Minimal Stringy Non-Relativistic Supergravity

08-05-2023, Non-Relativistic Strings and Beyond, Nordita

Jan Rosseel, Ruđer Bošković Institute

Mostly based on work with
E. Bergshoeff, J. Lahnsteiner, L. Romano, C. Şimşek: arXiv:2107.14636
E. Bergshoeff, J. Lahnsteiner, L. Romano: arXiv:2204.04089

Introduction

- Motivational question: What is the gravitational effective field theory description of non-relativistic string theory? (Gomis, Ooguri; Danielsson, Guijosa, Kruczenski)
- Addressed bosonically via worldsheet theory β-function calculations. (Gomis, Oh, Yan, Yu) Gravitational field theory with underlying String Newton-Cartan geometry. (see e.g., Andringa, Bergshoeff, Gomis, de Roo; Bergshoeff, Gomis, JR, Şimşek, Yan; Bergshoeff, van Helden, Lahnsteiner, Romano, JR; Bidussi, Harmark, Hartong, Obers, Oling)
- Alternative approaches:
  - ◇ Non-relativistic limit (Bergshoeff, Lahnsteiner, Romano, JR, Şimşek)
  - ◇ Non-Riemannian backgrounds in Double Field Theory (e.g.; Gallegos, Gürsoy, Verma, Zinnato; Morand, Park; Cho, Park; Park, Sugimoto; Ko, Melby-Thompson, Meyer, Park; Berman, Blair, Otsuki; Blair, Oling, Park)
  - ◇ Strings in Torsional Newton-Cartan backgrounds (Harmark, Hartong, Menculini, Obers, Oling Yan; Gallegos, Gürsoy, Zinnato)
- To discuss non-relativistic superstrings, need to construct non-relativistic supergravity theories.

Introduction

- Not so much work done yet on non-relativistic supergravity. Mostly in $3D$ for theories with underlying Newton-Cartan geometry:
    - gauging of $3D$, $\mathcal{N} = 2$ super-Bargmann algebra (Andringa, Bergshoeff, Sezgin, JR)
    - superconformal tensor calculus methods (Bergshoeff, JR, Zojer)
    - Chern-Simons supergravity (Bergshoeff, JR; Ozdemir, Ozkan, Tunca, Zorba; Concha, Ipinza, Ravera, Rodriguez)
    - via non-relativistic limit (Bergshoeff, JR, Zojer)
- Goal of this talk: obtain non-relativistic supergravity in $10D$, for non-relativistic string theory.
- Method used: careful taking of a non-relativistic limit:
    - does not rely on tricks that only work in specific dimensions
    - closely related to β-function results
- Starting point: relativistic $10D$, $\mathcal{N} = (1, 0)$ supergravity. Simplest case common to all superstring theories.

Relativistic $10D, \mathcal{N} = (1,0)$ Supergravity

- Relativistic $10D, \mathcal{N} = (1,0)$ supergravity: (Bergshoeff, de Roo, de Wit, van Nieuwenhuizen; Chamseddine)
  - Field content
    - Bosonic: Vielbein $E_\mu{}^{\hat{A}}$, Kalb-Ramond two-form $B_{\mu\nu}$, dilaton Φ
    - Fermionic: gravitino Ψ_μ (left-handed Majorana-Weyl), dilatino $\boldsymbol{\lambda}$ (right-handed Majorana-Weyl)
  - Symmetries:
    - local $\mathrm{SO}(1,9)$ Lorentz transformations with parameter $\Lambda^{\hat{A}\hat{B}}$ $(\hat{A}, \hat{B} = 0, 1, \cdots, 9)$
    - one-form gauge symmetry $\delta B_{\mu\nu} = 2\,\partial_{[\mu}\Theta_{\nu]}$
    - local supersymmetry with parameter ε (left-handed Majorana-Weyl)

