Nonrelativistic expansion of string theory

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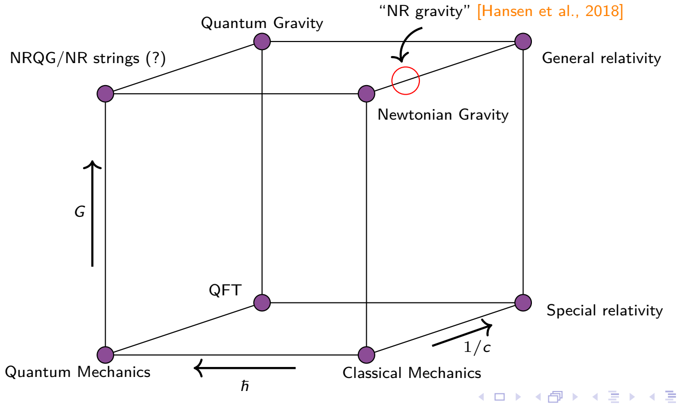
Non-Relativistic Strings and Beyond, Nordita 2023

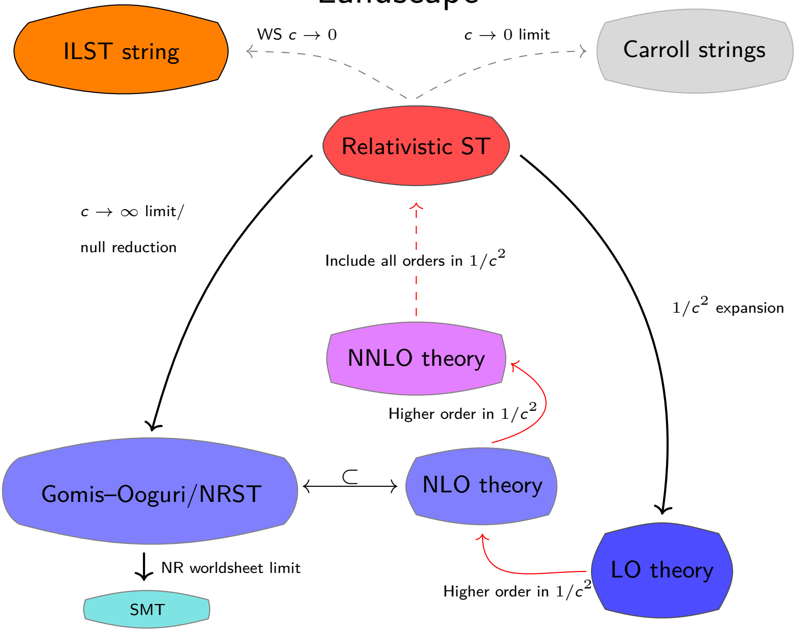
Nonrelativistic strings

- Nonrelativistic string theory has a long and illustrious history, starting with GO and Danielsson et al.
  [Gomis–Ooguri, '00; Danielsson et al., '00]

- This talk: NR string theory from "post-Newtonian" $1/c^2$ expansions (cf. Jørgen's talk)
  [Hansen et al., '18 & '20]



"Landscape"


ILST string
WS $c \to 0$
$c \to 0$ limit
Carroll strings
Relativistic ST
$c \to \infty$ limit/
null reduction
Include all orders in $1/c^2$
$1/c^2$ expansion
NNLO theory
Higher order in $1/c^2$
Gomis–Ooguri/NRST
$\subset$
NLO theory
LO theory
NR worldsheet limit
SMT
Higher order in $1/c^2$

The $_{\text{(string)}}$ $1/c^2$ expansion of string theory

An expansion requires a dimensionless parameter.

