Black Holes in Non-Relativistic Holography

Based mainly on `2207.12477` with N. Dorey and B. Zhao
(and a little on `2302.14850` with N. Dorey)

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The point of this talk

1. There is a natural **bottom-up framework** that relates:

Non-relativistic conformal field theory in $(d-2)$ spatial dimensions ⟷ **Gravity** in spacetimes asymptotic to X_{d+1}

2. There exists an **explicit such dual pair**, that can be stated as

Superconformal **quantum mechanics** on instanton moduli space ⟷ **M-theory** on an $X_7 \times S^4$ background

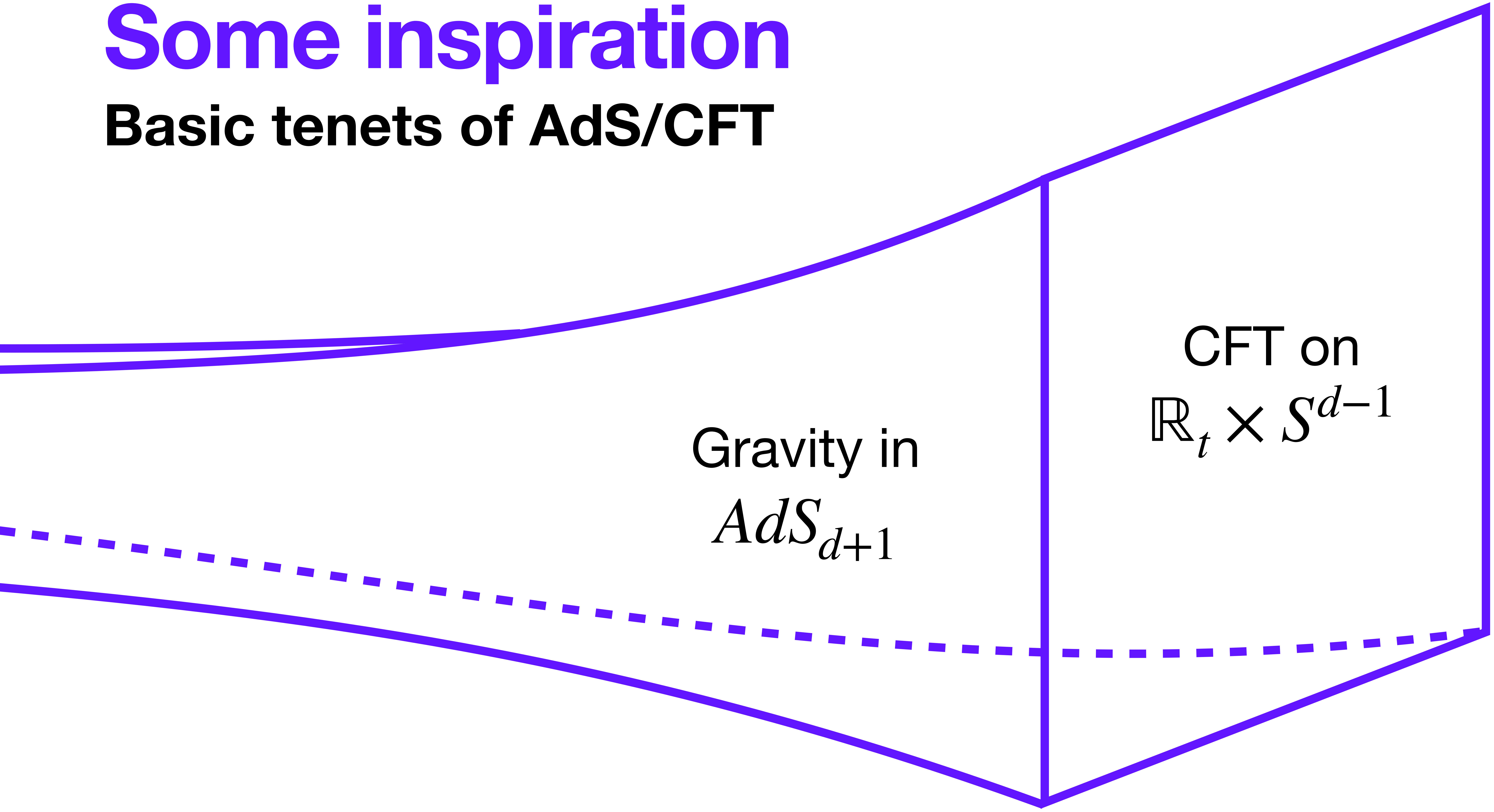
3. In this setup, we derive a **quantitative relationship** of the form

