KLT for Nonrelativisitic String Theory

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Content

The goal of this talk is to inspire further research into amplitudes for NRST.

1. Review KLT factorization in rel. strings
2. KLT of wound strings
   - Pictorial understanding
   - Example with four closed string tachyons
   - General factorization
3. In a background Kalb-Ramond field
   - Modifications
4. Nonrelativistic KLT relations
   - Derive it in the limit of relativistic string theory
   - Derive it from the NRST action

Review of KLT Factorization

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Why think about KLT? It has also lead to many insights into QFT amplitudes in the tension $\to \infty$ limit.

- BCJ relations
- Born-Infeld, NLSM, and special Galileons fit into the QFT type of KLT relations via CHY.

Review of KLT Factorization: Standard Relativistic KLT

Mandelstam Variables:

$$s = -(k_1 + k_2)^2\,, \quad t = -(k_1 + k_3)^2\,, \quad u = -(k_1 + k_4)^2$$

$$\text{define} \quad \alpha_x = 1 + \frac{\alpha'}{4} x.$$

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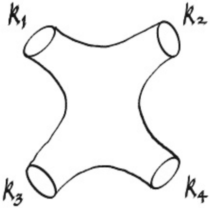

$$\mathcal{A}_{\text{closed}} \sim \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)\Gamma(-\alpha_u)}{\Gamma(1+\alpha_s)\Gamma(1+\alpha_t)\Gamma(1+\alpha_u)}$$

$$= \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s - \alpha_t)} \frac{1}{\Gamma(-\alpha_t)\Gamma(1+\alpha_t)} \frac{\Gamma(-\alpha_t)\Gamma(-\alpha_u)}{\Gamma(-\alpha_t - \alpha_u)}$$

$$= B(\alpha_s, \alpha_t) \frac{\sin(\pi\alpha_t)}{\pi} B(\alpha_t, \alpha_u)$$

$$\mathcal{A}_{\text{closed}} \sim \mathcal{A}_{\text{open}}(s,t) \frac{\sin(\pi\alpha_t)}{\pi} \mathcal{A}_{\text{open}}(t,u)$$

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Derive a KLT-like relation for $\mathcal{N}$ -point scattering amplitudes of closed strings when a spatial direction is compactified on a circle.

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Puzzle:

Open string state can either carry momentum or winding but not both!

$$\text{Dirichlet:} \quad X^1 = x^1 + 2\alpha' \frac{n}{R} \tau + \dots$$

$$\text{Neumann:} \quad X^1 = x^1 + 2wR\,\sigma + \dots$$

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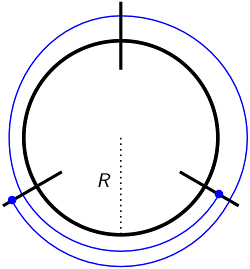
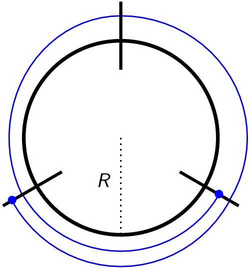
- We introduce a total of $n_+ + 1$ many D-branes in the compactified direction, with n_+ the total incoming momentum from the closed strings.
- The D-branes are equally separated by a distance which is T-dual to the circumference of the compactified circle, i.e. $L = \frac{2\pi\alpha'}{R}$.

KLT of Wound Strings: cont.



- For the i-th open string, the "fractional part" of the winding number of an open string stretched between two D-branes is given by n_i on the closed string side (in the above picture $n_i = 1$).

KLT of Wound Strings: cont.



- For the i-th open string, the "fractional part" of the winding number of an open string stretched between two D-branes is given by n_i on the closed string side (in the above picture $n_i = 1$).
- The open string also carries an integer winding number w_i (in the above picture $w_i = 1$). It soaks up the closed string winding.

