sigma_g \cong \mathbb{C} \times \Sigma_g$$ ("homological")

Question: $9 \text{ vs } 10 = 10\%$?

$\dim \mathrm{Sk}(T^3) = 9$ whereas analogous computation in Vafa-Witten theory gives $\dim \mathcal{H}_{\mathrm{VW}}(T^3) = 10$

A.Carrega
P.M.Gilmer

S.G., P.Koroteev, S.Nawata, D.Pei, I.Saberi



V.Turaev
J.Przytycki
⋮

$$\mathrm{Sk}(M_3) = \frac{\mathbb{C}[q^{\pm\frac{1}{2}}](\text{isotopy classes of framed links in } M_3)}{\left(\diagup\!\!\!\diagdown = q^{-1/2} \right) \left(+ q^{1/2} \asymp, \bigcirc = -q - q^{-1} \right)}$$

Surprise: New 4-manifold invariant

S.G., P.-S.Hsin, D.Pei

$$Z(M_4; q_1, q_2)$$

$q_2 \to 1$

$q_1 = q_2 = q$

$$Z_{\mathrm{VW}}(M_4) = q_1^{\frac{\chi}{12}} \sum_{n \in \mathbb{Z}_{\geq 0}} a_n q_1^n$$

$$Z_{T[M_4]}(q)$$

Elliptic genus of $T[M_4]$

6d theory on $T^2 \times M_4$

C.Vafa, E.Witten

A.Gadde, S.G., P.Putrov

B.Feigin, S.G.

:

for $G = SU(2)$ has

modular weight $w = -\frac{\chi}{2}$

$$\frac{\chi}{2} + \frac{3}{2}\sigma$$

Also different as "decorated TQFTs"