The non-relativistic limit

- Starting point for the non-rel. limit:

$$\hat{A} \quad \rightarrow \quad A = 0,1 \text{ or } +,- \quad \text{(longitudinal)} \quad \text{and} \quad a = 2,3,\cdots,10 \quad \text{(transversal)}$$

  and the invertible field redefinitions

$$E_\mu{}^A = c\,\tau_\mu{}^A \quad (E_A{}^\mu = c^{-1}\tau_A{}^\mu)\,, \qquad E_\mu{}^a = e_\mu{}^a \quad (E_a{}^\mu = e_a{}^\mu)\,,$$

$$\Phi = \phi + \log(c)\,, \qquad B_{\mu\nu} = -c^2\epsilon_{AB}\tau_\mu{}^A\tau_\nu{}^B + b_{\mu\nu}\,,$$

$$\Pi_\pm\Psi_\mu = c^{\pm 1/2}\psi_{\mu\pm}\,, \qquad \Pi_\pm\boldsymbol{\lambda} = c^{\pm 1/2}\lambda_\pm \quad \text{with} \quad \Pi_\pm = \tfrac{1}{2}(\mathbb{1} \pm \Gamma_{01})\,.$$

- Limit of any quantity (action, equations of motion, symmetry transformation rules):
  - ◇ Apply field redefinition and expand in powers of c^{-2}
  - ◇ Limit = leading order term in this expansion

The non-relativistic bosonic transformation rules

- Plugging the field redefinitions, along with

$$\Lambda^{AB} = \lambda_M \epsilon^{AB}\,, \qquad \Lambda^{Aa} = c^{-1}\lambda^{Aa}\,, \qquad \Lambda^{ab} = \lambda^{ab}\,, \qquad \Theta_\mu = \theta_\mu\,,$$

in the bosonic transformation rules, one finds the expansion

$$\delta_{\text{bos}} = \delta_{\text{bos}}^{(0)} + c^{-2}\delta_{\text{bos}}^{(-2)}\,.$$

In particular

$$\delta_{\text{bos}}\tau_\mu{}^A = \lambda_M \epsilon^A{}_B \tau_\mu{}^B + c^{-2}\lambda^A{}_a e_\mu{}^a\,, \qquad \delta_{\text{bos}} e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda_A{}^a \tau_\mu{}^A\,,$$

$$\delta_{\text{bos}} b_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]} - 2\epsilon_{AB}\lambda^A{}_a \tau_{[\mu}{}^B e_{\nu]}{}^a\,, \qquad \delta_{\text{bos}}\phi = 0\,,$$

$$\delta_{\text{bos}}\psi_{\mu+} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} - 2\lambda_M\right)\psi_{\mu+} + \tfrac{c^{-2}}{2}\lambda^{Aa}\Gamma_{Aa}\psi_{\mu-}\,,$$

$$\delta_{\text{bos}}\psi_{\mu-} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} + 2\lambda_M\right)\psi_{\mu-} + \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\psi_{\mu+}\,,$$

$$\delta_{\text{bos}}\lambda_+ = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} - 2\lambda_M\right)\lambda_+ + \tfrac{c^{-2}}{2}\lambda^{Aa}\Gamma_{Aa}\lambda_-\,,$$

$$\delta_{\text{bos}}\lambda_- = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} + 2\lambda_M\right)\psi_{\mu+} + \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\lambda_+\,.$$

- Well-defined $c \to \infty$ limit.

The non-relativistic bosonic transformation rules

- Plugging the field redefinitions, along with

$$\Lambda^{AB} = \lambda_M \epsilon^{AB}\,, \qquad \Lambda^{Aa} = c^{-1}\lambda^{Aa}\,, \qquad \Lambda^{ab} = \lambda^{ab}\,, \qquad \Theta_\mu = \theta_\mu\,,$$

in the bosonic transformation rules, one finds the expansion

$$\delta_{\text{bos}} = \delta^{(0)}_{\text{bos}} + c^{-2}\delta^{(-2)}_{\text{bos}}\,.$$

