Integrability in non-relativistic string theory

Andrea Fontanella (Perimeter Institute)

Non-Relativistic Strings and Beyond,
Workshop at Nordita, 8-11 May 2023

Based on:

Light-Cone Gauge in Non-Relativistic $AdS_5 \times S^5$ String Theory
with J.M. Nieto and A. Torrielli arXiv:2102.00008,

Classical string solutions in non-relativistic $AdS_5 \times S^5$: closed and twisted sectors
with J.M. Nieto arXiv:2109.13240,

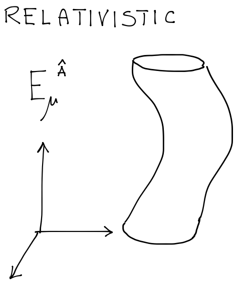
Coset space actions for nonrelativistic strings
with S. van Tongeren arXiv:2203.07386,

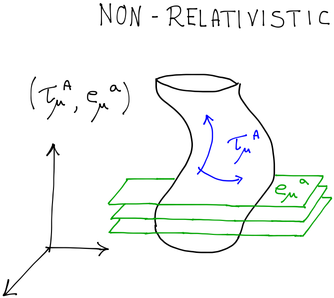
Extending the non-relativistic string AdS coset
with J.M. Nieto arXiv:2208.02295,

Non-relativistic string monodromies
with J.M. Nieto and O. Ohlsson Sax arXiv:2211.04479

Non-AdS holography

- We are motivated to study non-AdS holography
- Strings with: 1) NR target space, 2) relativistic world-sheet
- β-function vanishes [Gomis, J. Oh, Z. Yan, 2019][Gallegos, Gursoy, Zinnato, 2019]





Foliation "2+8" $\hat{A} = (A, a)$

$A = 0, 1 \qquad a = 2, ..., 9$

NR string theory

Strings in Flat space

- Relativistic action describes free-fields (conformal gauge)
- Spectrum is easy to solve. Not sensible to choice of vacuum.
- holography?

NR limit [Gomis, Ooguri, 2000][Danielsson, Guijosa, Kruczenski, 2000]

$$\tau_\mu{}^A \ : \ \mathbb{R}^{1,1} \qquad\qquad e_\mu{}^a \ : \ \mathbb{R}^8$$

- NR action describes again free fields (static gauge, eliminating λ_A)
- Integrable (obvious way)
- NR spectrum derived by taking a limit of the relativistic one

Strings in $\mathrm{AdS}_5 \times S^5$

- Relativistic action highly non-trivial, but integrable!
- Spectrum was found by using integrability techniques.

  1) S-matrix $\rightarrow$ Bethe ansatz $\rightarrow$ spectrum

  2) Lax pair + string solution $\rightarrow$ spectral curve $\rightarrow$ spectrum

  Depends on choice of vacuum (BMN, GKP, ...).
- Dual field theory: $\mathcal{N} = 4$ SYM, checked "$E(\sqrt{\lambda}, m) = \Delta(\lambda, m)$"

NR limit: GGK theory [Gomis, Gomis, Kamimura, 2005]

$$\tau_\mu{}^A \ : \ \mathrm{AdS}_2 \qquad\qquad e_\mu{}^a \ : \ f(z)\mathbb{R}^3 \times \mathbb{R}^5$$

- NR action describes 3 massive + 5 massless free fields in AdS_2 (static gauge, eliminating λ_A)
- A priori, not obviously integrable
- NR spectrum still unknown

Questions

- Is GGK theory integrable?
- What is its energy spectrum?

Outline

1. Vacuums of the GGK theory (NR analogue of BMN, GKP)
2. Semiclassical expansion of GGK action
3. Coset formulation of GGK action and Lax pair
4. Spectral curve

GGK theory

Start from (bosonic) relativistic strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Polyakov action

$$S = \int \mathrm{d}^2\sigma \, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$$

requires choice of coordinates X^μ.