Degeneracy of **BPS states** ⟷ **Bekenstein-Hawking entropy** of a **BPS black hole**

Part I: General aspects of non-relativistic holography

Some inspiration

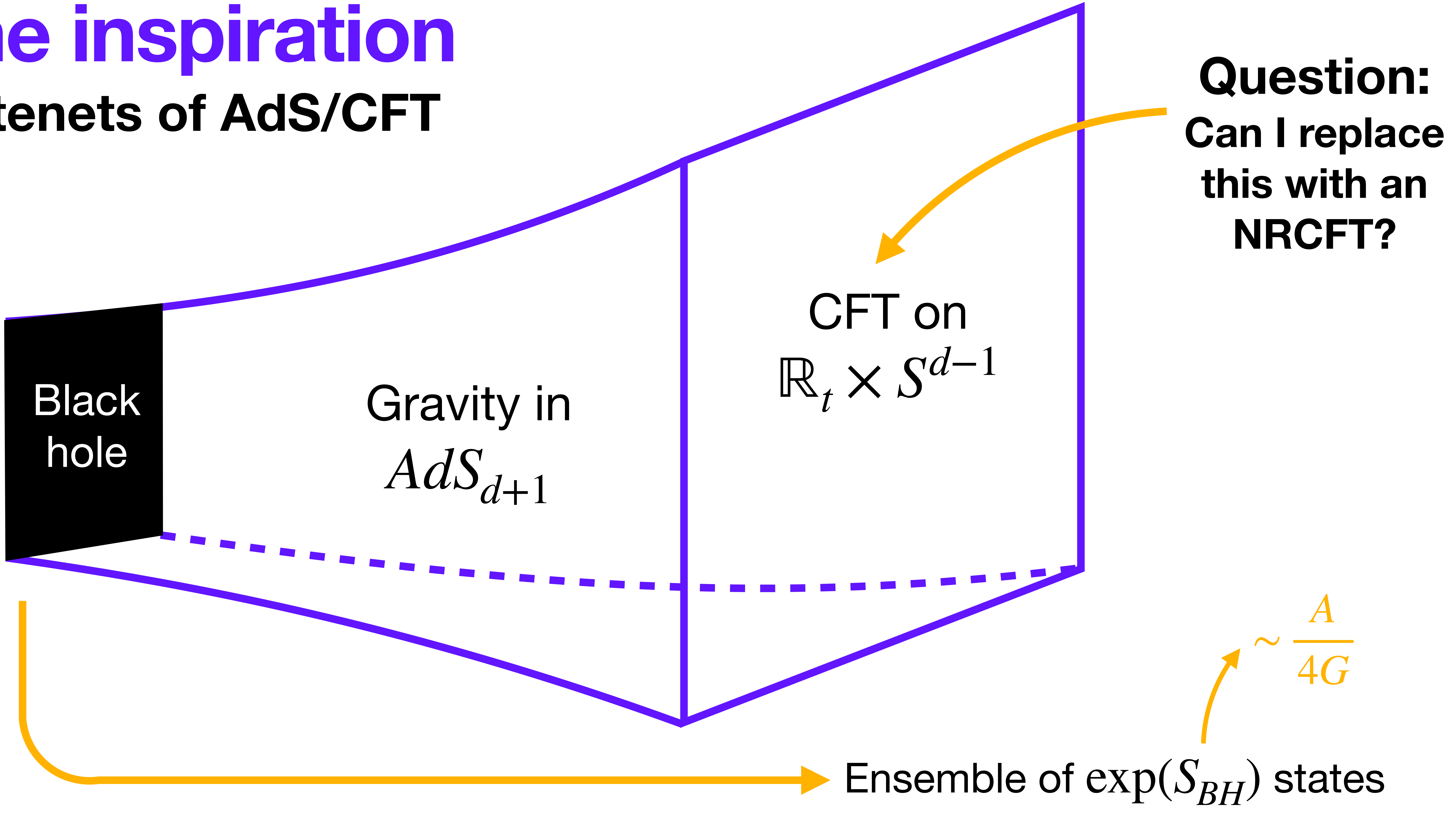
Basic tenets of AdS/CFT



Killing symmetries ⟷ Conformal Killing symmetries

Some inspiration

Basic tenets of AdS/CFT



An NRCFT primer

Symmetries and spectrum

- Symmetries given by the **Schrödinger group Schr$(d-2)$** (with $z=2$)

Hamiltonian	Dilatation	Special	Spatial translations and rotations	Galilean boost	Particle number
H	D	C			K