Surprise: New 4-manifold invariant

S.G., P.-S.Hsin, D.Pei

$$Z(M_4; q_1, q_2)$$

$q_2 \to 1$ $\qquad$ $q_1 = q_2 = q$

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Elliptic genus of $T[M_4]$

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⋮

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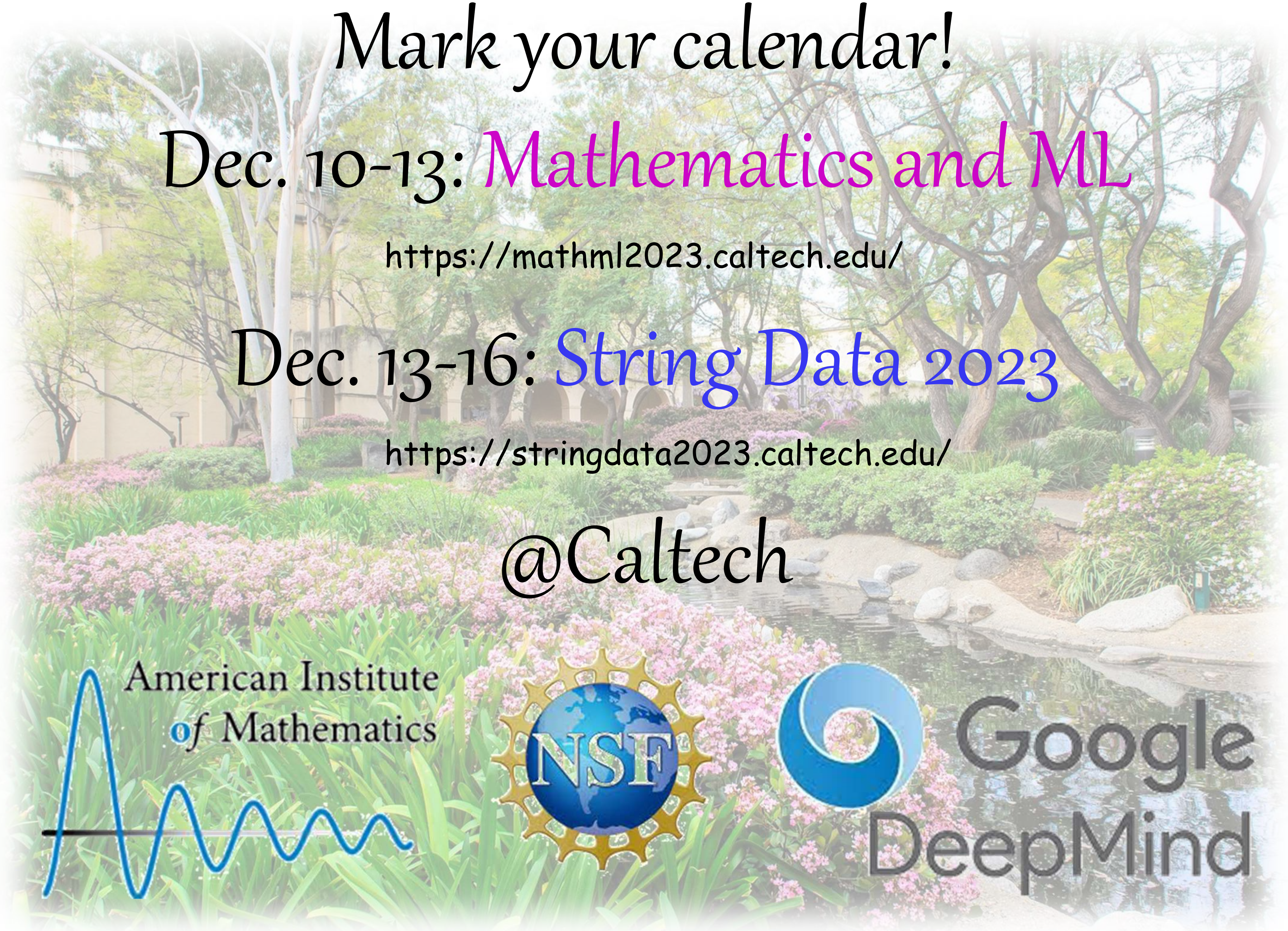
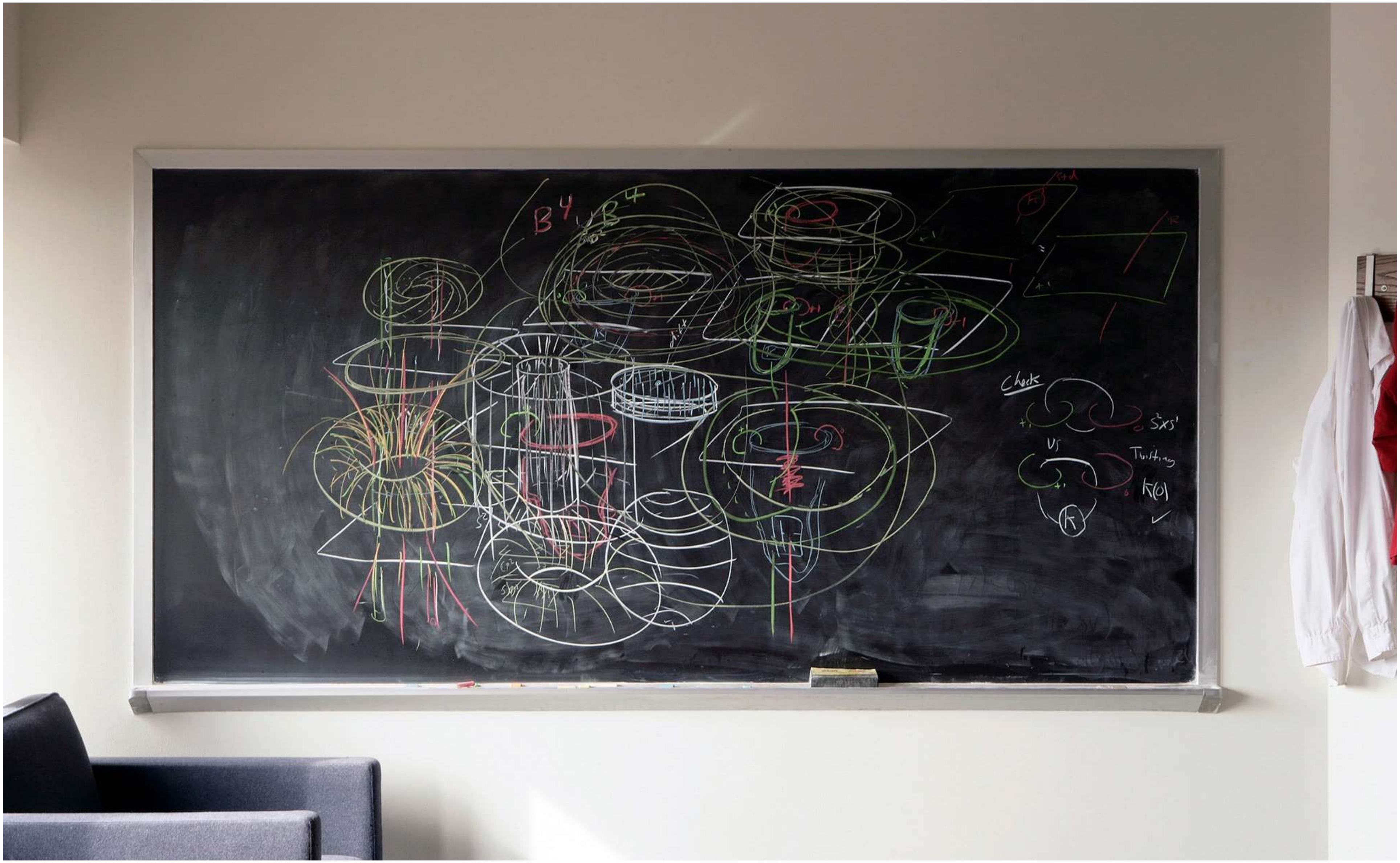
modular weight $w = -\frac{\chi}{2}$ $\qquad\qquad$ $\frac{\chi}{2} + \frac{3}{2}\sigma$