KLT of Wound Strings: cont

In summary we have the following mapping:

closed string	**open string**
momentum	fractional winding
winding	integer winding

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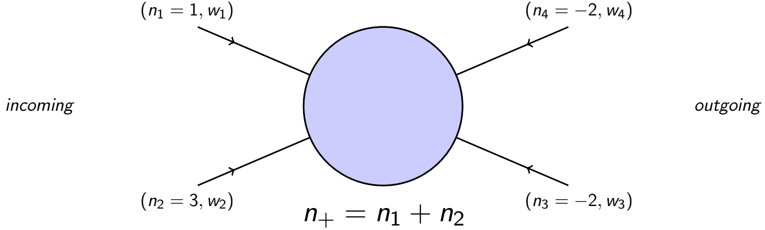
KLT of Wound Strings: closed string amp

To be more explicit, consider the data of a 4-point closed string amp.



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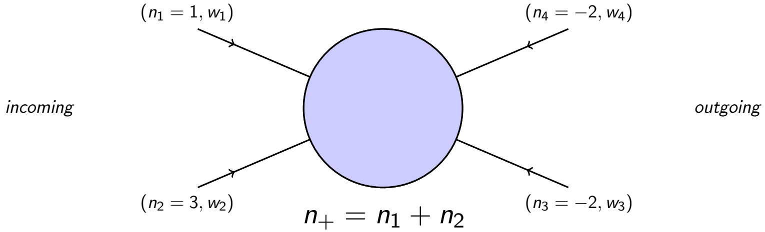


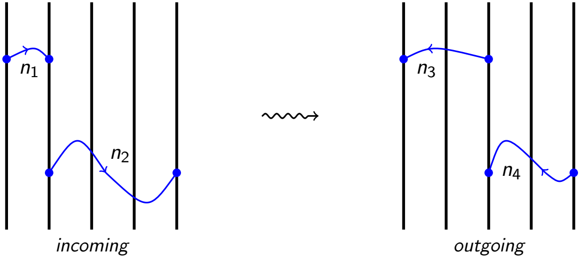
$$\mathcal{V} \sim e^{iK_\mathrm{L}X_\mathrm{L} + iK_\mathrm{R}X_\mathrm{R}}, \quad \begin{cases} K_\mathrm{L} = (k^i, \frac{n}{R} - \frac{wR}{\alpha'}) \\ K_\mathrm{R} = (k^i, \frac{n}{R} + \frac{wR}{\alpha'}) \end{cases}$$

$$X_\mathrm{L} = x_\mathrm{L} - \frac{i}{2}\alpha' \left(\frac{n}{R} - \frac{wR}{\alpha'} \right) \log z + \ldots$$

$$X_\mathrm{R} = x_\mathrm{R} - \frac{i}{2}\alpha' \left(\frac{n}{R} + \frac{wR}{\alpha'} \right) \log \bar{z} + \ldots$$

KLT of Wound Strings: open string amp

On the open string side, the amplitude is mapped to:



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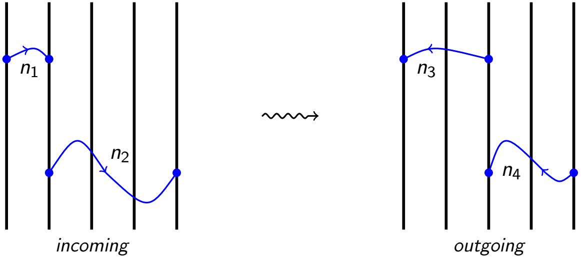

- n_i above are fractional windings. Came from closed string Kaluza-Klein number.
- During the scattering process, the two strings join into one, then the single intermediate string splits at the third D-brane.

This is how we realize *conservation of momentum* for closed strings.