$\mathfrak{sl}(2,\mathbb{R})$

- **Operator state map** relates: [Nishida, Son]

Operators of definite scaling dimension ⟷ Eigenstates of the oscillator Hamiltonian $\Delta := H + C$

- A sector of fixed particle number described by a **conformal quantum mechanics**

An NRCFT primer

Relation to higher-dimensional CFT

- **Schr**$(d-2)$ is the **centraliser of null translations** in the conformal group $SO(2,d)$ in $(1,d-1)$ dimensions

$\Longrightarrow$ Can **recast** NRCFT as (generically non-local) **CFT on null-compactified** $\mathbb{R}^{1,d-1}$

Particle number $\longrightarrow$ Momentum on compact null circle

- Operator-state map rephrased as a **conformal map**
  - Operators on null-compactified $\mathbb{R}^{1,d-1}$ $\longleftrightarrow$ States on (time slices of) Y_d
  - Y_d is just the **null-compactified pp-wave spacetime!**

$$ds^2 = -2d\xi\, dt - x^i x^i dt^2 + dx^i dx^i, \qquad \xi \sim \xi + 2\pi, \qquad \Delta = i\partial_t$$

The bulk geometry

Some nice coordinates

- A simple step: pp-wave is **conformally flat**

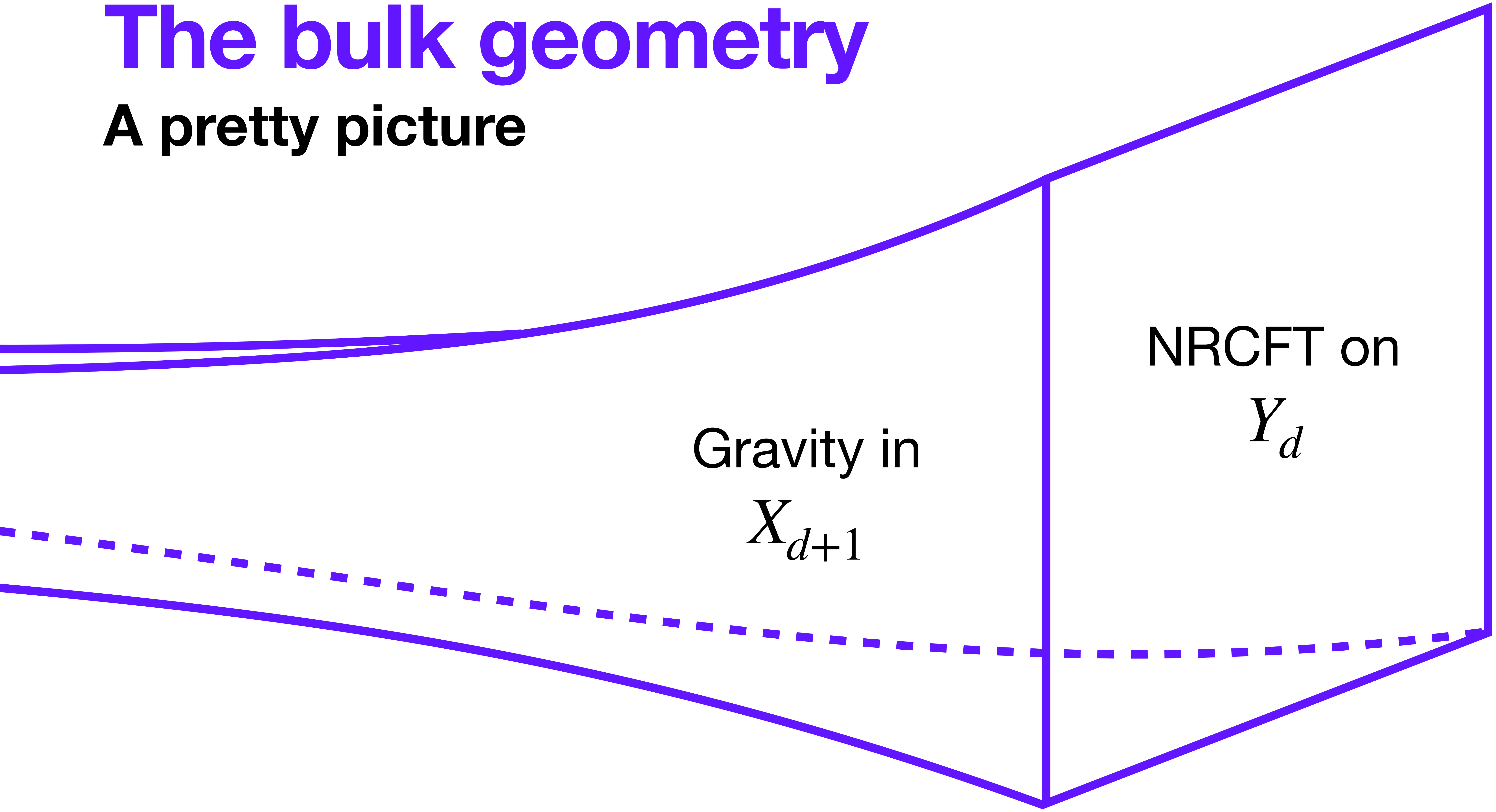
  $\implies$ There exist coordinates on AdS_{d+1} with **pp-wave conformal boundary**