In particular

$$\delta^{(0)}_{\text{bos}} \tau_\mu{}^A = \lambda_M \epsilon^A{}_B \tau_\mu{}^B\,, \qquad \delta^{(0)}_{\text{bos}} e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda_A{}^a \tau_\mu{}^A\,,$$

$$\delta^{(0)}_{\text{bos}} b_{\mu\nu} = 2\partial_{[\mu}\theta_{\nu]} - 2\epsilon_{AB}\lambda^A{}_a \tau_{[\mu}{}^B e_{\nu]}{}^a\,, \qquad \delta^{(0)}_{\text{bos}}\phi = 0\,,$$

$$\delta^{(0)}_{\text{bos}}\psi_{\mu+} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} - 2\lambda_M\right)\psi_{\mu+}\,,$$

$$\delta^{(0)}_{\text{bos}}\psi_{\mu-} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} + 2\lambda_M\right)\psi_{\mu-} + \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\psi_{\mu+}\,,$$

$$\delta^{(0)}_{\text{bos}}\lambda_{+} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} - 2\lambda_M\right)\lambda_{+}\,,$$

$$\delta^{(0)}_{\text{bos}}\lambda_{-} = \tfrac{1}{4}\left(\lambda^{ab}\Gamma_{ab} + 2\lambda_M\right)\psi_{\mu+} + \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\lambda_{+}\,.$$

- Well-defined $c \to \infty$ limit.

The non-relativistic bosonic transformation rules

- After the limit $\tau_\mu{}^A$, $e_\mu{}^a$, $b_{\mu\nu}$ and the fermions transform under

$$(\mathrm{SO}(1,1) \times \mathrm{SO}(8)) \ltimes \mathbb{R}^{16} \,,$$

  where:

  - $\mathrm{SO}(1,1)$ = longitudinal Lorentz transformations (parameters λ^{AB})
  - $\mathrm{SO}(8)$ = transversal spatial rotations (parameters λ^{ab})
  - $\mathbb{R}^{16}$ = String Galilean boosts (parameters λ^{Aa})

$$\delta_{\rm bos}\tau_\mu{}^A = \lambda_M \epsilon^A{}_B \tau_\mu{}^B \,, \qquad \delta_{\rm bos} e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda_A{}^a \tau_\mu{}^A \,,$$
$$\delta_{\rm bos} b_{\mu\nu} = -\,2\epsilon_{AB}\lambda^A{}_a \tau_{[\mu}{}^B e_{\nu]}{}^a \,.$$

- Fermions $\psi_{\mu\pm}$, $\lambda_\pm$ have characteristic non-relativistic boost transformations:

$$\delta\psi_{\mu-} = \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\psi_{\mu+} \,, \qquad \delta\psi_{\mu+} = 0 \,, \qquad \delta\lambda_- = \tfrac{1}{2}\lambda^{Aa}\Gamma_{Aa}\lambda_+ \,, \qquad \delta\lambda_+ = 0 \,.$$

The appearance of String Newton-Cartan geometry

- $(\mathrm{SO}(1,1) \times \mathrm{SO}(8)) \ltimes \mathbb{R}^{16} =$ structure group of String Newton-Cartan geometry. (see e.g., Andringa, Bergshoeff, Gomis, de Roo; Bergshoeff, Gomis, JR, Şimşek, Yan; Bergshoeff, van Helden, Lahnsteiner, Romano, JR; Bidussi, Harmark, Hartong, Obers, Oling)

- Metric structure of String Newton-Cartan geometry:

$$\text{rank-2: } \tau_{\mu\nu} = \tau_\mu{}^A \tau_\nu{}^B \eta_{AB}\,, \qquad \text{rank-8: } h^{\mu\nu} = e_a{}^\mu e_b{}^\nu \delta^{ab}\,.$$

- $b_{\mu\nu}$ couples to non-relativistic string worldsheet via Wess-Zumino action

$$\frac{T}{2} \int \mathrm{d}^2\sigma\, \epsilon^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, b_{\mu\nu}\,.$$

  Similar to how central charge gauge field m_μ of Newton-Cartan geometry couples to a non-relativistic particle worldline:

$$\int \mathrm{d}\tau\, \dot{X}^\mu\, m_\mu\,.$$

- Newton-Cartan geometry: Newton potential = m_0.