The closed string tachyon is described by the following vertex operator:

$$\mathcal{V}_{\mathcal{C}}(z\,,\bar{z}) = g_{\mathsf{c}}\exp\left[\tfrac{i}{2}\,\pi R\,w\,(\hat{p}_{\mathsf{L}}+\hat{p}_{\mathsf{R}})\right]\,:e^{iK_{\mathsf{L}}\cdot\mathbb{X}_{\mathsf{L}}(z)+iK_{\mathsf{R}}\cdot\mathbb{X}_{\mathsf{R}}(\bar{z})}:\,,$$

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$$\mathbb{X}_{\rm L}^{M} = \left(X^{\mu},\, X_{\rm L}\right), \qquad K_{\rm L}^{M} = \left(k^{\mu},\, p_{\rm L}\right),$$
$$\mathbb{X}_{\rm R}^{M} = \left(X^{\mu},\, X_{\rm R}\right), \qquad K_{\rm R}^{M} = \left(k^{\mu},\, p_{\rm R}\right).$$

The extra phase factor is known as the *cocycle*. It is there to remove the phases from crossing certain branch cuts when vertex operators are interchanged.

4-point KLT: cont.

The amplitude for four closed string tachyons on a spherical worldsheet is

$$\mathcal{A}_c^{(4)} = e^{-\chi\,\Phi_0} \int_{\mathbb{C}} d^2 z_2 \left\langle \prod_{i=1}^{4} :\mathcal{V}_{\mathcal{C}_i}(z_i\,,\overline{z}_i): \right\rangle_{S^2}$$

$$\propto i\,(2\pi)^{25}\,\delta^{(25)}\big(k_1 + \cdots + k_4\big)\,\delta_{n_1+\cdots+n_4,\,0}\,\delta_{w_1+\cdots+w_4,\,0}\,\mathcal{M}_c^{(4)}\,.$$

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where

$$\mathcal{M}_{\mathrm{c}}^{(4)} = \frac{1}{\alpha'}\,\exp\Big(i\pi\sum\nolimits_{\substack{i,j=1\\ i<j}}^{4} n_i\,w_j\Big)\,\mathcal{I}^{(4)}\,.$$

We can try to factorize $\mathcal{I}^{(4)}$, by manipulating complex integrals.

4-point KLT: cont.

We find that

$$\mathcal{I}^{(4)} = \frac{1}{2}\, i\, \mathcal{I}_{\mathsf{L}}^{(4)}\, \mathcal{I}_{\mathsf{R}}^{(4)}\,,$$

where

$$\mathcal{I}_{\mathsf{L}}^{(4)} = B(\alpha_{s_{\mathsf{L}}}, \alpha_{u_{\mathsf{L}}})\,, \quad \mathcal{I}_{\mathsf{R}}^{(4)} = -2\, i \sin(\pi\, \alpha_{s_{\mathsf{R}}})\, B(\alpha_{s_{\mathsf{R}}}, \alpha_{t_{\mathsf{R}}})\,.$$

Upgraded Mandelstam variables taking into account windings:

$$s_{\mathsf{L,R}} = -(K_{\mathsf{L,R1}} + K_{\mathsf{L,R2}})^2\,, \qquad t_{\mathsf{L,R}} = -(K_{\mathsf{L,R1}} + K_{\mathsf{L,R3}})^2\,,$$
$$u_{\mathsf{L,R}} = -(K_{\mathsf{L,R1}} + K_{\mathsf{L,R4}})^2\,.$$

4-point KLT: cont.

We find that $\mathcal{M}_{\mathsf{c}}^{(4)}$ from $\mathcal{A}_{\mathsf{c}}^{(4)}$ is factorizable as

$$\mathcal{M}_{\mathsf{c}}^{(4)}(1,2,3,4) =$$
$$= -\frac{1}{\alpha'}\, C(1,2,3,4)\, \sin\left(\tfrac{1}{2}\,\pi\alpha' K_{\mathsf{R}1}\cdot K_{\mathsf{R}2}\right) \mathcal{M}_{\mathsf{L}}(1,2,3,4)\, \mathcal{M}_{\mathsf{R}}(2,1,3,4)$$
$$= -\frac{1}{\alpha'}\, C(1,2,3,4)\, \sin\left(\tfrac{1}{2}\,\pi\alpha' K_{\mathsf{R}2}\cdot K_{\mathsf{R}3}\right) \mathcal{M}_{\mathsf{L}}(1,2,3,4)\, \mathcal{M}_{\mathsf{R}}(1,3,2,4)\,,$$

This is invariant under switching "L" and "R", on the open string side.