$$ds^2_{AdS} = \frac{dr^2}{g^2 r^2} + r^2 \left(-2d\xi\, dt - x^i x^i dt^2 + dx^i dx^i\right) - \frac{dt^2}{g^2}$$

- **Define X_{d+1}** simply by identifying $\xi \sim \xi + 2\pi$ throughout the bulk
  - Conformal boundary is Y_d

The bulk geometry

A pretty picture



c.f. a bunch of 2008/2009 papers:

[Goldberger]

[Son]

[Balasubramanian, McGreevy]

[Barbon, Fuertes]

[Adams, Balasubramanian, McGreevy]

[Herzog, Rangamani, Ross]

[Maldacena, Martelli, Tachikawa]

[Balasubramanian, de Boer, Sheikh-Jabbari, Simón]

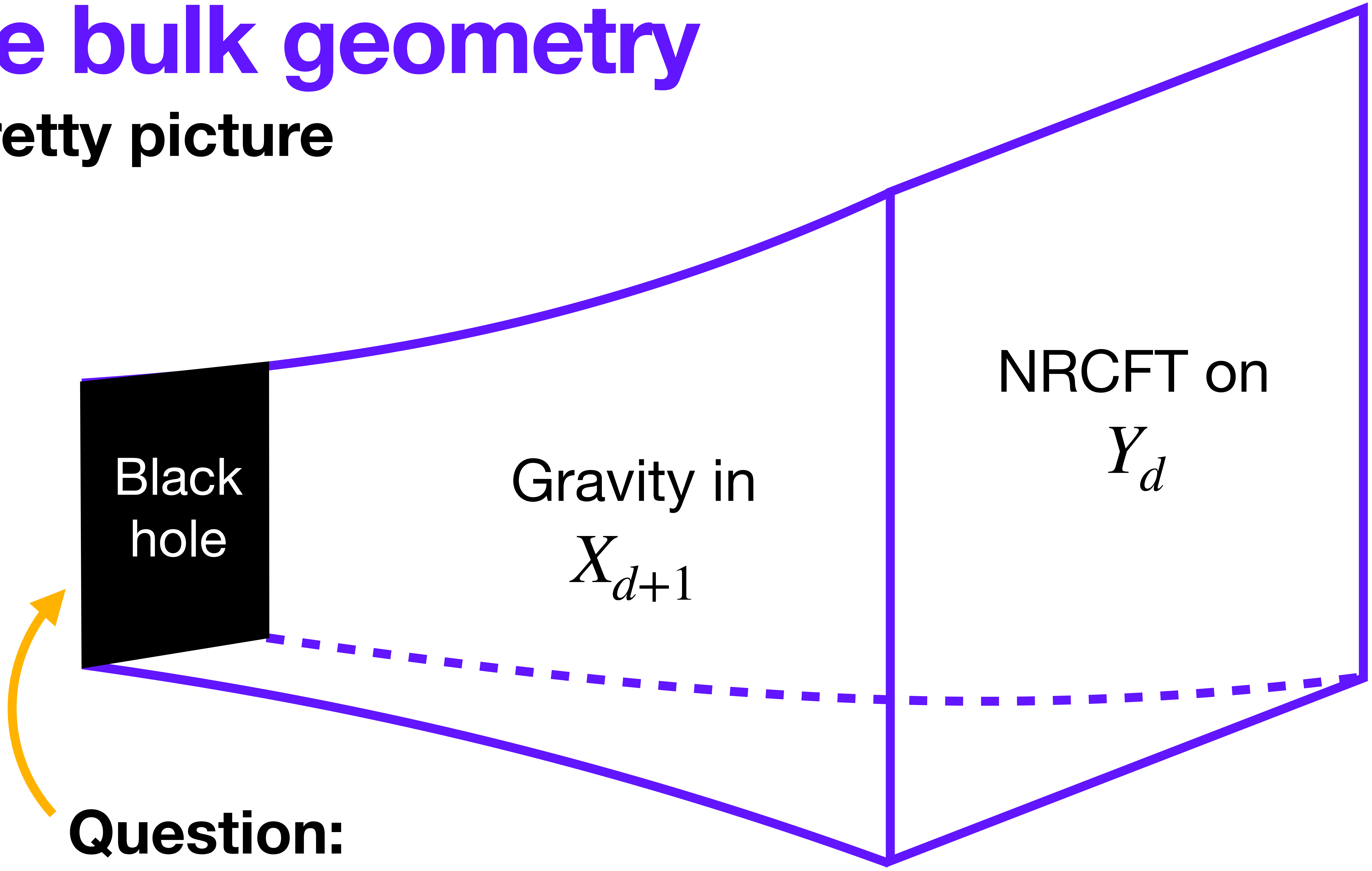
⋮

Schrödinger group!

Killing symmetries ⟷ Conformal Killing symmetries

The bulk geometry

A pretty picture



Question:

How do I construct this?
(in Einstein gravity + …)

How to build a black hole

- The pp-wave spacetime arises in the **Penrose limit** of $\mathbb{R}_t \times S^{d-1}$
- Follows that an asymptotically AdS_{d+1} black hole admits a limit which, after a null orbifold, is an **asymptotically X_{d+1} black hole** [Maldacena, Martelli, Tachikawa]
- Only works for **rotating** black holes
  - One angular momentum becomes very large, in a way coordinated with the Penrose limit
  - This **precisely coincides** with the "**ultra-spinning**" limit [Mouland, Dorey] [Klemm] [Hennigar, Kubiznak, Mann, Musoke], c.f. [Emperan, Myers],…

- An **ultra-spinning black hole** is dual to an ensemble of states in **conformal QM**