But because

$$\mathrm{AdS}_5 \times S^5 = \frac{SO(2,4) \times SO(6)}{SO(1,4) \times SO(5)} \equiv \frac{G}{H}$$

the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric is captured by a Maurer-Cartan 1-form

Polyakov action can be written in "coordinate free" language

[Metsaev, Tseytlin, 1998]

$$S = \int \mathrm{d}^2\sigma \, \gamma^{\alpha\beta} \langle PA, PA \rangle \,, \qquad A = g^{-1}\mathrm{d}g \,, \qquad g \in G$$

$P : \mathfrak{g} \to \mathfrak{g} \setminus \mathfrak{h}$ (projector) $\quad \mathfrak{g} = \{P_a, J_{ab}\} \quad \mathfrak{h} = \{J_{ab}\} \quad \mathfrak{g} \setminus \mathfrak{h} = \{P_a\}$

$$A_\mu = e_\mu{}^a P_a + \omega_\mu{}^{ab} J_{ab} \qquad \langle P_a, P_b \rangle = \eta_{ab}$$

Choice of g (e.g. $g = e^{\xi^a P_a}$) $\implies$ Polyakov in coords $X^\mu = \{\xi^a\}$

NR limit in GGK theory: take the Inonu-Wigner contraction

$$\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \longrightarrow \text{string Newton-Hooke}_5 \oplus \text{Eucl}_5$$

translations = "2 + 8" $\quad P_0 \to \frac{P_0}{c},\ P_1 \to \frac{P_1}{c}$ $\quad$ (+ boost)

rescaling of generators $\longleftrightarrow$ rescaling of coords

- Coordinate rescaling $\implies$ ${E_\mu}^{\hat{A}}$ expansion

  longitudinal $\quad {E_\mu}^A = c{\tau_\mu}^A + \frac{1}{c}{m_\mu}^A + \mathcal{O}(c^{-3}) \quad A = 0, 1$

  transverse $\quad {E_\mu}^a = {e_\mu}^a + \mathcal{O}(c^{-2}) \quad a = 2, ..., 9$

$$g_{\mu\nu} = c^2 {\tau_\mu}^A {\tau_\nu}^B \eta_{AB} + H_{\mu\nu} + \mathcal{O}(c^{-2})$$

- couple closed critical B-field $\quad B_{\mu\nu} = c^2 \varepsilon_{AB} {\tau_\mu}^A {\tau_\nu}^B$
- Trick: $\quad c^2$ part of $(g_{\mu\nu} + B_{\mu\nu}) = \lambda_A \mathcal{F}^A + \frac{1}{c^2}\lambda_A \lambda^A$
- Take $c \to \infty$

$$S^{\rm NR} = \int \mathrm{d}^2\sigma \left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu H_{\mu\nu} + \varepsilon^{\alpha\beta}(\lambda_+ \theta_\alpha{}^+ \tau_\mu{}^+ + \lambda_- \theta_\alpha{}^- \tau_\mu{}^-)\partial_\beta X^\mu \right)$$

- world-sheet Zweibein: $\theta_\alpha{}^\pm \equiv \theta_\alpha{}^0 \pm \theta_\alpha{}^1$ $\quad [\, h_{\alpha\beta} = -\theta_{(\alpha}{}^+\theta_{\beta)}{}^-]$
- $H_{\mu\nu} \equiv 2\tau_{(\mu}{}^A m_{\nu)}{}^B \eta_{AB} + e_\mu{}^a e_\nu{}^b \delta_{ab}$ $\quad \tau_\alpha{}^\pm \equiv \tau_\alpha{}^0 \pm \tau_\alpha{}^1$
- structure "$(\partial X)^2 + \partial X$"

string Newton-Cartan $\mathrm{AdS}_5 \times S^5$ data:

$\tau_\mu{}^A \quad \longrightarrow \quad \mathrm{AdS}_2\ (t, z)$

$e_\mu{}^a \quad \longrightarrow \quad f(z)\ \mathbb{R}^3 \times \mathbb{R}^5$

$m_\mu{}^A \quad \longrightarrow \quad$ coordinate dependent vielbein

In Cartesian and GGK coordinates: $m_\mu{}^A \neq 0$
In polar coordinates: $m_\mu{}^A = 0$

$H_{\mu\nu}$ does not transform covariantly under diffeos (rank of $H_{\mu\nu}$ changes)

Solving the equations of motion

- Fix conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$
- Solve e.o.m. for the fields (X^μ, λ_A) + Virasoro constraints
- Impose closed boundary conditions to cancel surface term.