G - valued

$$B \in \Omega^{2,+}(M_4, \mathfrak{g}) \quad C \in \Omega^0(M_4, \mathfrak{g}) \quad \phi, \overline{\phi} \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})$$


Mark your calendar!
Dec. 10-13: Mathematics and ML
https://mathml2023.caltech.edu/
Dec. 13-16: String Data 2023
https://stringdata2023.caltech.edu/
@Caltech
American Institute of Mathematics
NSF
Google DeepMind



Chalkboard of David Gabai at Princeton
Jessica Wynne

Steve Smale

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
TOP	1	1	1	1	1	1	1	1	1	1	1	1
PL	1	1	1	?	1	1	1	1	1	1	1	1
DIFF	1	1	1	?	1	1	28	2	8	6	992	1

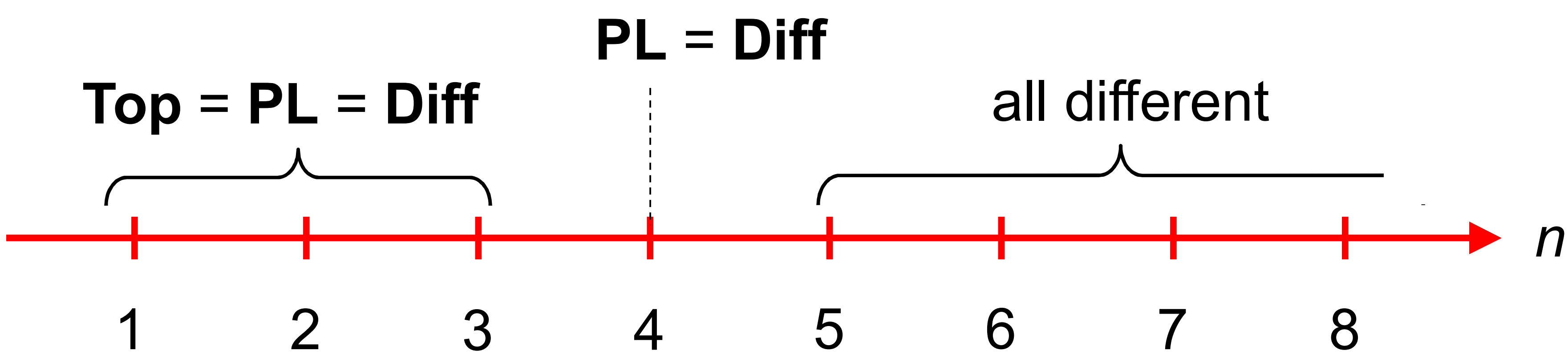
Number of homotopy *n*-spheres in each category.