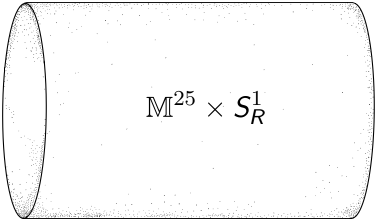
- This requires a compact direction in target space

  $\epsilon = \alpha' \hbar/(cR^2) = \dfrac{\alpha'_{\text{eff}}\hbar}{c^2 R^2_{\text{eff}}}$ with $cT = T_{\text{eff}}$, $R/c = R_{\text{eff}}$

  Hence:

$1/c^2$-expansion $\leftrightarrow$ expansion around decompactification limit

- Nonrelativistic interpretation: $v_{\text{com}} \sim \sqrt{\hbar\alpha'_{\text{eff}}}/R_{\text{eff}} \ll c$



Expansion of the spectrum and longitudinal T-duality

$$\boxed{M^2 = \frac{\hbar^2 n^2}{c^2 R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha' c}(N + \tilde{N} - 2\hbar)}$$

- This leads to $E = \sqrt{M^2 c^4 + p^2 c^2} = c^2 E_{\mathrm{LO}} + E_{\mathrm{NLO}} + \cdots$, with

$$E = \overbrace{\frac{c^2 w R_{\mathrm{eff}}}{\alpha'_{\mathrm{eff}}}}^{\sim mc^2} + \frac{N_{(0)} + \tilde{N}_{(0)}}{w R_{\mathrm{eff}}} + \frac{\alpha'_{\mathrm{eff}}}{2 w R_{\mathrm{eff}}} p_{(0)}^2 + \mathcal{O}(c^{-2})$$

- Spectrum can be written as

$$M^2 = \frac{n^2 \tilde{R}^2}{\alpha'^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha' c}(N + \tilde{N} - 2\hbar)$$

- T-duality amounts to: $R \leftrightarrow \dfrac{\hbar \alpha'}{c R} =: \tilde{R}$ and $w \leftrightarrow n$

  $\Rightarrow$ T-dual parameter $\tilde{\epsilon} = \dfrac{\alpha' \hbar}{c \tilde{R}^2}$.

- T-duality switches between expansions in ϵ and $\tilde{\epsilon}$: $\boxed{\epsilon \leftrightarrow \tilde{\epsilon}}$

Nonrelativistic expansion of string actions (a taste of...)

- The Polyakov Lagrangian on flat target space is

$$\mathcal{L}_{\mathsf{P}} = -\frac{cT}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \eta_{MN}$$

- Here, $\eta_{MN} = \mathrm{diag}(\overbrace{-c^2, c^2}^{A}, \overbrace{1, \ldots}^{i})$ and (cf. Jelle's talk)

$$X = x + c^{-2}y + \cdots\,, \qquad \overbrace{\gamma_{\alpha\beta} = \gamma_{(0)\alpha\beta} + c^{-2}\gamma_{(2)\alpha\beta} + \cdots}^{\text{gauge fix } \gamma_{\alpha\beta}=\eta_{\alpha\beta}}$$

- Splitting the index as $M = (A, i)$ gives the LO and NLO Lagrangians ($\mathcal{L}_{\mathsf{P}} = c^2\mathcal{L}_{\mathsf{P\text{-}LO}} + \mathcal{L}_{\mathsf{P\text{-}NLO}} + \cdots$)

Equivalent to GO w/o instanton term with $\partial_\pm y^\mp \sim \lambda_\pm$

$$\mathcal{L}_{\mathsf{P\text{-}LO}} = -\frac{T_{\mathsf{eff}}}{2}\partial_\alpha x^A \partial^\alpha x_A\,, \qquad \mathcal{L}_{\mathsf{P\text{-}NLO}} = -\frac{T_{\mathsf{eff}}}{2}\left[\partial_\alpha x^i \partial^\alpha x^i + 2\partial_\alpha y^A \partial^\alpha x_A\right]$$