Instrumental to start with: $\mathcal{E}^{\lambda_\pm} \equiv \varepsilon^{\alpha\beta}\theta_\alpha{}^\pm \tau_\mu{}^\pm \partial_\beta X^\mu = 0$

Equations "living" in AdS_2. Find $T = T(t),\ Z = Z(z)$, so they become

$$\dot{T} + Z' = 0 \qquad T' + \dot{Z} = 0$$

Can be solved in general, $T = f_+(\sigma_+) + f_-(\sigma_-) \quad Z = f_+(\sigma_+) - f_-(\sigma_-) + \text{const.}$

Fixing $f_\pm$ (residual $\mathrm{Diff}_+ \oplus \mathrm{Diff}_-$)

$$T = \kappa\tau \qquad Z = \kappa\sigma$$

- Cartesian $\quad t = \kappa\tau \quad z = -2\tan(\kappa\sigma/2) \quad \kappa \in \mathbb{Z}$
- Polar $\quad t = \kappa\tau \quad \rho = \mathrm{gd}^{-1}(\kappa\sigma) \quad \kappa \in \mathbb{Z}$
- GGK $\quad x^0 = \mathrm{gd}(\kappa\tau) \quad x^1 = \kappa\sigma \quad$ compactified x^1

NR string must have **winding**!

Two simplest vacua admitted by GGK theory: [AF, Nieto 2021]

- *Static string* (zero Energy)

$$t = \kappa\tau \qquad z = -2\tan(\kappa\sigma/2) \qquad \text{others} = 0$$

- *BMN-like* (Energy E, linear momentum J)

$$t = \kappa\tau \qquad z = -2\tan(\kappa\sigma/2) \qquad \phi = \nu\tau \qquad \lambda_\pm = \pm\frac{\nu^2}{\kappa}\cos(\kappa\sigma)$$

Dispersion relation $E \sim J^2$

Found also a generalisation of the GKP (rotating in transverse AdS)

How does the action expands about the BMN-like vacuum?

It is not a simple vacuum... [AF, Nieto, Torrielli 2021]

Relativistic BMN: $t = \phi = \kappa\tau$

Fix light-cone gauge $X_{\pm} = t \pm \phi$ and expand $X = X_{\rm cl} + \frac{1}{\sqrt{T}}\delta X \quad (T \gg 1)$

Relativistic:

$$S^{\rm rel} \sim \text{free } (\delta X)^2 + \frac{1}{T}(\delta X)^4 + \ldots$$

GGK action:

Also need $R \gg 1$, then

$$S^{NR} \sim \text{free } (\delta X)^2 + \frac{1}{\sqrt{T}} f(\tfrac{\sigma}{R})(\delta X)^3 + \ldots$$

Reason: 1) z is not isometry $z = z_{\rm cl} + z_{\rm fl}$ 2) z is not in the lightcone $X_{\pm}$

T-duality

Adapt GGK coords to $\mathrm{AdS}_5 \times S^5$ where

x^1 isometry $\qquad$ x^0 not isometry

(x^0, x^1) span AdS_2

T-duality along (the compactified) x^1, apply Busher's rules.

T-dual $g_{\mu\nu}$: time dependent pp-wave $\times\ \mathbb{R}^5$

$$\mathrm{d}s^2 = 2\mathrm{d}x^0\mathrm{d}\tilde{x}^1 - \frac{2x^a x^a}{\cosh^2 x^0}\mathrm{d}x^0\mathrm{d}x^0 + \mathrm{d}x^a\mathrm{d}x^a + \mathrm{d}x^m\mathrm{d}x^m$$

+dilaton and RR flux (sol. of Type IIA sugra) [Gomis, Gomis, Kamimura, 2005]

Quantization NR strings in $AdS_5 \times S^5$ is T-dual to discrete light-cone quantisation (DLCQ) of time-dependent pp-wave.

Issue: x^0 is not isometry of T-dual $g_{\mu\nu}$.

$\rightarrow$ Define light-cone coords $X^{\pm}$ useful for quantization?

Coset construction of GGK action

A step back: Relativistic

$\mathrm{AdS}_5 \times S^5 = G/H$, Polyakov action in coordinate free language

$$S = \int \mathrm{d}^2\sigma \gamma^{\alpha\beta} \langle A_\alpha^{(1)}, A_\beta^{(1)} \rangle$$

$\mathbb{Z}_2$ automorphism $\quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \qquad \mathfrak{g}^{(0)} \equiv \mathfrak{h} \qquad \mathfrak{g}^{(1)} \equiv \mathfrak{p}$

- E.o.m. $\quad \partial_\alpha(\gamma^{\alpha\beta} A_\beta^{(1)}) + \gamma^{\alpha\beta}[A_\alpha^{(0)}, A_\beta^{(1)}] = 0$
- Lax pair [Bena, Polchinski, Roiban, 2003]

  $$\mathscr{L} = A^{(0)} + \ell_1 A^{(1)} - \ell_2 \star A^{(1)} \qquad \ell_1^2 - \ell_2^2 = 1 \quad \to \quad \xi$$

  ξ is the spectral parameter
- Equivalence: $\quad \mathrm{d}\mathscr{L} + \mathscr{L} \wedge \mathscr{L} = 0 \quad \Longleftrightarrow \quad$ E.o.m.