Part II: An explicit case study

Plug and play

- One way to get a dual pair: **plug in your favourite AdS/CFT dual pair!**
- The NRCFT is precisely the **Discrete Lightcone Quantisation (DLCQ)** of the original CFT
- Notoriously subtle: to get something sensible, one must "integrate out" **zero modes**, or else find some way to deal with them
- One setting where we have control: **M5-branes!** [Aharony et al.]

6d $U(N)$ $(2,0)$ **superconformal field theory**	⟷	**M-theory** on an $AdS_7 \times S^4$ background

The boundary theory

Basic features

- In a sector of fixed particle number K, NRCFT becomes a $(0+1)$-dim σ-model

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu}(X)\dot{X}^{\mu}\dot{X}^{\nu} + g_{\mu\nu}(X)\psi^{\dagger\mu}D_t\psi^{\nu} - \frac{1}{2}R_{\mu\nu\rho\sigma}(X)\psi^{\dagger\mu}\psi^{\dagger\nu}\psi^{\rho}\psi^{\sigma}$$

- Target space is $\mathscr{M}_{K,N}$, the moduli space of K **Yang-Mills instantons** in $SU(N)$

Hyper-Kähler cone

Maximal SUSY (Hyper-Kähler) — **Conformal symmetry** (cone)

- In summary, the theory is an $\mathscr{N} = (4,4)$ **superconformal quantum mechanics**

Actually super-Schrödinger!

The boundary theory

Organising states

- **States labelled** by eigenvalues under…

Oscillator Hamiltonian → Scaling dimension Δ

R- and Global symmetries $SO(5) \times SU(2) \times SU(2) \times SU(N)$ → Charges J_1, J_2, Q_1, Q_2, n_a