The plan

1. String $1/c^2$ expansion of closed bosonic string theory
2. Phase space formulation
3. Open strings

String Newton–Cartan geometry from large-c

Geometry	Field content
String $1/c^2$ expansion of Lorentzian geometry	LO: $\tau_M{}^A$, $H^\perp_{MN}$ up to NLO: $\tau_M{}^A$, $H^\perp_{MN}$, $m_M{}^A$, $\phi^\perp_{MN}$ up to NNLO: $\tau_M{}^A$, $H^\perp_{MN}$, $m_M{}^A$, $\phi^\perp_{MN}$, $B_M{}^A$ etc.
Type II SNC	$\tau_M{}^A$, $H^\perp_{MN}$, $m_M{}^A$, $\phi^\perp_{MN}$ (LO and NLO fields from above)
Type I SNC	$\tau_M{}^A$, $H^\perp_{MN}$, $m_M{}^A$

- Decompose metric and its inverse as [Hansen et al., '18; Hansen et al. '20]

$$G_{MN} = c^2 \eta_{AB} T_M{}^A T_N{}^B + \Pi^\perp_{MN}, \qquad G^{MN} = c^{-2} \eta^{AB} T^M{}_A T^N{}_B + \Pi^{\perp MN}$$

where $A, B = 0, 1$ and $T_M{}^A \Pi^{\perp MN} = T^M{}_A \Pi^\perp_{MN} = 0$

- Expand as

$$T_M{}^A = \tau_M{}^A + c^{-2} m_M{}^A + \mathcal{O}(c^{-4}) , \qquad \Pi^\perp_{MN} = H^\perp_{MN} + \mathcal{O}(c^{-2})$$

to get [Andringa et al., '12; Bergshoeff et al. '18]

$$G_{MN} = c^2 \tau_{MN} + H_{MN} + \mathcal{O}(c^{-2})$$

with

$$\tau_{MN} = \eta_{AB} \tau_M{}^A \tau_N{}^B , \qquad H_{MN} = H^\perp_{MN} + 2\eta_{AB} \tau_{(M}{}^A m_{N)}{}^B$$

Codimension-2 foliations and the "strong foliation constraint"

- Historically the "SFC" played an important role in NRST

[Bergshoeff et al., 2018]

$$d\tau^A = \omega^A{}_B \wedge \tau^B$$

The $1/c^2$ expansion comes equipped with its own foliation constraint:

- The beta function of ST at LO in α' is Einstein's Eq. $R_{MN} = 0$, which at LO in $1/c^2$ becomes the Frobenius condition

$$H^{\perp QS} H^{\perp RT} \left(d\tau^A\right)_{QR} \left(d\tau^B\right)_{ST} = 0 \iff \boxed{d\tau^A = \alpha^A{}_B \wedge \tau^B}$$

- Reduces to SFC when $\alpha^A{}_B = \omega^A{}_B$

Expansion generalities

- String Lagrangian schematically of the form $\mathcal{L}[X;c]$ and expands in $1/c^2$ as [Hansen et al., '19]

$$\mathcal{L}[X;c] = c^2 \overset{(-2)}{\mathcal{L}}(X) + \overset{(0)}{\mathcal{L}}(X) + \mathcal{O}(c^{-2})$$

- A further (functional) Taylor expansion in $X = x + c^{-2} y + \cdots$ leads to

$$\begin{aligned} \mathcal{L}[X;c] &= c^2 \overset{(-2)}{\mathcal{L}}(x) + \left[\overset{(0)}{\mathcal{L}}(x) + y^M \frac{\delta \overset{(-2)}{\mathcal{L}}(x)}{\delta x^M} \right] + \mathcal{O}(c^{-2}) \\ &= c^2 \mathcal{L}_{\mathsf{LO}}(x) + \mathcal{L}_{\mathsf{NLO}}(x,y) + \mathcal{O}(c^{-2}) \end{aligned}$$

$\Rightarrow$ subleading fields "remember" lower-order e.o.m.s

Expansion of the Polyakov Lagrangian

- The Polyakov Lagrangian is

$$\mathcal{L}_{\mathrm{P}} = -\frac{cT}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}$$

EXPANSION RECIPE: [Hansen et al., '19; Harmark et al. '19; Hansen et al., '20]