From Lax pair one can construct infinitely many conserved charges (see later...)

GGK theory

Question:

$$\text{string Newton-Cartan } \mathrm{AdS}_5 \times S^5 = \frac{\text{Isometry}}{\text{Isotropy}} \qquad ?$$

Strictly speaking:

- Isometry is ∞-dim [Bagchi, Gopakumar, 2009]
- Isotropy is "String Newton-Cartan" algebra (finite-dim)
- but we need: dim(Isometry) - dim(Isotropy) = 10

We need to find an alternative route...

$$\text{SNC } \mathrm{AdS}_5 \times S^5 = \frac{\text{Lie Algebra Expansion}[\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)]}{\text{String Bargmann} \oplus \{Z_{ab}, Z_{a'b'}\}}$$

$$\begin{aligned}
J_{AB} &\to \varepsilon_{AB}(M + \epsilon^2 Z) & P_A &\to H_A + \epsilon^2 Z_A \\
J_{Aa} &\to \epsilon G_{Aa} & P_a &\to \epsilon P_a \\
J_{ab} &\to J_{ab} + \epsilon^2 Z_{ab} & P_{a'} &\to \epsilon P_{a'} \\
J_{a'b'} &\to J_{a'b'} + \epsilon^2 Z_{a'b'} & A = 0,1 \quad a &= 2,3,4 \quad a' = 1, ..., 5
\end{aligned}$$

Coset numerator $\mathfrak{g} = \{H_A, P_a, P_{a'}, M, G_{Aa}, J_{ab}, J_{a'b'}, Z_A, Z, Z_{ab}, Z_{a'b'}\}$

Coset denominator $\mathfrak{h} = \mathfrak{g} \setminus \{H_A, P_a, P_{a'}\}$

dim($\mathfrak{g}$) - dim($\mathfrak{h}$) = 10

Maurer-Cartan captures the SNC data of $\mathrm{AdS}_5 \times S^5$:

$$A_\mu = \tau_\mu{}^A H_A + e_\mu{}^a P_a + e_\mu{}^{a'} P_{a'} + m_\mu{}^A Z_A + ...$$

What about the Lagrange multipliers d.o.f. λ_A *?*

Define a generalised Maurer-Cartan

$$J \equiv A + \star_2 \Lambda \qquad \Lambda_\alpha = \lambda_- \theta_\alpha{}^- Z_+ + \lambda_+ \theta_\alpha{}^+ Z_-$$

NR coset action

[AF, van Tongeren 2022][AF, Nieto, 2022]

$$S^{\mathrm{NR}} = \int \mathrm{d}^2\sigma \gamma^{\alpha\beta} \langle J_\alpha{}^{(1)}, J_\beta{}^{(1)} \rangle$$

In [AF, van Tongeren 2022] we did not include the ideal $\{Z_{ab}, Z_{a'b'}\}$

Result:

- $\langle \cdot, \cdot \rangle$ is degenerate in $\mathfrak{g}$ (but not in the physical space $\mathfrak{g} \setminus \mathfrak{h}$)
- adjoint invariant can be demanded at most under $\mathfrak{h} \setminus \{Z_A\}$

In [AF, Nieto, 2022] we included $\{Z_{ab}, Z_{a'b'}\}$

Result:

- $\langle \cdot, \cdot \rangle$ is non-degenerate in $\mathfrak{g}$
- it is adjoint invariant under $\mathfrak{g}$
- spinorial rep. of $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ inherited by Lie algebra expan.

Equations of motion and Lax pair

- Equations of motion

$$\partial_\alpha(\gamma^{\alpha\beta}J^{(1)}_\beta) + \gamma^{\alpha\beta}[J^{(0)}_\alpha, J^{(1)}_\beta] = 0 \qquad \mathcal{E}^{\lambda_\pm} = \varepsilon^{\alpha\beta}\theta_\alpha{}^{\pm}A^{H_\pm}_\beta = 0$$

They contain more e.o.m. than d.o.f., but they are not all independent. Noether identities (due to gauge invariance) relates them.