- Chosen **BPS bound** takes the form

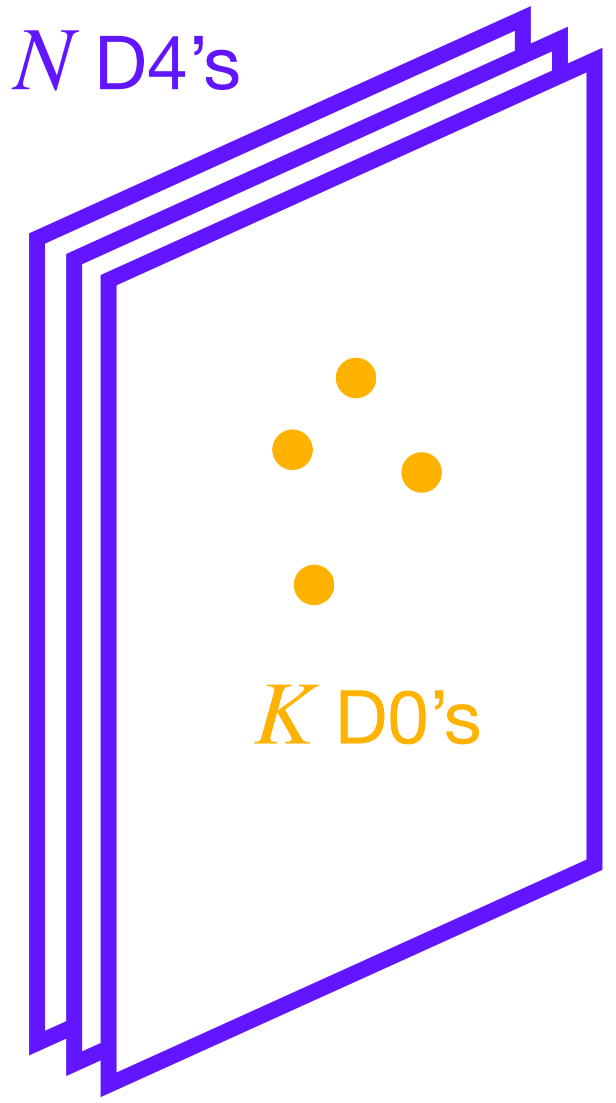
$$\{\mathbb{Q}, \mathbb{S}\} = \mathcal{U} = \Delta - J_1 - J_2 - 2Q_1 - 2Q_2 \geq 0 \qquad (\mathbb{Q}^\dagger = \mathbb{S})$$

The boundary theory

A stringy perspective

- QM describes slow motion of K instanton-particles in an auxiliary 5d $\mathcal{N}=2\ U(N)$ Yang-Mills theory

A pretty IIA picture



- Think like a D0 $\longrightarrow$ QM on $\mathscr{M}_{K,N}$ is the strongly-coupled fixed-point of a **$U(K)$ matrix quantum mechanics**!

(Resolution = Turning on FI parameters)

($U(K)$ BFSS
+ N fund. hypers)

Statement of the duality

Superconformal Quantum Mechanics on $\mathscr{M}_{K,N}$

⟷

Self-contained!

M-theory on an asymptotically $X_7 \times S^4$ spacetime

Cartan generators
$\Delta, J_1, J_2, Q_1, Q_2$

$J_\pm = J_1 \pm J_2$
$Q_\pm = Q_1 \pm Q_2$

Restrict to $SU(N)$ singlet sector (i.e. treat as a gauge symmetry)

N units of flux on S^4

K units of momentum on circle in X_7

$$R_{AdS} = 2R_{S^4} = 2(\pi N)^{1/3} l_p$$

Δ, J_1, J_2 isometries of X_7

Q_1, Q_2 isometries of S^4

Ultra-spinning black holes

Construction from known solution

- So, we just do BH$_{AdS_7}[E, J_1, J_2, J_3, Q_1, Q_2] \longrightarrow$ BH$_{X_7}[\Delta, J_1, J_2, Q_1, Q_2, K]$, right?
- Almost: Most general known AdS$_7$ solution has $Q_1 = Q_2 = Q$ [Chow]

$\longrightarrow$ X_7 **BH:** Energy Δ, angular mom. J_1, J_2, charges $Q = Q_1 = Q_2$, momentum K

- Admits a BPS limit: $\Delta = J_+ + 2Q_+$. **Supersymmetric** and **extremal**
  - BPS BH labelled by K, L, $J\_$. Remaining charge $F = -2Q$ fixed ← "Non-linear constraint"

  (L: $J_+ + Q_+$)

- Compute $\mathcal{S}_{BH}(K, L, J\_) = \dfrac{A}{4G}$; e.g. $\mathcal{S}_{BH}(K, L, 0) = 2\pi\sqrt{K}\sqrt{\dfrac{N^3}{12}}\left[\sqrt{1+\sqrt{1+\dfrac{6L^2}{KN^3}}} - \sqrt{2}\right]$

Ultra-spinning black holes

But what good is supergravity?