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This is invariant under switching "L" and "R", on the open string side.

Recap

In a spacetime with a spatial direction compactified over a circle of radius R

Closed string amplitude $\to$ Open string on an array of D-branes transverse to the compactified direction.

$\mathcal{N}$-point KLT

An $\mathcal{N}$-point tree-level closed string amplitude takes the form:

$$\mathcal{A}_{\mathsf{c}}^{(\mathcal{N})} \propto (2\pi)^{25}\,\delta^{(25)}\Big(\sum_{i=1}^{\mathcal{N}} k_i\Big)\,\delta_{\sum_{i=1}^{\mathcal{N}} n_i\,,\,0}\,\delta_{\sum_{i=1}^{\mathcal{N}} w_i\,,\,0}\,\big(i\mathcal{M}_{\mathsf{c}}^{(\mathcal{N})}\big)\,,$$

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We have an $\mathcal{N}$-point KLT:

$$\begin{aligned}\mathcal{M}_{\mathsf{c}}(1,\ldots,\mathcal{N}) = (-1)^{\mathcal{N}-3}\sum_{\rho,\,\sigma} & C\big(1\,,\sigma(2\,,\ldots,\mathcal{N}-2)\,,\mathcal{N}-1\,,\mathcal{N}\,\big)\\ & \times\mathcal{S}_{\mathsf{R}}\big[\sigma(2\,,\ldots,\mathcal{N}-2)\,\big|\,\rho(2\,,\ldots,\mathcal{N}-2)\big]_{K_{\mathsf{R}(\mathcal{N}-1)}}\\ & \times\mathcal{M}_{\mathsf{L}}\big(1\,,\,\sigma(2\,,\ldots,\mathcal{N}-2)\,,\mathcal{N}-1\,,\mathcal{N}\,\big)\\ & \times\mathcal{M}_{\mathsf{R}}\big(1\,,\mathcal{N}-1\,,\,\rho(2\,,\ldots,\mathcal{N}-2)\,,\mathcal{N}\,\big)\,.\end{aligned}$$

With $\mathcal{S}_{\mathsf{R}}$ the generalized momentum kernel. Cocycle here is important for KLT to hold and respect permutation invariance.

Background Kalb-Ramond Field and KLT

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The sigma model is

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(\partial_\alpha X^\mu \, \partial^\alpha X_\mu + 2i \, \epsilon^{\alpha\beta} \, \partial_\alpha X^0 \, \partial_\beta X^1 \, B \right).$$

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To realize such a KLT relation:

- D-brane configuration previously discussed
- D-branes also need to carry *electric gauge potentials*.

Background Kalb-Ramond Field and KLT: cont.

The canonical momentum conjugate to X^0 is

$$K_0 = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, \left(\partial_t X^0 - B \, \partial_\sigma X^1\right) = \varepsilon - \frac{wR}{\alpha'} \, B \, ,$$

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where

$$\varepsilon = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, \partial_t X^0 \, .$$

The factorization of winding closed string amplitudes takes the same form of before, but now the closed string variables are

$$K_{\rm L} = \left(\varepsilon - \frac{wR}{\alpha'} \, B \, , \, \frac{n}{R} - \frac{wR}{\alpha'} \, , \, k_{A'}\right) ,$$

$$K_{\rm R} = \left(\varepsilon - \frac{wR}{\alpha'} \, B \, , \, \frac{n}{R} + \frac{wR}{\alpha'} \, , \, k_{A'}\right) .$$

Background Kalb-Ramond Field and KLT: open strings

The associated open string amplitude now takes the form of$\mathcal{M}_{\mathsf{L}}$ and $\mathcal{M}_{\mathsf{R}}$, but with the open string variables are

$$\widetilde{K}_{\mathsf{L}} = \left(\varepsilon - \frac{wR}{\alpha'} B - \frac{nB}{R} , \frac{n}{R} - \frac{wR}{\alpha'} , k_{A'} \right) ,$$

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This is almost the same as $K_{\mathsf{L},\mathsf{R}}$ but with an extra shift nB/R of the energy.