- Lax pair

$$\mathscr{L}^{\rm NR} = A^{(0)} + \ell_1 A^{(1)} - \ell_2 \star J^{(1)} \qquad \ell_1^2 - \ell_2^2 = 1 \quad \to \quad \xi$$

on solutions of $\mathcal{E}^{\lambda_\pm} = 0$ [AF, van Tongeren 2022]

Spectral curve

- Alternative route to the spectrum, it captures TBA equations
- Compute the eigenvalues of monodromy matrix

$$\mathcal{M} = \mathrm{P}\exp\left(\int_0^{2\pi} \mathrm{d}^2\sigma \ \mathscr{L}_\sigma^{\mathrm{NR}}(\xi)\right)$$

  $\xi =$ spectral parameter

- Theorem: On solutions of $\mathcal{E}^{\lambda_\pm} = 0$ all eigenvalues are ξ-independent

  [AF, Nieto, Ohlsson Sax 2022]

  Checked on 2 finite reps: spinorial and adjoint

- $\mathcal{M}$ evaluated on BMN-like sol. is **non-diagonalisable**

$$\mathcal{M}|_{\text{BMN-like}} = S \begin{pmatrix} {\color{blue}\times} & {\color{red}\times} & {\color{red}\times} & & & \\ & {\color{blue}\times} & {\color{red}\times} & & & \\ & & {\color{blue}\times} & & & \\ & & & {\color{blue}\times} & {\color{red}\times} & \\ & & & & {\color{blue}\times} & \\ & & & & & {\color{blue}\times} \end{pmatrix} S^{-1}$$

${\color{blue}\times}$ no ξ-dep. ${\color{red}\times}$ yes ξ-dep. $\Longrightarrow$ spectral curve defined by "${\color{red}\times}$"

- Reason of non-diagonalisability:

  $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ is semi-simple, but its Lie algebra expansion considered is not

- semi-simple part of string Newton-Hooke$_5$$\oplus$Eucl$_5$ has generators with diagonalisable rep

Same apply for **relativistic** string in flat space.

- Poincaré algebra is not semi-simple.
- Eigenvalues of monodromy on any solution do not depend on ξ

Diagonalisable: $\mathcal{M} = S e^{p_i(\xi) C_i} S^{-1}$ $\quad C_i \in \text{Cartan}$

Non-diagonalisable: $\mathcal{M} = S e^{q_i(\xi) W_i} S^{-1}$ $\quad W_i \in \text{MAS}$

(MAS = maximal abelian subalgebra)

[AF, Nieto, Ohlsson Sax 2022]

Classical Integrability

Trace of monodromy is conserved in time evolution

$$\partial_\tau \mathrm{Tr}[\mathcal{M}(\xi)] = 0$$

Expanding $\mathrm{Tr}[\mathcal{M}(\xi)]$ around a point of analyticity, e.g. $\xi = 0$

$$\mathrm{Tr}[\mathcal{M}(\xi)] = H_0 + \xi H_1 + \xi^2 H_2 + \ldots \qquad \partial_\tau H_n = 0 \qquad \forall\ n$$

$\{H_n\}$ is the infinite set of conserved charges.

If in involution, the theory is classically integrable.

From previous theorem, $\partial_\xi \mathrm{Tr}\mathcal{M} = 0$. No conserved charges?

- The theorem holds for 2 finite dim. representation.
- But e.g. Poincaré algebra admits an infinite dim rep. where P is diagonal

$$P|p\rangle = p|p\rangle$$

  $\mathcal{M}$ is diagonal and eigenvalues depend on ξ.
- expect something similar for the GGK theory

Summary of NR strings in $\mathrm{AdS}_5 \times S^5$

- NR string needs winding (consequence of $\mathcal{E}^{\lambda_\pm} = 0$)
- having winding spoils semiclassical expansion of the action in light-cone gauge (z is not an isometry)
- found a coset formulation + Lax pair
- Monodromy is non-diagonalisable, its eigenvalues are ξ-independent
- proposed an alternative definition of spectral curve

Open problems

- SUSY coset action
- Generalised spectral curve
- S-matrix (in progress with Marius de Leeuw)
- Identify the "dual" limit on $\mathcal{N} = 4$ SYM on "boundary geometry"
  Gauge invariant dual operators to NR string solutions?

Thanks for your attention!

More questions?

✉ afontanella@perimeterinstitute.ca