  String Newton-Cartan geometry: Newton potential = b_{01}.

The appearance of String Newton-Cartan geometry

- Metrics-compatible connection introduced via 3 spin connections:

$$\omega_\mu \quad (\lambda_M)\,, \qquad \omega_\mu{}^{Aa} \quad (\lambda^{Aa})\,, \qquad \omega_\mu{}^{ab} \quad (\lambda^{ab})\,.$$

that obey generalized first Cartan structure equations

$$\begin{cases} 2\,\partial_{[\mu}\tau_{\nu]}{}^A + 2\,\epsilon^A{}_B\,\omega_{[\mu}\,\tau_{\nu]}{}^B = T_{\mu\nu}{}^A\,, \\ 2\,\partial_{[\mu}e_{\nu]}{}^a + 2\,\omega_{[\mu}{}^a{}_{|b|}\,e_{\nu]}{}^b - 2\,\omega_{[\mu|A|}{}^a\,\tau_{\nu]}{}^A = T_{\mu\nu}{}^a\,, \\ 3\,\partial_{[\mu}b_{\nu\rho]} - 6\,\epsilon_{AB}\,\omega_{[\mu}{}^A{}_a\tau_\nu{}^B e_{\rho]}{}^a = T^{(b)}_{\mu\nu\rho}\,. \end{cases}$$

Equations that contain ω-components express that ω-components depend on $\tau_\mu{}^A$, $e_\mu{}^a$, $b_{\mu\nu}$.

- Equations that do not contain ω-components $\rightarrow$ intrinsic torsion. E.g.:

$$2e_a{}^\mu e_b{}^\nu \partial_{[\mu}\tau_{\nu]}{}^A = e_a{}^\mu e_b{}^\nu T_{\mu\nu}{}^A \equiv T_{ab}{}^A\,,$$

$$2e_a{}^\mu \tau_{(A}{}^\nu \partial_{[\mu}\tau_{\nu]B)} = e_a{}^\mu \tau^\nu_{(A} T_{|\mu\nu|B)} \equiv T_{a(A\,,B)}\,.$$

See talk by Kevin van Helden.

The limit of the action

- After plugging in the field redefinitions, the action S can generically can expanded as:

$$S = c^2 S^{(2)} + S^{(0)} + c^{-2} S^{(-2)} + c^{-4} S^{(-4)}\,, \qquad S^{(i)} = S^{(i)}[\tau, e, b, \phi, \psi_\pm, \lambda_\pm]\,.$$

  Due to non-trivial cancellations (e.g. between Einstein-Hilbert term and kinetic term of $B_{\mu\nu}$), one finds

$$S^{(2)} \equiv 0\,.$$

- Limit of the action $S^{(0)}$ is invariant under $\delta^{(0)}_{\text{bos}}$:

$$\delta_{\text{bos}} S = \delta^{(0)}_{\text{bos}} S^{(0)} + c^{-2}\left(\delta^{(0)}_{\text{bos}} S^{(-2)} + \delta^{(-2)}_{\text{bos}} S^{(0)}\right) + \mathcal{O}(c^{-4})$$

$$\delta_{\text{bos}} S = 0 \qquad \Rightarrow \qquad \delta^{(0)}_{\text{bos}} S^{(0)} = 0\,.$$

  In particular, $S^{(0)}$ is invariant under $(\mathrm{SO}(1,1) \times \mathrm{SO}(8)) \ltimes \mathbb{R}^{16}$.