John Milnor

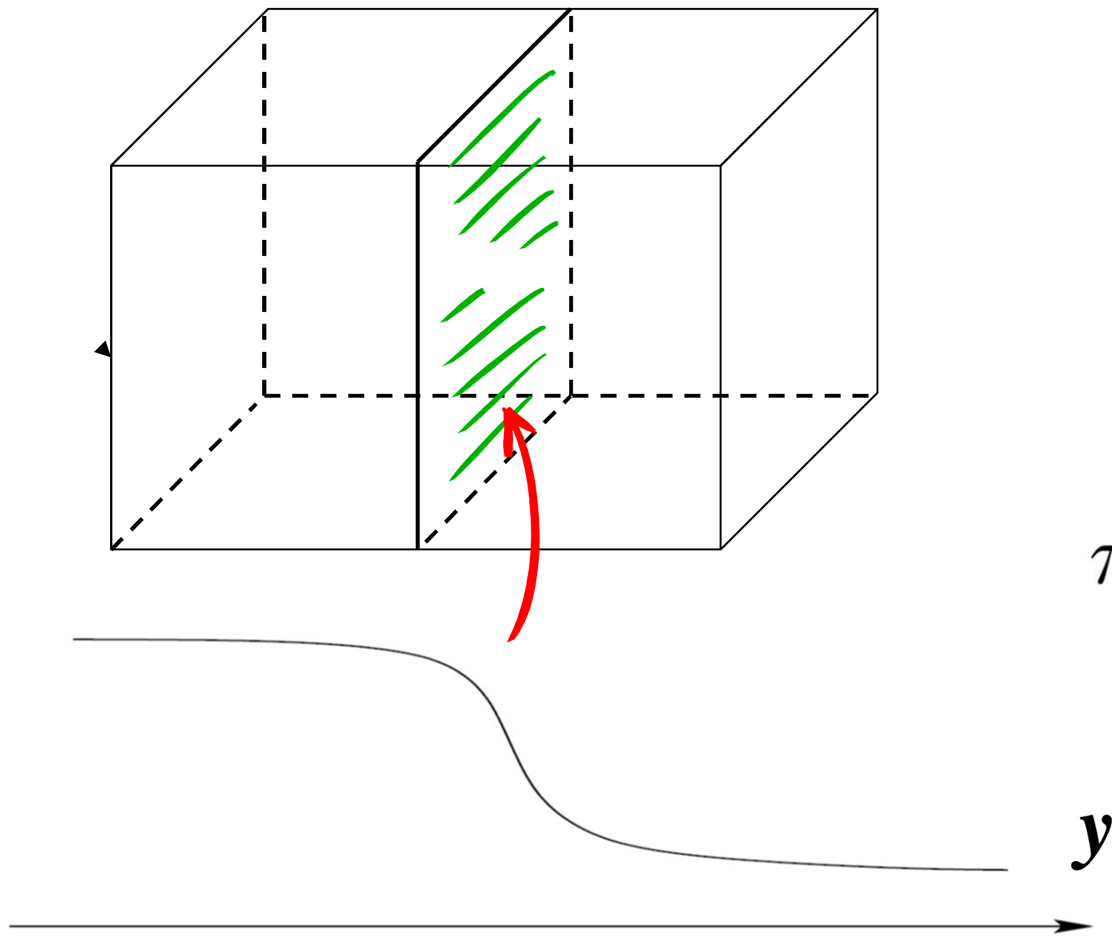
The generalized Poincare conjecture:

late 1950s

- **Top**: true for all *n*
- **PL**: true for all $n \neq 4$ (*n* = 4 currently not known)
- **Diff**: true for *n* = 1, 2, 3, 5, and 6

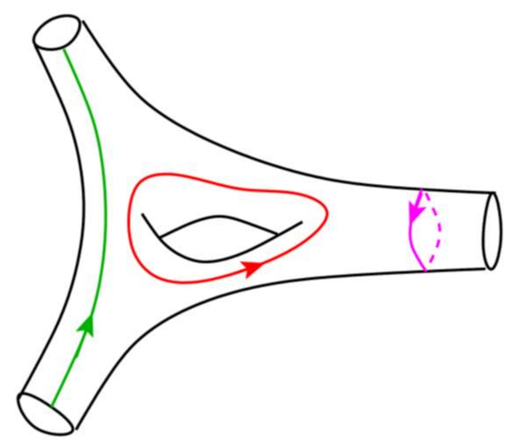


Parameter walls / interfaces / surface operators in 3d:



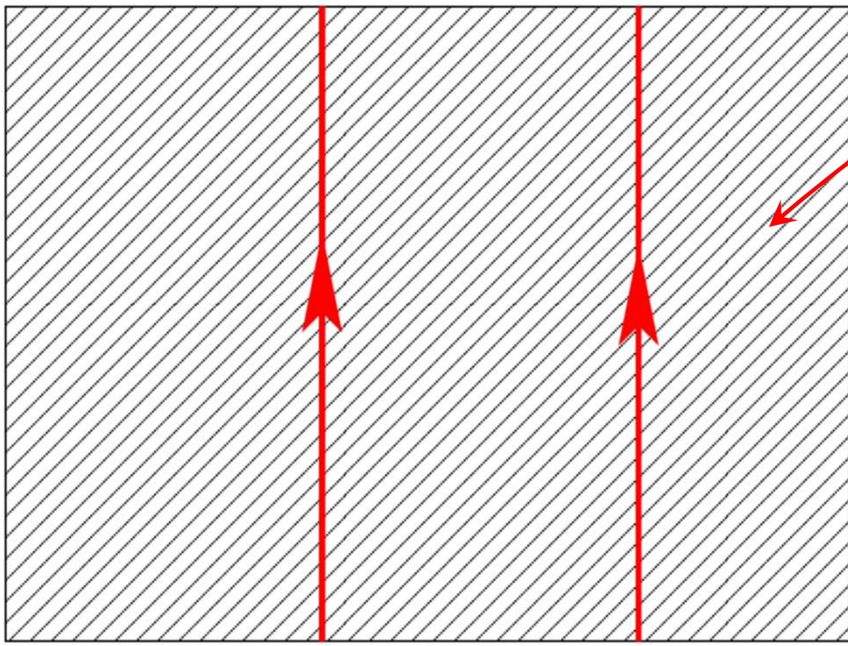
Kahler / Stab

$$\pi_1(\{\text{parameters}\})$$



A.Gadde, S.G., P.Putrov (2013)

cf. (codim-1) lines in 2d A-model / B-model:



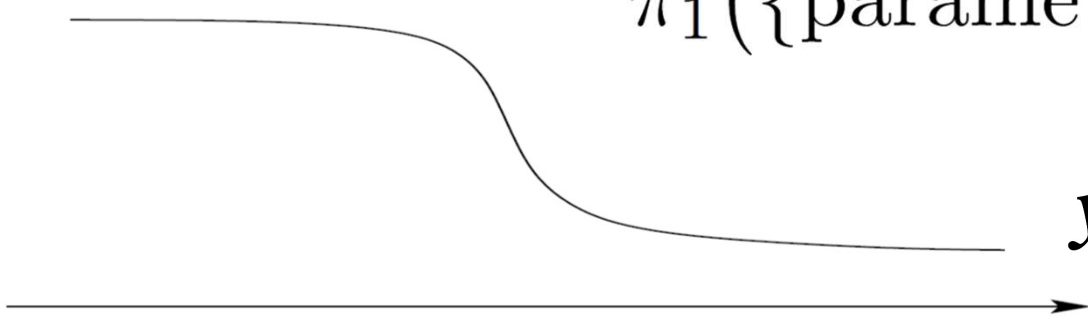
A-model / B-model to

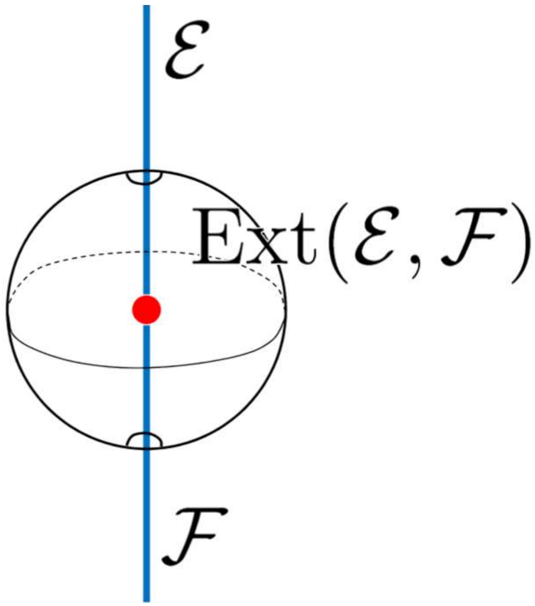
$$Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_2} \cong T^*\mathbb{C}\mathbf{P}^1$$