- To be a good approximation to M-theory, we need:
  - **Weak curvature** in Planck units $\longrightarrow N \gg 1$
  - The **circle in X_7** should become **large and spacelike** in the bulk [Dine et al.] $\longrightarrow K \gg N^{7/3}$ c.f. [Maldacena, Martelli, Tachikawa]
- **In summary: $K \gg N^{7/3} \gg 1$**
  - Charges scale like $\Delta, J_1, J_2, Q \sim \sqrt{KN^3}$. Also the entropy $\mathcal{S}_{BH} \sim \sqrt{KN^3}$

Part III: Counting black hole microstates

A quantitative test

What to compute

- QM must provide a **microscopic derivation of black holes' entropy**
- Let $d(K, \Delta, J_1, J_2, Q_1, Q_2)$ denote the **degeneracy of QM states** with these charges.
- In the **supergravity regime** $K \gg N^{7/3} \gg 1$, we must find

$$\log d(K, \Delta, J_1, J_2, Q_1, Q_2) \sim \mathcal{S}_{BH}(K, \Delta, J_1, J_2, Q_1, Q_2)$$

- Specialise to **BPS states** $\Delta = J_+ + 2Q_+$ with $Q_- = 0$, and swap (J_1, J_2, Q_1, Q_2) for

$$L \; (= J_+ + Q_+) \qquad J_- \qquad F \; (= -2Q_2)$$

Should find $\log d_{BPS}(K, L, J_-, F) \sim \mathcal{S}_{BH}^{BPS}(K, L, J_-, F)$

A quantitative test

Counting BPS states in the QM

- Compute a maximally-refined **superconformal index**
  - Counts BPS states, with alternating sign for B/F, graded by all charges that commute with $\mathbb{Q}, \mathbb{S}$

$$\mathcal{I}_K(t, x, y, w_a) = \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F t^L x^{J_-} y^{Q_-} \prod_a w_a^{n_a} \right]$$

Hilbert space on $\mathcal{M}_{K,N}$

- Rephrase superconformal symmetry **geometrically**. $\mathcal{I}_K$ is a **topological invariant** of $\mathcal{M}_{K,N}$: a particular equivariant Euler characteristic in sheaf cohomology [Barns-Graham, Dorey]

➡ Compute using localisation theorems in **equivariant K-theory**

A quantitative test

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$$= \sum_{||\lambda||=K} \prod_{i,j} \prod_{s\in Y(\lambda_i)} \mathrm{Pexp}\left(\frac{z_i}{z_j} t^{g_{ij}(s)} x^{f_{ij}(s)} [ty][t/y]\right)$$

A quantitative test

Counting BPS states in the QM

- Compute a maximally-refined **superconformal index**
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- Point is: **it is known**
- Coincides precisely with the **K-instanton contribution to the Nekrasov partition function** of a 5d $\mathcal{N} = 2^*$ theory
  - Agrees with proposal of [Kim, Kim, Koh, Lee, Lee '11]
- **Alternative approach**: Extract $\mathcal{I}_K$ from index of 6d $(2{,}0)$ theory [Dorey, Mouland '23]

Extracting the BPS degeneracy (a sketch)

Setting up the problem

Index of $\mathscr{M}_{K,N}$ theory

- Define the **total index** $\mathscr{I}(\beta; t, x, y) = \sum_{K=0}^{\infty} e^{-\beta K} \int d\mu_{Haar}[w] \mathscr{I}_K(t, x, y, w_a)$

Projects onto $SU(N)$ singlets

- $\mathscr{C}(K, L, J\_)$ is the coefficient of $e^{-\beta K} t^L x^{J_-}$ $\longrightarrow$ Extract by **contour integral**

$$\mathscr{C}(K, L, J\_) = \frac{1}{(2\pi i)^4} \oint \frac{dq}{q^{K+1}} \frac{dt}{t^{L+1}} \frac{dx}{x^{J_-+1}} \frac{dy}{y} \mathscr{I}(\beta; t, x, y), \qquad q = e^{-\beta}$$

$$= \sum_{F \in \mathbb{Z}} (-1)^F d_{BPS}(K, L, J\_, F)$$

Disaster!

- Want asymptotics as $K \to \infty$
- Hardy-Ramanujan $\longrightarrow$ Integral dominated region near **essential singularity** $\beta \to 0$

Extracting the BPS degeneracy (a sketch)

Exploiting S-duality

- Reinterpret $\mathscr{I}(\beta; t, x, y)$ in terms of inst. part of a **Nekrasov partition function** $\mathscr{Z}_{Nek}$
- $\mathscr{Z}_{Nek}$ understood as twisted Euclidean partition function of 6d $(2,0)$ theory on $T^2 \times \mathbb{R}^4$
    - $\beta = 2\pi i \tau$, with τ **the complex structure on** T^2
- **S-duality** maps $\tau \to -1/\tau$. Get $\beta \to 0$ behaviour from $\beta \to \infty$ behaviour!
- To do this properly, need the modular properties of $\mathscr{Z}_{Nek}$
    - Use conjectured equivalence $\mathscr{Z}_{Nek} = \mathscr{Z}_{Ell}$ to the index that counts bound states of the self-dual string in the $(2,0)$ theory ("M-strings") [Vafa et al.]
    - Manifest modular transformation $\longrightarrow$ $\mathscr{I}(\beta; t, x, y)$ **as** $\beta \to 0$ **accessible!**