Emergent dilatations

- $S^{(0)}$ is also invariant under an emergent dilatation symmetry: (Bergshoeff, Lahnsteiner, Romano, JR, Şimşek)

$$\delta_D \tau_\mu{}^A = \lambda_D \tau_\mu{}^A \,, \qquad \delta_D \phi = \lambda_D \,,$$
$$\delta_D \psi_{\mu\pm} = \pm \tfrac{1}{2} \lambda_D \psi_{\mu\pm} \,, \qquad \delta_D \lambda_\pm = \pm \tfrac{1}{2} \lambda_D \lambda_\pm \,.$$

Target space equivalent of symmetry of worldsheet action of bosonic non-rel. string in arbitrary String Newton-Cartan background. (Bergshoeff, Gomis, JR, Şimşek, Yan)

- Consequence: need to introduce dependent dilatation gauge field and change one of the first Cartan structure equations to

$$2\, \partial_{[\mu} \tau_{\nu]}{}^A + 2\, \epsilon^A{}_B\, \omega_{[\mu}\, \tau_{\nu]}{}^B - 2\, b_{[\mu} \tau_{\nu]}{}^A = T_{\mu\nu}{}^A \,.$$

$T_{aA}{}^A$ is no longer intrinsic. Dilatation covariant intrinsic torsion components

$$T_{ab}{}^A \qquad \text{and} \qquad T_{a\{A\,,B\}} \,.$$

The limit of the supersymmetry transformations

- With $\Pi_\pm \varepsilon = c^{\pm 1/2}\epsilon_\pm$, the expansion of the supersymmetry transformation rules is:

$$\delta_Q = c^2 \delta_Q^{(2)} + \delta_Q^{(0)} + c^{-2}\delta_Q^{(-2)} \,.$$

  $\delta_Q^{(0)}$ has the right structure for non-relativistic supersymmetry transformation rules. Divergence $\delta_Q^{(2)}$ is only non-zero when acting on the fermions:

$$\delta_Q^{(2)}\psi_{\mu+} = \tfrac{1}{4}\,\tau_\mu{}^+ T_{ab}{}^- \Gamma^{ab}\epsilon_+\,, \qquad \delta_Q^{(2)}\psi_{\mu-} = \tfrac{1}{4}\,\tau_\mu{}^+\left(T_{ab}{}^-\Gamma^{ab}\epsilon_- + T_{a+}{}^-\Gamma_-{}^a\epsilon_+\right),$$

$$\delta_Q^{(2)}\lambda_+ = 0\,, \qquad \delta_Q^{(2)}\lambda_- = -\tfrac{1}{4}T_{ab}{}^-\Gamma^{ab}{}_-\epsilon_+\,,$$

  with $\{T_{ab}{}^-, T_{a+}{}^-\}$ a subset of the dilatation covariant intrinsic torsion components.

- Presence of divergences has two consequences:
  - ◇ Need for intrinsic torsion constraints to ensure supersymmetry invariance of $S^{(0)}$
  - ◇ Emergence of fermionic Stueckelberg symmetries

Intrinsic torsion constraints

- $S^{(0)}$ is not automatically invariant under $\delta_Q^{(0)}$:

$$\delta_Q S = c^2 \delta_Q^{(2)} S^{(0)} + \left(\delta_Q^{(0)} S^{(0)} + \delta_Q^{(2)} S^{(-2)}\right) + \mathcal{O}(c^{-2})$$

$$\delta_Q S = 0 \qquad \Rightarrow \qquad \delta_Q^{(0)} S^{(0)} = -\delta_Q^{(2)} S^{(-2)}\,.$$

Note however that all terms in $\delta_Q^{(2)} S^{(-2)}$ are proportional to $T_{a+}{}^{-}, T_{ab}{}^{-}$.

$\Rightarrow$ $S^{(0)}$ is invariant under non-rel. supersymmetry $\delta_Q^{(0)}$ if

$$T_{a+}{}^{-} = 0\,, \quad \text{and} \quad T_{ab}{}^{-} = 0 \qquad \Leftrightarrow \qquad \tau_{[\mu}{}^{-} \partial_\nu \tau_{\rho]}{}^{-} = 0\,.$$

- Can be imposed in a supersymmetric way since

$$\delta_Q^{(0)} \tau_\mu{}^{-} = 0 \qquad \Rightarrow \qquad \begin{cases} \delta_Q^{(0)} T_{a+}{}^{-} = 0 \\ \delta_Q^{(0)} T_{ab}{}^{-} = 0 \end{cases}.$$

- Also ensures that the commutator of two $\delta_Q^{(0)}$ transformations on $\tau_\mu{}^A$ properly closes.