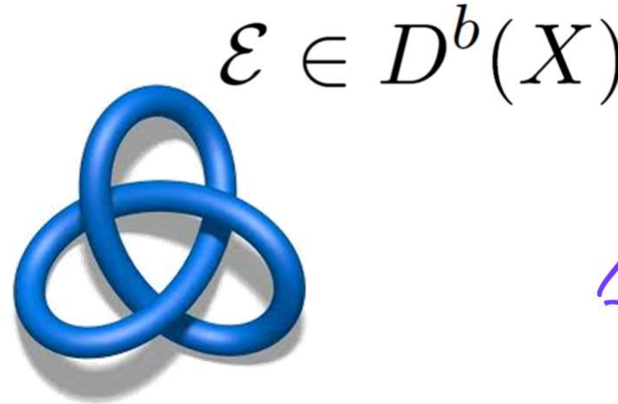
$$x^2 + y^2 + z^2 = \text{const}$$

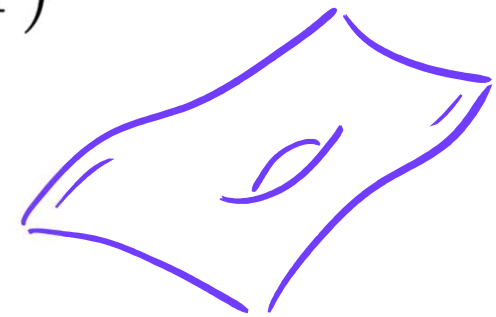
S.G, E.Witten ('06)

$$\pi_1(\{\text{parameters}\}) = ?$$









"local"
operators

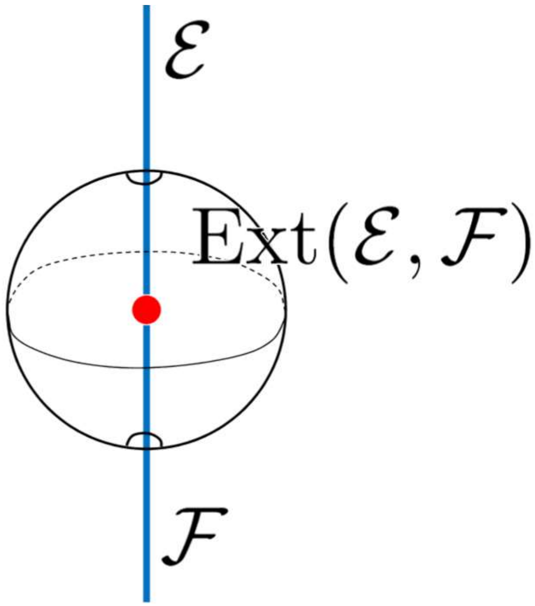
line
operators

surface
operators

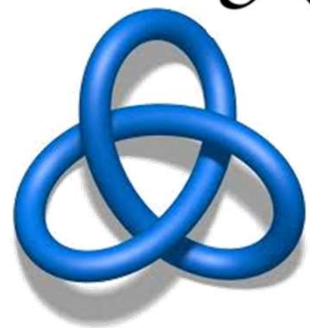
Symmetries
decorated TQFT

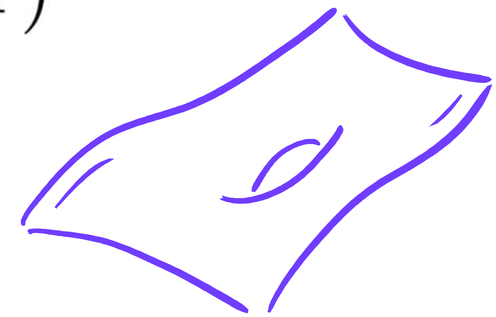
Higher groups

Logarithmic
(non-semisimple)