To compensate for this extra term, we are required to assign to the s-th brane an electric potential:

$$V_s = 2\pi s B \, \frac{\alpha'}{R} \, .$$

Background Kalb-Ramond Field and KLT: open strings

Suppose an open string ends at s-th and $s+n$-th D-brane, the boundary action takes the form

$$
\begin{aligned}
S_{\text{bdry}} &= \frac{i}{2\pi\alpha'}\int_{\partial\Sigma} dy\, A_\mu\, \partial_y X^\mu \\
&= \frac{i}{2\pi\alpha'}\int_0^\infty dy\, V_{s+n}\, \partial_y X^0 + \frac{i}{2\pi\alpha'}\int_{-\infty}^0 dy\, V_s\, \partial_y X^0\,.
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As a result, we find

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S_{\text{bdry}} = \frac{inB}{R}\int_{\mathbb{R}} d\tau\, \partial_\tau X^0 \,,
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this boundary terms contributes an extra shift in energy, with

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The potential that shifts the energy takes $\widetilde{K}_{\text{L,R}} \to K_{\text{L,R}}$.

Connection to Nonrelativistic String Theory

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$$\varepsilon = \frac{\alpha'_{\text{eff}}}{2wR} \left[k_{A'} k_{A'} + \frac{2}{\alpha'_{\text{eff}}} \left(N + \widetilde{N} - 2 \right) \right].$$

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- Describes strings interacting in a string-Galilean invariant spacetime.
- All physical states have non-zero winding number. Zero-winding sector has no graviton.
- Not GR at low energy, instantaneous Newtonian potential from exchange of off-shell states.

Nonrelativistic String Amps

Another important feature about nonrel. amps is that they are finite. Gives quantization of nonrelativistic spacetime geometry-so called *string Newton-Cartan* geometry.

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Example:

Consider the kinematical quantity

$$\alpha' K_{\mathsf{L}i} \cdot K_{\mathsf{L}j} = \alpha' \left[-\varepsilon_i \, \varepsilon_j + \alpha' \left(\frac{n_i}{R} - \frac{w_i R}{\alpha'} \right) \left(\frac{n_j}{R} - \frac{w_j R}{\alpha'} \right) + k_i^{A'} k_j^{A'} \right]$$

$$\supset \frac{w_i w_j R^2}{\alpha'} \quad \text{is singular when } \alpha' \to 0.$$

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$$\supset \frac{w_i w_j R^2}{\alpha'} \quad \text{is singular when } \alpha' \to 0.$$

But in the presence of the critical B-field, this term is cancelled.

Nonrelativistic KLT

The nonrelativistic analogue of the KLT relation can be obtained by taking $\alpha' \to 0$ limit of the relativistic KLT relation for winding string amplitudes in the presence of a critical B-field.

Nonrelativistic KLT

The nonrelativistic analogue of the KLT relation can be obtained by taking $\alpha' \to 0$ limit of the relativistic KLT relation for winding string amplitudes in the presence of a critical B-field. We also need to take the following rescalings for the nonrelativistic limit to be nonsingular:

$$K_{\rm L,R} = \left(\varepsilon - \frac{wR}{\alpha'}\,B\,,\,\frac{n}{R} \mp \frac{wR}{\alpha'}\,,\,\sqrt{\frac{\alpha'_{\rm eff}}{\alpha'}}\,k_{A'}\right)$$

$$A_\mu = \frac{\alpha'}{\alpha'_{\rm eff}}\,a_\mu\,,$$

$$V_s = \frac{\alpha'}{\alpha'_{\rm eff}}\,v_s\,, \qquad v_s = 2\pi s B\,\frac{\alpha'_{\rm eff}}{R}\,, \quad B = -1$$

Proposition

$v_s = -2\pi s\,\alpha'_{\rm eff}/R$ gains the interpretation as the electric potential on the s-th D-brane. Constant electric field between the D-branes.