Emergent fermionic Stueckelberg symmetries

- Note that $S^{(0)}$ is invariant under $\delta_Q^{(2)}$, before imposing the intrinsic torsion constraints:

$$\delta_Q S = c^2 \delta_Q^{(2)} S^{(0)} + \left(\delta_Q^{(0)} S^{(0)} + \delta_Q^{(2)} S^{(-2)}\right) + \mathcal{O}(c^{-2})$$
$$\delta_Q S = 0 \qquad \Rightarrow \qquad \delta_Q^{(2)} S^{(0)} = 0\,.$$

  Structure of $\delta_Q^{(2)}$ suggests invariance of $S^{(0)}$ under the following fermionic S- and T-symmetries:

$$\delta_S \psi_{\mu+} = \tfrac{1}{2}\tau_\mu{}^+ \Gamma_+ \eta_- \,, \qquad \delta_S \lambda_- = \eta_- \,,$$
$$\delta_T \psi_{\mu-} = \tau_\mu{}^+ \rho_- \,.$$

- The commutator of the S-symmetry with $\delta_Q^{(0)}$ contains the dilatation symmetry.

- The non-relativistic action $S^{(0)}$ is thus invariant under 3 extra symmetries that are emergent and were not present in the relativistic starting point.

Multiplet shortening

- One can redefine the non-relativistic fields $\{\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \psi_{\mu\pm}, \lambda_\pm\}$ to $\{\tilde{\tau}_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \phi, \tilde{\psi}_{\mu+}, \tilde{\psi}_-, \tilde{\psi}_{\mu-}, \tilde{\lambda}_\pm\}$ via

$$\tilde{\tau}_\mu{}^A \equiv \mathrm{e}^{-\phi}\tau_\mu{}^A , \quad \tilde{\psi}_{\mu+} \equiv \mathrm{e}^{-\phi/2}\left(\psi_{\mu+} - \tfrac{1}{2}\tau_\mu{}^+\Gamma_+\lambda_-\right) , \quad \tilde{\psi}_- \equiv \mathrm{e}^{3\phi/2}\tau_+{}^\mu\psi_{\mu-} ,$$
$$\tilde{\psi}_{\mu-} \equiv \mathrm{e}^{\phi/2}\left(\psi_{\mu-} - \tau_\mu{}^+\tilde{\psi}_-\right) , \qquad \tilde{\lambda}_\pm = \mathrm{e}^{\mp\phi/2}\lambda_\pm .$$

- These redefined fields can be divided in two sets:
  - ◇ $\{\tilde{\tau}_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \tilde{\psi}_{\mu+}, \tilde{\psi}_{\mu-}, \tilde{\lambda}_+\}$: invariant under the dilatation, S- and T-symmetries.
  - ◇ $\{\phi, \tilde{\psi}_-, \tilde{\lambda}_-\}$: transform as Stueckelberg fields under the dilatation, S- and T-symmetries:

$$\delta\phi = \lambda_D , \qquad \delta\tilde{\psi}_- = \mathrm{e}^{3\phi/2}\rho_- , \qquad \delta\tilde{\lambda}_- = \mathrm{e}^{\phi/2}\eta_- .$$

- Invariance of $S^{(0)}$ under δ_D, δ_S and δ_T implies that ϕ, $\tilde{\psi}_-$ and $\tilde{\lambda}_-$ do not occur in it.
  $\Rightarrow$ $S^{(0)}$ is an action for a shortened multiplet $\{\tilde{\tau}_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \tilde{\psi}_{\mu+}, \tilde{\psi}_{\mu-}, \tilde{\lambda}_+\}$.