- Mantra: expand everything

$$X^M = x^M + c^{-2}y^M + \mathcal{O}(c^{-4})$$

$$\gamma_{\alpha\beta} = \gamma_{(0)\alpha\beta} + c^{-2}\gamma_{(2)\alpha\beta} + \mathcal{O}(c^{-4})$$

$$G_{\alpha\beta}(X) = \partial_\alpha X^M \partial_\beta X^N G_{MN}(X) = c^2\tau_{\alpha\beta}(x) + H_{\alpha\beta}(x,y) + \mathcal{O}(c^{-2})$$

where

$$\tau_{\alpha\beta}(x) = \partial_\alpha x^M \partial_\beta x^N \tau_{MN}(x)$$

$$H_{\alpha\beta}(x,y) = \partial_\alpha x^M \partial_\beta x^N H_{MN}(x) + 2\partial_{(\alpha} x^M \partial_{\beta)} y^N \tau_{MN}(x) + \partial_\alpha x^M \partial_\beta x^N y^L \partial_L \tau_{MN}(x)$$

Expanding the Polyakov Lagrangian to NLO

- This procedure leads to an expansion of the form

$$\mathcal{L}_{\mathsf{P}} = c^2 \mathcal{L}_{\text{P-LO}} + \mathcal{L}_{\text{P-NLO}} + \mathcal{O}(c^{-2})$$

where

$$\mathcal{L}_{\text{P-LO}} = -\frac{T_{\text{eff}}}{2}\sqrt{-\gamma_{(0)}}\gamma_{(0)}^{\alpha\beta}\tau_{\alpha\beta}$$

$$\mathcal{L}_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\sqrt{-\gamma_{(0)}}\gamma_{(0)}^{\alpha\beta}H_{\alpha\beta} + \frac{T_{\text{eff}}}{4}\sqrt{-\gamma_{(0)}}G_{(0)}^{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\gamma_{(2)\gamma\delta} + y^M\frac{\delta\mathcal{L}_{\text{P-LO}}}{\delta x^M}$$

with $G_{(0)}^{\alpha\beta\gamma\delta} = \gamma_{(0)}^{\alpha\gamma}\gamma_{(0)}^{\delta\beta} + \gamma_{(0)}^{\alpha\delta}\gamma_{(0)}^{\gamma\beta} - \gamma_{(0)}^{\alpha\beta}\gamma_{(0)}^{\gamma\delta}$ the WDW metric

- The LO e.o.m. is

$$0 = \varepsilon^{\alpha\beta}\partial_\alpha x^K \partial_\beta x^L \varepsilon_{AB}(d(\tau^A \wedge \tau^B))_{MKL} + \text{“Virasoro”}$$

- If the Frobenius condition is satisfied, a sufficient* condition for the above to vanish *identically* is if $\alpha_M{}^A{}_B$ is *traceless*, of which the SFC is a special case

SNC string $\subset$ NLO string

$$\boxed{\mathcal{L}_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\sqrt{-\gamma_{(0)}}\gamma_{(0)}^{\alpha\beta}H_{\alpha\beta} + \frac{T_{\text{eff}}}{4}\sqrt{-\gamma_{(0)}}G_{(0)}^{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\gamma_{(2)\gamma\delta} + y^M\frac{\delta\mathcal{L}_{\text{P-LO}}}{\delta x^M}}$$