Nonrelativistic KLT from first principles

Nonrelativistic string theory in flat spacetime is defined by the action

$$S = \frac{1}{4\pi\alpha'_{\text{eff}}} \int_\Sigma d^2\sigma \left(\partial_\alpha X^{A'} \, \partial^\alpha X^{A'} + \lambda \, \bar{\partial} X + \bar{\lambda} \, \partial \overline{X} \right),$$

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The vertex operator for the closed string tachyon:

$$\mathcal{V} = \exp\big(i\pi \, n \, \hat{w}\big) \; :\exp\big(i \, K_{A'} \, X^{A'} + i \, p_A X^A + i \, q^A X'_A\big): .$$

- X^1 is compactified, and $q = -\bar{q} = wR/\alpha'_{\text{eff}}$. $p_0 = \varepsilon$ is the energy.
- Since X^1 is compactified, $p_1 = n/R$ is quantized, with $n \in \mathbb{Z}$ the KK number.

A Change of Variables

There is a clever change of variables to make NRST computations much simpler.

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$$S = \frac{1}{4\pi\alpha'_{\text{eff}}} \int_{\mathbb{C}} d^2 z \left(2\, \partial_z X^{A'}\, \partial_{\bar{z}} X^{A'} + \lambda_z\, \partial_{\bar{z}} X + \lambda_{\bar{z}}\, \partial_z \overline{X} \right),$$

- Introduce a local redefinition of the one-form fields λ and $\bar{\lambda}$,

$$\lambda_z = -2\, \partial_z X' \,, \qquad \lambda_{\bar{z}} = 2\, \partial_{\bar{z}} \overline{X}' \,.$$

- The auxiliary coordinates $X' = X'(z)$ and $\overline{X}' = \overline{X}'(\bar{z})$ are T-dual to X and $\overline{X}$, respectively.

A Change of Variables: cont.

We further define

$$\varphi_{\mathrm{L}}^{0}(z) = \tfrac{1}{2}(X + X'), \qquad \varphi_{\mathrm{R}}^{0}(\bar{z}) = \tfrac{1}{2}(\overline{X} - \overline{X}'),$$
$$\varphi_{\mathrm{L}}^{1}(z) = \tfrac{1}{2}(X - X'), \qquad \varphi_{\mathrm{R}}^{1}(\bar{z}) = \tfrac{1}{2}(\overline{X} + \overline{X}'),$$
$$X^{A'} = \varphi^{A'}(z) + \bar{\varphi}^{A'}(\bar{z}) \quad \mu = 0, 1, \ldots d - 1\,.$$

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$$X^{A'} = \varphi^{A'}(z) + \bar\varphi^{A'}(\bar z) \quad \mu = 0, 1, \ldots d-1\,.$$

The OPEs are

$$\varphi^\mu_{\mathsf{L}}(z_1)\,\varphi^\nu_{\mathsf{L}}(z_2) \sim -\frac{\alpha'_{\text{eff}}}{2}\,\eta^{\mu\nu}\,\ln\big(z_1 - z_2\big)\,,$$

$$\varphi^\mu_{\mathsf{R}}(\bar z_1)\,\varphi^\nu_{\mathsf{R}}(\bar z_2) \sim -\frac{\alpha'_{\text{eff}}}{2}\,\eta^{\mu\nu}\,\ln\big(\bar z_1 - \bar z_2\big)\,,$$

A Change of Variables: cont.

In doing this transformation, the OPEs look like those arising from a theory

$$S_\varphi = \frac{1}{4\pi\alpha'_{\text{eff}}} \int d^2\sigma \, \partial_\alpha \varphi^\mu \, \partial^\alpha \varphi_\mu \, ,$$

for relativistic strings. We then perform two steps:

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for relativistic strings. We then perform two steps:

- We now rewrite the vertex operator for the tachyon in NRST.
- For OPE computations, physical quantities in NRST can be obtained from relativistic string theory via our mapping.

The T-dual Frame

Can we make this hidden relativistic nature more manifest?

- Transform the theory into a T-dual frame, i.e. the DLCQ of relativistic string theory.