Field equations

Summary

- Shortened multiplet: $\{\tilde{\tau}_\mu{}^A, e_\mu{}^a, b_{\mu\nu}, \tilde{\psi}_{\mu+}, \tilde{\psi}_{\mu-}, \tilde{\lambda}_+\}$
- Pseudo-action $S^{(0)}[\tilde{\tau}, e, b, \tilde{\psi}_\pm, \tilde{\lambda}_+]$ to be supplemented with intrinsic torsion constraints:

$$e_a{}^\mu \tilde{\tau}_+{}^\nu \partial_{[\mu}\tilde{\tau}_{\nu]}{}^- = 0\,, \qquad\qquad e_a{}^\mu e_b{}^\nu \partial_{[\mu}\tilde{\tau}_{\nu]}{}^- = 0\,.$$

- The field equations derived from $S^{(0)}$ can be obtained by taking the non-relativistic limit of the relativistic field equations.
- However: multiplet shortening $\Rightarrow$ less independent field equations coming from $S^{(0)}$ than there are relativistic ones.
- 2 fermionic and 1 bosonic equations of motion for the Stueckelberg fields ϕ, $\tilde{\psi}_-$ and $\tilde{\lambda}_-$ missing.

Field equations

- It is possible to take the non-relativistic limit of the field equations directly, such that one ends up with as many independent field equations for $\tau_\mu{}^A$, $e_\mu{}^a$, $b_{\mu\nu}$, ϕ, $\psi_{\mu\pm}$, $\lambda_\pm$ as in the relativistic parent theory.

- The limit of the field equations taken in this direct way consists of those derived from $S^{(0)}$, as well as 1 bosonic and 2 fermionic 'missing' non-relativistic equations of motion.

- The full set of field equations is
  - ◇ dilatation covariant
  - ◇ covariant under the S- and T-symmetries
  - ◇ supersymmetric upon imposing the intrinsic torsion constraints $T_{ab}{}^- = 0 = T_{a+}{}^-$.

- The linearization of the missing bosonic equation of motion contains:

$$\partial_a \partial^a b_{01} + \cdots = 0\,.$$

$\Rightarrow$ covariant generalization of the Poisson equation.

 The longitudinal T-duality viewpoint

The longitudinal T-duality viewpoint

- Longitudinal T-duality: non-relativistic string theory on a spatial longitudinal circle is dual to relativistic string theory compactified on a null direction.

- From the supergravity point of view: reduction of non-relativistic $10D, \mathcal{N} = (1,0)$ supergravity = reduction of relativistic $10D, \mathcal{N} = (1,0)$ supergravity along a null Killing vector (Bergshoeff, Lahnsteiner, Romano, JR)

$10D$

$(\tau_\mu{}^A,\ e_\mu{}^a,\ b_{\mu\nu}, \phi)$ with $\tau_{[\mu}{}^- \partial_\nu \tau_{\rho]}{}^- = 0$ $\longleftrightarrow$ (T) $\longleftrightarrow$ $(\tilde{E}_\mu{}^{\hat{A}},\ \tilde{B}_{\mu\nu},\ \tilde{\Phi})$ with $\partial_{[\mu}(\tilde{G}_{\nu]z} + \tilde{B}_{\nu]z}) = 0$

reduction along spatial longitudinal direction z $\searrow$ $\swarrow$ reduction along lightlike direction z

$9D$

Newton-Cartan geometry $(\tau_i,\ e_i{}^a,\ m_i)$ + matter $(\phi,\ n_i,\ m_{ij})$
with $\partial_{[i}\tau_{j]} = -\partial_{[i}n_{j]}$

The longitudinal T-duality viewpoint

- Note:

$$\tau_{[\mu}{}^{-}\partial_\nu\tau_{\rho]}{}^{-} = 0 \qquad \overset{\mathrm{T}}{\longleftrightarrow} \qquad \partial_{[\mu}(\tilde{G}_{\nu]z} + \tilde{B}_{\nu]z}) = 0\,.$$