- Decompose

$$\gamma_{(0)\alpha\beta} = \eta_{ab}e_\alpha{}^a e_\beta{}^b\ , \qquad \gamma_{(2)\alpha\beta} = 2e_{(\alpha}{}^a e_{\beta)}{}^b A_{ab}$$

where a, b are tangent space WS indices

- This results in the Lagrangian

$$\mathcal{L}_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\varepsilon^{\alpha\beta}\left[\lambda_{++}e_\alpha{}^+\tau_\beta{}^+ + \lambda_{--}e_\alpha{}^-\tau_\beta{}^-\right] - \frac{T_{\text{eff}}}{2}\sqrt{-\gamma_{(0)}}\gamma_{(0)}^{\alpha\beta}H_{\alpha\beta} + y^M\frac{\delta\mathcal{L}_{\text{NG-LO}}}{\delta x^M}$$

where $\tau_\beta{}^\pm = \tau_\beta{}^0 \pm \tau_\beta{}^1$ and

$$\lambda_{\pm\pm} = 4e^\alpha{}_\mp\left(\mp\tau_\alpha{}^0 + \tau_\alpha{}^1\right)A_{\pm\pm} \mp 2e^\gamma{}_\pm(\tau_N{}^\mp\partial_\gamma y^N + \partial_\gamma x^M y^N \partial_N \tau_M{}^\mp)$$

- For $\alpha^A{}_A = 0$ this reproduces the SNC string

[H_2O, '17; Bergshoeff et al., '18; Harmark et al. '19]

Phase space formulation and Poisson brackets

- The relativistic phase Lagrangian is

$$\mathcal{H}_\pm = \frac{1}{4cT}(P \pm cTX')^M (P \pm cTX')^N \eta_{MN}$$

$$L = \oint d\sigma^1 \big[\dot{X}^M P_M - \overbrace{\vartheta^- \mathcal{H}_- - \vartheta^+ \mathcal{H}_+}\big]$$

- Momentum expands as $P_A = c^2 P_{(-2)A} + P_{(0)A} + c^{-2} P_{(2)A} + \cdots$ and $P_i = P_{(0)i} + c^{-2} P_{(2)i} + \cdots$
- This gives

$$L_{\mathrm{LO}} = \oint d\sigma^1 \left[\dot{x}^A P_{(-2)A} + \text{“LO constraints”}\right] ,$$

$$L_{\mathrm{NLO}} = \oint d\sigma^1 \left[\dot{x}^i P_{(0)i} + \dot{x}^A P_{(0)A} + \dot{y}^A P_{(-2)A} + \text{“LO \& NLO constraints”}\right] ,$$

$$L_{\mathrm{NNLO}} = \oint d\sigma^1 \big[\dot{x}^A P_{(2)A} + \dot{y}^A P_{(0)A} + \dot{z}^A P_{(-2)A} + \dot{x}^i P_{(2)i} + \dot{y}^i P_{(0)i} + \cdots\big]$$

- Hence: **Poisson brackets change at each order**

The normal ordering constant

- Gauge-fixed Poisson brackets are

$$\{\alpha^i_k, \alpha^j_{-k}\} = -ik\delta^{ij} \text{ at NLO}, \qquad \{\alpha^i_k, \beta^j_{-k}\} = -\frac{ik}{2}\delta^{ij} \text{ at NNLO}$$

where $x^i \sim \sum(i/k)\alpha^i_k e^{-ik\sigma^-} + \cdots$ and $y^i \sim \sum(i/k)\beta^i_k e^{-ik\sigma^-} + \cdots$

- Standard approach: $[\hat{q}, \hat{p}] = i\hbar\{q, p\}$ $(*)$
- However, subleading Lagrangians have non-standard dimensions; $\mathcal{L} = c^2\mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} + c^{-2}\mathcal{L}_{\text{NNLO}} + \cdots$

$\rightsquigarrow$ Replace $(*)$ with

$$[\hat{a}, \hat{b}] = ik_{[a][b]}\{a, b\}$$

where $k_{[a][b]}$ is some combination of fundamental constants with dimensions of $[a][b]$

The normal ordering constant (cont'd.)