Extracting the BPS degeneracy (a sketch)

Final result by saddle-point

Chemical potentials e.g. $t = e^{(\epsilon_1+\epsilon_2)/2}$

- Taking also $N \gg 1$, finally obtain $\log \mathcal{I}(\beta; t, x, y) \sim -\frac{N^3}{24}\frac{\Delta_1^2\Delta_2^2}{\epsilon_1\epsilon_2\beta}$

Plug in, and saddle-point →

$$\log|\mathscr{C}(K, L, J\_)| \sim \mathcal{S}_{BH}(K, L, J\_, F)$$

Leading growth of index coefficients exactly matches Bekenstein-Hawking entropy

- Recall, $\mathscr{C}(K, L, J\_) = \sum_F (-1)^F d_{BPS}(K, L, J\_, F)$
  - Our BH solution (GR arguments fix $F = F(K, L, J\_)$) **dominates the index**, like in all other known holographic microstate counting examples
  - If there are more BHs, they contribute subleadingly to $\mathscr{C}$ ($\mathcal{S}$ subleading? B/F pairs?)

Summary, directions, and a question

In summary

- Discussed the basic rules of **holography for non-relativistic conformal field theories**, and how go construct the corresponding **ultra-spinning black holes**
- Provided an explicit such dual pair, given by the **superconformal quantum mechanics** of Yang-Mills instantons
- Successfully recovered the **Bekenstein-Hawking entropy** of supersymmetric ultra-spinning black holes by **counting BPS states**, using the superconformal index of the QM
- As far as we know, this is the **first such match for a system with a finite number of degrees of freedom!**

Further topics

- SCQM offers new playground to probe quantum gravity: corrections to entropy, finite temperature, dynamical processes,…
- How can we **practically probe** things like this?
  - Have a precise index! More that can be done with it
  - Excitingly, the matrix model is **amenable to simulation** on classical and quantum computers $\longrightarrow$ numerically access dynamical (i.e. unprotected) observables [Rinaldi et al.] [Filev, O'Connor]

- **BPS black holes factorise:** BPS ultra-spinning black holes are in a particular sense the **"fundamental building blocks"** that make up general BPS AdS black holes

  Results in 4d and 6d [Dorey, Mouland, '23] - **Ask me about this!**

A question for you!

Does the bulk admit a regime of semiclassical gravity?

- Even for $K \gg N^{7/3} \gg 1$, only **black hole states** ($\Delta \sim \sqrt{KN^3}$) admit a description in 11-dim (Einstein) supergravity
  - e.g. the **QM vacuum** naively corresponds to empty $X_7 \times S^4$, but we can't trust the supergravity approximation to M-theory on this background!

Null compactification of AdS_7

- Want: A **semiclassical gravity** description for (some regime of) M-theory on $X_7 \times S^4$
- On generic DLCQ backgrounds we have, roughly:

DLCQ of ST ⟷ (**T-Duality**) ⟷ Non-relativistic ST

- Can we relate $X_7 \times S^4$ to a **stable vacuum** of strongly-coupled non-relativistic IIA string theory ("non-relativistic M-theory")?

Thanks!

Backup slides

Holomorphic factorisation and ultra-spinning black holes

[Dorey, Mouland '23]

- A quite general story: SCFT indices and partition functions **factorise** into "**blocks**" that are "**glued**" together. We focussed on 4d and 6d SC indices. Schematically,

$$\mathcal{I}_{4d} \sim \int d(\text{gauge})\, \mathcal{Z}_{hol}\mathcal{Z}_{hol}, \qquad \mathcal{I}_{6d} \sim \int d(\text{gauge})\, \mathcal{Z}_{Nek}\mathcal{Z}_{Nek}\mathcal{Z}_{Nek}$$

- Geometrically understood in SCFT: **gluing of disc partition functions**
- Can **derive** DLCQ index in a certain **limit** of the superconformal index (Technically the **lens space** index)

➡ Index of DLCQ quantum mechanics coincides with a **single holomorphic block**!

- **BPS ultra-spinning black holes** play the role of **holomorphic blocks** in the gravity dual

- Suggests they provide a concrete realisation of "**gravitational blocks**" [Zaffaroni et al.]