- The non-relativistic $10D, \mathcal{N} = (1,0)$ multiplet is shortened. What about the T-dual side?
- Since z is a null direction, one has $\tilde{G}_{zz} = 0$. Not supersymmetric! Supersymmetric set of constraints:

$$\tilde{G}_{zz} = 0\,, \qquad \tilde{\Psi}_z = 0\,, \qquad \partial_{[\mu}(\tilde{G}_{\nu]z} + \tilde{B}_{\nu]z}) = 0\,.$$

- Multiplet shortening on both sides of the duality:

	Non-Rel.	Rel. with null Killing vector
$\mathbf{1}$:	δ_D	$\tilde{G}_{zz} = 0\,,$
$\mathbf{8}+\mathbf{8}$:	$\delta_S + \delta_T$	$\tilde{\Psi}_z = 0\,,$
$\mathbf{36}$:	$\tau_{[\mu}{}^{-}\partial_\nu\tau_{\rho]}{}^{-} = 0\,,$	$\partial_{[\mu}(\tilde{G}_{\nu]z} + \tilde{B}_{\nu]z}) = 0\,.$

Half-supersymmetric solutions

- Longitudinal T-duality can be used to generate solutions of non-rel. supergravity from those of rel. supergravity that satisfy the necessary constraints.

- Example 1: pp-wave, corresponding to state with winding $w = 0$, momentum $n = 1$

$$d\tilde{s}^2_{pp} = -2\,dt(dz + K\,dt) + dz^2_{(8)}\,, \qquad \tilde{B} = 0\,, \qquad \mathrm{e}^{\tilde{\Phi}} = g_s\,,$$

dualizes to a non-rel. fundamental string solution:

$$ds^2_\tau = -dt^2 + dz^2\,, \qquad ds^2_e = dz^2_{(8)}\,, \qquad b = K\,dt \wedge dz\,, \qquad \mathrm{e}^{\phi} = g_s\,,$$

corresponding to state with $w = 1$ and $n = 0$ and dispersion relation $E \propto k^a k_a$.

- From $9D$ point of view: flat Newton-Cartan geometry with Newton potential $m_t = K$, sourced by massive particle.

Half-supersymmetric solutions

- Example 2: rel. fundamental string, corresponding to state with $w = 1$, $n = 0$

$$d\tilde{s}^2_{F1} = -2\,H^{-1} dt dz + dz^2_{(8)}\,, \qquad \tilde{B} = (H^{-1} - 1)\,dt \wedge dz\,, \qquad \mathrm{e}^{\tilde{\Phi}} = g_s\,H^{-1/2}\,,$$

dualizes to 'unwound' string solution with $w = 0$, $n = 1$

$$ds^2_\tau = -H^{-2}\,dt^2 + \big(dz + (H^{-1} - 1)dt\big)^2\,, \qquad ds^2_e = dz^2_{(8)}\,, \qquad \mathrm{e}^{\phi} = g_s\,H^{-1/2}\,.$$

- Note that the rel. anti-fundamental string does not satisfy the constraint $\partial_{[\mu}(\tilde{G}_{\nu]z} + \tilde{B}_{\nu]z}) = 0$!

- Can also be applied to obtain non-relativistic NS5-brane and KK5-monopole solutions from their relativistic counterparts.

Outlook

- Minimal $10D$ supergravity can be constructed via a stringy non-relativistic limit.
- Interesting features that regularize the limit
  - emergent dilatation and fermionic shift symmetries
  - supersymmetric set of intrinsic torsion constraints
- Result consistent with what one expects from longitudinal T-duality.
- Outlook:
  - inclusion of Yang-Mills multiplet
  - type II non-relativistic supergravity (see talk by U. Zorba)
  - $11D$ non-relativistic supergravity (see talk by J. Lahnsteiner)
  - stringy non-relativistic expansions (see talks by J. Musaeus and E. Have)
  - DFT techniques (see talk by K. Morand)
  - solutions?
  - ...