@NLO:

- NLO vacuum $\alpha^i_{k>0}|0\rangle_{\mathrm{NLO}} = 0$, with $[\alpha^i_k] = \mathrm{length}\times\sqrt{\mathrm{mass/time}}$
- Commutator is $[\alpha^i_k, \alpha^j_{-k}] = \hbar k \delta^{ij}$, # op. is $N_{(0)} = \frac{1}{2}\sum_{k\neq 0}\alpha^i_{-k}\alpha^i_k$
- Normal ordering gives

$$\frac{1}{2}\sum_{k\neq 0}\alpha^i_{-k}\alpha^i_k = \sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^i_k + \frac{\hbar d}{2}\sum_{k=1}^{\infty} k = \sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^i_k - \frac{\hbar d}{24}$$

@NNLO:

- NNLO vacuum $\alpha^i_k|0\rangle_{\mathrm{NNLO}} = \beta^i_k|0\rangle_{\mathrm{NNLO}} = 0$ for all $k > 0$, with $[\beta] = \sqrt{\mathrm{mass}}\times\mathrm{length}\times(\mathrm{time})^{-5/2}$
- Commutator is now $[\alpha^i_k, \beta^j_{-k}] = c^2\frac{\hbar k}{2}\delta^{ij}$, subleading # op. is $N_{(2)} = \sum_{k=1}^{\infty}\alpha^i_{-k}\beta^i_k + \sum_{k=1}^{\infty}\alpha^i_k\beta^i_{-k}$ (with $N = N_{(0)} + c^{-2}N_{(2)} + \cdots$)
- Reproduces the same normal ordering constant:

$$\sum_{k=1}^{\infty}\alpha^i_k\beta^i_{-k} = \sum_{k=1}^{\infty}\beta^i_{-k}\alpha^i_k + c^2\frac{\hbar d}{2}\sum_{k=1}^{\infty} k = \sum_{k=1}^{\infty}\beta^i_{-k}\alpha^i_k - c^2\frac{\hbar d}{24}\,.$$

The $1/c^2$ expansion of open strings

- v-direction a DD-direction
  $\rightsquigarrow$ theory defined on a D24-brane [Gomis et. al., '20]
- NLO action in flat TS given by

$$S_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\int d^2\sigma \left[\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i + y^A \frac{\delta S_{\text{P-LO}}}{\delta x^A}\right] - 2\int d\sigma^0 \left[y^t \partial_0 x_t\right]_{\sigma^1=0}^{\sigma^1=\pi}$$

- BCs: $\eta_{AB} x'^A \delta y^B\big|_{\text{ends}} = \eta_{AB} y'^A \delta x^B\big|_{\text{ends}} = \eta_{ij} x'^i \delta x^j\big|_{\text{ends}} = 0$
  $\Rightarrow$ x^A and y^A satisfy *same* BCs

$\rightsquigarrow$ Revisit T-duality, which in the longitudinal sector is related the DLCQ of string theory and NCOS

[Gomis et. al., '20]

$\rightsquigarrow$ String $1/c^2$ expansion of the DBI action and D-branes

Summary and Future Directions

What we have achieved

- Framework for expanding string theory to any desired order in $1/c^2$
- Does not need near-critical Kalb–Ramond background
- Reproduces NRST at NLO as well as the GO spectrum (when taking into account WZ term)

What lies ahead

- beta functions for NRST & $1/c^2$ expansion of NS-NS gravity: the missing Poisson equation? (cf. Jan's talk)
- Open strings and D-branes
- Explore the "landscape" of non-Lorentzian string theories and their possible holographic dualities

Thank you for your attention

Extra slides: Spectrum on flat target space

Fixing the gauge

- Flat string NC geometry: ${\tau_M}^0 = \delta^t_M$, ${\tau_M}^1 = \delta^v_M$, ${m_M}^A = 0$, $H_{MN} = \delta^i_M \delta^i_N$ with $i = 2, \ldots, 26$
- Expand rel. WS gauge redundancy

$$\Xi^\alpha = \xi^\alpha_{(0)} + c^{-2}\xi^\alpha_{(2)} + \mathcal{O}(c^{-4}) \;, \qquad \omega = \omega_{(0)} + c^{-2}\omega_{(2)} + \mathcal{O}(c^{-4})$$