$\mathcal{E} \in D^b(X)$





"local" operators

line operators

surface operators

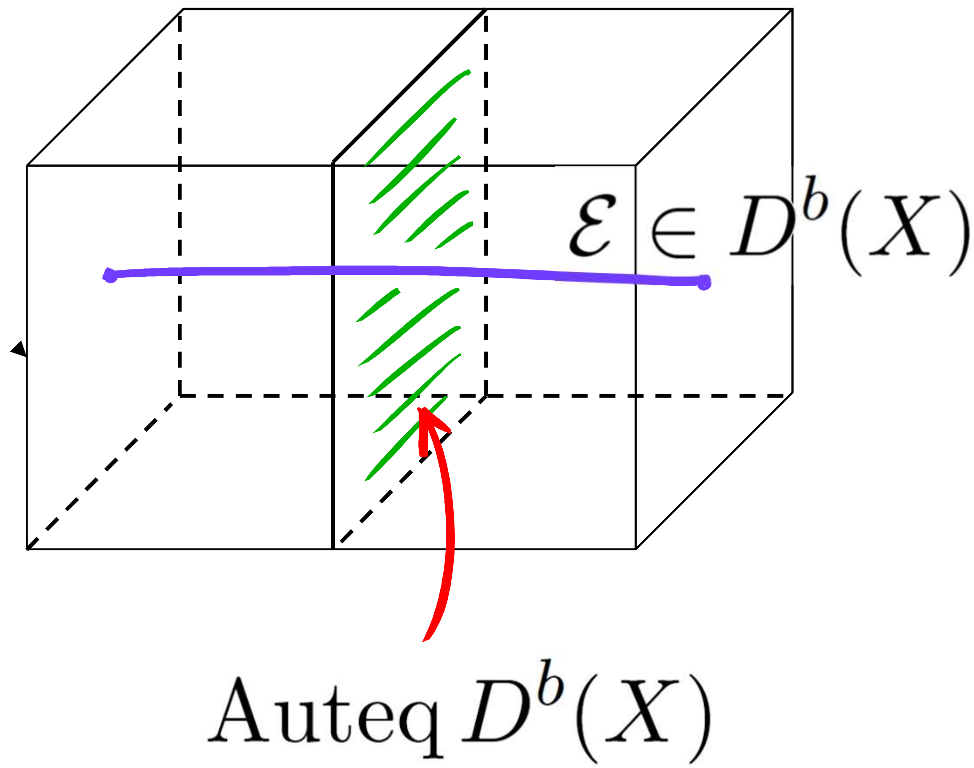
$$\mathrm{Auteq}\, D^b(X) \;\cong\; \pi_1\big(\{\text{parameters}\}\big)$$

<u>Example:</u> $X = K3$

$$\mathrm{Pic}\, X \rtimes \mathrm{Aut}\, X \times \mathbb{Z}[1]$$

H.Uehara
:

Modern low-dimensional topology:



<u>Example:</u> $X = K3$

$$\operatorname{Pic} X \rtimes \operatorname{Aut} X \times \mathbb{Z}[1]$$

H.Uehara
:

$\mathcal{M}_H^{\mathcal{E}}(G,\Sigma)$: $L_i = K^{R_i/2} \quad (R_i \in \mathbb{Z})$

$L_1 \otimes \mathrm{ad}_P \oplus L_2 \otimes \mathrm{ad}_P \oplus L_3 \otimes \mathrm{ad}_P$ valued Higgs bundles

or, $\mathcal{E} = L_1 \oplus L_2 \oplus L_3$ valued, for short

Moreover, $U(1)_t \circlearrowright \mathcal{M}_H(G,\Sigma)$

$(A,\Phi) \mapsto (A, e^{i\theta}\Phi)$ circle action

Similarly, $U(1)_x \times U(1)_y \times U(1)_t \circlearrowright \mathcal{M}_H^{\mathcal{E}}(G,\Sigma)$

⇨ explicit expression for $\mathcal{H}_{\mathrm{VW}}(\Sigma_g \times S^1)$
and its equivariant character

S.G., A.Sheshmani, S.-T.Yau
cf. V.Munoz