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- We send $\gamma = 1/\sqrt{1 - v^2} \to \infty$ and R_0 the radius of a spatially compact circle to 0, but hold $\widetilde{R} \equiv 2\,\gamma R_0$ fixed.

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- We send $\gamma = 1/\sqrt{1-v^2} \to \infty$ and R_0 the radius of a spatially compact circle to 0, but hold $\widetilde{R} \equiv 2\,\gamma R_0$ fixed.
- DLCQ has a dispersion relation with Galilei boost symmetry.
- We show that the double scaling limit of closed string amplitude gives DLCQ amplitude, which are T-dual to nonrelativistic closed string amplitude.

Nonrelativistic KLT from first principles: cont.

In terms of $\varphi^{\mu}_{\mathsf{L},\,\mathsf{R}}$ in closed string vertex operator becomes

$$\mathcal{V} = \exp\left[\tfrac{i}{4}\,\pi\alpha'_{\text{eff}}\big(K_{\mathsf{L}} - K_{\mathsf{R}}\big)\cdot\big(\hat{K}_{\mathsf{L}} + \hat{K}_{\mathsf{R}}\big)\right]\exp\big(iK_{\mathsf{L}}\cdot\varphi_{\mathsf{L}} + iK_{\mathsf{R}}\cdot\varphi_{\mathsf{R}}\big)\,,$$

where

$$K_{\mathsf{L}\,\mu} = \big(p+q\,,\;p-q\,,\;k_{A'}\big)\,,\qquad K_{\mathsf{R}\,\mu} = \big(\bar{p}-\bar{q}\,,\;\bar{p}+\bar{q}\,,\;k_{A'}\big)\,.$$

Nonrelativistic KLT from first principles: cont.

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For a open string tachyons state, with the ends of the open string anchored on two D-branes located in the compactified direction X^1, the associated vertex operator is

$$\mathcal{V}_{\mathrm{open}} = :\exp\big[\,i\,K_{\mathrm{L}}\cdot\varphi_{\mathrm{L}}(y) + i\,K_{\mathrm{R}}\cdot\varphi_{\mathrm{R}}(y)\,\big]:\,,$$

$$K_{\mathrm{L}\mu} = K_{\mathrm{R}\mu} = K_{\mu} \equiv \left(\frac{\varepsilon}{2} + \frac{WR}{\alpha'_{\mathrm{eff}}}\,,\,\frac{\varepsilon}{2} - \frac{WR}{\alpha'_{\mathrm{eff}}}\,,k_{A'}\right),\quad W = w + \frac{L}{2\pi R}\,.$$

Nonrelativistic KLT from first principles: cont.

We compute the $\mathcal{N}$-point closed string scattering amplitude without using the $\alpha' \to 0$ limit. It takes the Virasoro-Shapiro form,

$$\mathcal{M}_{\mathsf{c}}^{\mathcal{N}} = \frac{C(1,\ldots,\mathcal{N})}{\alpha' g_{\mathsf{s}}^2} \int_{\mathbb{C}^{\mathcal{N}-3}} d^2 z_2 \cdots d^2 z_{\mathcal{N}-2} \prod_{\substack{i,j=1 \\ i<j}}^{\mathcal{N}-1} z_{ji}^{\frac{1}{2}\alpha' K_{\mathsf{L}i}\cdot K_{\mathsf{L}j}} \, \bar{z}_{ji}^{\frac{1}{2}\alpha' K_{\mathsf{R}i}\cdot K_{\mathsf{R}j}}$$

the nonrelativistic formalism gives

$$\begin{aligned} \alpha' K_{\mathsf{L}i} \cdot K_{\mathsf{L}j} &= -\left(w_i\,\varepsilon_j + w_j\,\varepsilon_i\right) R + \alpha' k_i^{A'} k_j^{A'} - \left(n_i\,w_j + n_j\,w_i\right), \\ \alpha' K_{\mathsf{R}i} \cdot K_{\mathsf{R}j} &= -\left(w_i\,\varepsilon_j + w_j\,\varepsilon_i\right) R + \alpha' k_i^{A'} k_j^{A'} + \left(n_i\,w_j + n_j\,w_i\right). \end{aligned}$$

and you can get this from just using the change of variables, and knowing the relativistic answer.