- Fix LO and NLO redundancies partially by setting

$$\gamma_{(0)\alpha\beta} = \eta_{\alpha\beta} \;, \qquad \gamma_{(2)\alpha\beta} = 0$$

- This leaves residual redundancy (true at all orders)

$$\xi_{(0,2)}(\sigma) = \xi^-_{(0,2)}(\sigma^-)\partial_- + \xi^+_{(0,2)}(\sigma^+)\partial_+$$

Mode expansions and spectrum: LO

- The gauge-fixed P-LO Lagrangian is

$$\mathcal{L}_{\text{P-LO}} = \frac{T_{\text{eff}}}{2}\eta^{\alpha\beta}\partial_\alpha x^t \partial_\beta x^t - \frac{T_{\text{eff}}}{2}\eta^{\alpha\beta}\partial_\alpha x^v \partial_\beta x^v$$

with Virasoro constraints $\partial_\mp x^\pm = 0$, leading to

$$x^\pm = x_0^\pm + wR_{eff}\sigma^\pm + \overbrace{\sigma^\pm\text{-oscillations}}^{=0 \text{ by fixing res. red.}}$$

- The energy at LO is the 'stringy' rest mass

$$E_{\text{LO}} = -\oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-LO}}}{\partial(\partial_0 x^t)} = \frac{wR_{\text{eff}}}{\alpha'_{\text{eff}}}$$

Mode expansions and spectrum: NLO

- The gauge-fixed P-NLO Lagrangian on flat space is

$$\mathcal{L}_{\text{P-NLO}} = -\frac{T_{\text{eff}}}{2}\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i + T_{\text{eff}}\eta^{\alpha\beta}\partial_\alpha y^t \partial_\beta x^t - T_{\text{eff}}\eta^{\alpha\beta}\partial_\alpha y^v \partial_\beta x^v$$

with Virasoro constraints $\partial_\mp y^\pm = \frac{1}{wR_{\text{eff}}}\partial_\mp x^i \partial_\mp x^i$ and $\partial_\mp x^\pm = 0$

- e.o.m.: $\partial_+\partial_- y^t = \partial_+\partial_- y^v = \partial_+\partial_- x^i = 0$ leads to mode expansions

$$x^i = x_0^i + \frac{1}{2\pi T_{\text{eff}}} p_{(0)i}\sigma^0 + \frac{1}{\sqrt{4\pi T_{\text{eff}}}} \sum_{k\neq 0} \frac{i}{k}\left[\alpha_k^i e^{-ik\sigma^-} + \tilde{\alpha}_k^i e^{-ik\sigma^+}\right]$$

$$y^\pm = y_0^\pm - \frac{1}{2\pi T_{\text{eff}}} p_{(0)\mp}(\sigma^+ + \sigma^-) + \overbrace{\text{oscillations}}^{\text{remove osc. from } \partial_\pm y^\pm \text{ by fixing NLO red.}}$$

- This leads to $N_{(0)} - \tilde{N}_{(0)} = \hbar n w$ and

$$E_{\text{NLO}} = -\oint d\sigma^1 \frac{\partial \mathcal{L}_{\text{P-NLO}}}{\partial \partial_0 x^t} = -p_{(0)t} = \frac{N_{(0)} + \tilde{N}_{(0)}}{wR_{\text{eff}}} + \frac{\alpha'_{\text{eff}}}{2wR_{\text{eff}}} p_{(0)}^2$$

...and combining everything

... we get our previous result:

$$
\begin{aligned}
E &= c^2 E_{\text{LO}} + E_{\text{NLO}} + \mathcal{O}(c^{-2}) \\
&= \frac{c^2 w R_{\text{eff}}}{\alpha'_{\text{eff}}} + \frac{N_{(0)} + \tilde{N}_{(0)}}{w R_{\text{eff}}} + \frac{\alpha'_{\text{eff}}}{2 w R_{\text{eff}}} p_{(0)}^2 + \mathcal{O}(c^{-2})
\end{aligned}
$$