Nonrelativistic KLT from first principles: cont.

The $\mathcal{N}$-point KLT relation is given by

$$\mathcal{M}_{\mathrm{c}}^{\mathcal{N}} = \frac{(-1)^{\mathcal{N}-3}}{\alpha' g_s^2} \sum_{\rho,\,\sigma} C(1, \sigma, \mathcal{N}-1, \mathcal{N})\, \mathcal{S}_{\mathrm{L}}[\rho|\sigma]_{K_{\mathrm{L}1}} \times \mathcal{M}_{\mathrm{L}}\left(\rho, 1, \mathcal{N}-1, \mathcal{N}\right) \mathcal{M}_{\mathrm{R}}\left(1, \sigma, \mathcal{N}-1, \mathcal{N}\right),$$

$$\mathcal{M}_{\mathrm{L,\,R}}(1, \ldots, \mathcal{N}) = \int_{0<y_2<\cdots<y_{\mathcal{N}-1}<1} dy_2 \cdots dy_{\mathcal{N}-2} \prod_{\substack{i,j=1 \\ i<j}}^{\mathcal{N}-1} |y_{ij}|^{\frac{\alpha'}{2} K_{\mathrm{L,\,R}i} \cdot K_{\mathrm{L,\,R}j}}$$

- The form is the same as the relativistic KLT
- The kinematical data differs from before

$$K_{\mathrm{L,\,R}}^{M} = K^{M} = \left(\frac{\varepsilon}{2} + \frac{wR}{\alpha'}\,,\, \frac{\varepsilon}{2} - \frac{wR}{\alpha'}\,,\, k_{A'}\right).$$

- The KLT relation agrees with what is found from taking the limit.

One-loops Amplitudes in NRST

- Gomis-Ooguri computed the bosonic one-loop free energy at a finite temperature (thermodynamic partition function) of free closed strings.

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One-loops Amplitudes in NRST

- Gomis-Ooguri computed the bosonic one-loop free energy at a finite temperature (thermodynamic partition function) of free closed strings.
- They found that the result *localizes* to a set of discrete points in the fundamental domain of $\mathrm{SL}(2,\mathbb{C})$ labeled by winding sectors.
- They also found a Hagedorn temperature exists when performing a sum over the winding states.
- We showed that at one-loop, the bosonic open string amplitudes exhibit a similar localization in the moduli space (parametrized by $t \in \mathbb{R}^+$), and computed the free energy for open strings.

One-loops Amplitudes in NRST: cont.

We find that the vacuum amplitude takes the following form:

$$\mathcal{Z} = \sum_{m,w} \frac{1}{m} \left(\frac{2\pi RW}{\alpha'_{\text{eff}}\, \beta m} \right)^{12} \left[\eta\!\left(\frac{i\beta m}{4\pi RW} \right) \right]^{-24} .$$

Performing the sum over m, the Helmholtz free energy is:

$$\mathscr{F} = -T\mathcal{Z} = T \sum_{\varepsilon} D(\varepsilon) \ln \left(1 - e^{-\beta\, \varepsilon} \right) ,$$

$$\varepsilon = \frac{\alpha'_{\text{eff}}}{2wR} \left[k_{A'} k_{A'} + \frac{1}{\alpha'_{\text{eff}}} (N-1) \right] .$$

$D(\varepsilon)$ to denote the density of states associated with the energy ε . This is the nonrelativistic open string analog of the result in Gomis-Ooguri.

Summary

- We found a $\mathcal{N}$-point KLT factorization for closed string amplitudes with momentum and *winding*.
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  - We need to introduce electric potentials on the D-branes
- By taking the $\alpha' \to 0$ and critical B-field limit, we arrive at nonrelativistic string theory KLT.
- Taking the limit matches what we get if we work in a strictly nonrelativistic famework.

Fin.