Intrinsic torsion in Galilean and Carrollian spacetimes from a mathematician's perspective

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Non-Relativistic Strings and Beyond
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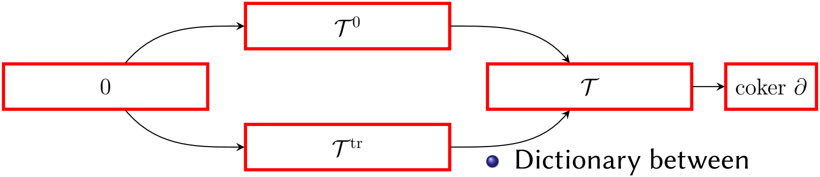
Non-relativistic spacetime

- Galilean & Carroll spacetimes
- Intrinsic torsion
    - Independent of connection
    - Characterises geometry

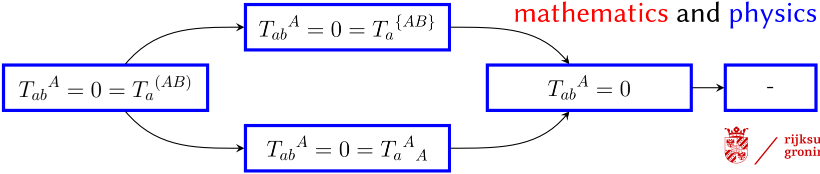
Preview

Theorem

*The subrepresentations of the space of all intrinsic torsions on a Galilean/Carroll (p-branes) spacetime are as in the diagrams below.**



- Dictionary between mathematics and physics



Inverse Vielbeine and (local) frame fields

M spacetime manifold, x^μ local coordinates on $U \subseteq M$

V fin. dim. vector space with basis $\mathbf{e}_{\hat{A}}$ and dual basis $\mathbf{e}^{\hat{A}}$

Frame $u : V \to T_pM$ $\rightsquigarrow$ **Frame bundle** $FM \ni u$

(Local) frame field:

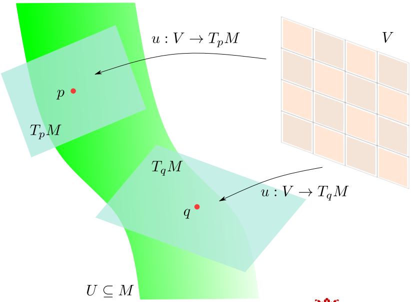
section $E : U \to FM$,

$$E = \mathbf{e}^{\hat{A}} E^\mu_{\hat{A}} \frac{\partial}{\partial x^\mu}$$

$\updownarrow$

Inverse Vielbein $E^\mu_{\hat{A}}$

$$E^\mu_{\hat{A}}(p) = dx^\mu(E(p)\mathbf{e}_{\hat{A}})$$



Vielbeine and (local) coframe fields

Solder form $\theta \in \Omega^1(FM, V), \theta_u(X_u) = u^{-1}(\pi_*(X_u))$

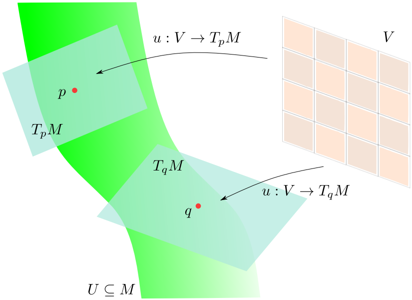
(Local) **coframe field**:

$$E^*\theta \in \Omega^1(U, V)$$
$$E^*\theta = \mathbf{e}_{\hat{A}} E^{\hat{A}}_\mu dx^\mu$$

$\updownarrow$

Vielbein $E^{\hat{A}}_\mu \in C^\infty(U)$

$$E^{\hat{A}}_\mu(p) = \mathbf{e}^{\hat{A}}((E^*\theta)_p \frac{\partial}{\partial x^\mu})$$



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Principal bundle

- Some frames are more equal than others:

$$\text{Structure group } G \subseteq \mathrm{GL}(V)$$

G is defined by letting $\delta \in \odot^2 W$ and $\eta \in \odot^2 \mathrm{Ann}(W)$ invariant for some 'transversal' subspace $W \subseteq V$

$\updownarrow$

Transformation rules for $E_\mu^{\hat{A}} = ({\tau_\mu}^A, {e_\mu}^a)$ under G:

$\delta {\tau_\mu}^A = {\lambda^A}_B {\tau_\mu}^B, \qquad \delta {e_\mu}^a = {\lambda^a}_b {e_\mu}^b - {\lambda^a}_A {\tau_\mu}^A.$

- A generalization of Galilei and Carroll structures!
- The new smooth structure is a principal G-bundle P

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Connection

Connection: How to relate copies of G above different points

Connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with $\mathfrak{g} = \mathrm{Lie}(G) = \langle J_{\hat{A}}{}^{\hat{B}} \rangle$

$$\rightsquigarrow \Omega := E^*\omega = \Omega_\mu dx^\mu = J_{\hat{A}}{}^{\hat{B}} \omega_\mu{}^{\hat{A}}{}_{\hat{B}} dx^\mu \in \Omega^1(U, \mathfrak{g})$$

$$\updownarrow$$

Structure group connection (with spin connection ω)

$$\Omega_\mu = \frac{1}{2}\omega_\mu{}^A{}_B J_A{}^B + \frac{1}{2}\omega_\mu{}^a{}_b J_a{}^b + \omega_\mu{}^a{}_A G_a{}^A$$

Torsion

Torsion: measures the failure of θ being parallel with respect to ω

Torsion 2-form $\Theta^\omega = d\theta + \omega \wedge \theta \in \Omega^2(P, V)$

$$\rightsquigarrow T^\omega := E^*\Theta^\omega = dE^*\theta + E^*\omega \wedge E^*\theta \in \Omega^2(U, V)$$

$$T^\omega = (T^\omega)_{\mu\nu}{}^{\hat{A}} \mathbf{e}_{\hat{A}} dx^\mu \wedge dx^\nu$$

$$\updownarrow$$

$$(T^\omega)_{\mu\nu}{}^{A} = 2\partial_{[\mu}\tau_{\nu]}{}^{A} - 2\omega_{[\mu}{}^{A}{}_{B}\tau_{\nu]}{}^{B}$$

$$(T^\omega)_{\mu\nu}{}^{a} = 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{a}{}_{b}e_{\nu]}{}^{b} - 2\omega_{[\mu}{}^{a}{}_{A}\tau_{\nu]}{}^{A}$$

Spencer differential

Locally, the Spencer differential ∂ is given by

$$\partial : \Omega(U, \mathfrak{g}) \to \Omega^2(U, V)$$
$$E^*\omega' - E^*\omega \mapsto T^{\omega'} - T^{\omega} = (E^*\omega' - E^*\omega) \wedge E^*\theta$$

Intrinsic torsion: Find coker $\partial = \Omega^2(U, V)/\text{im}\,\partial$.

$\updownarrow$

Intrinsic torsion: solve the spin connection ω in terms of the torsion components and see what torsion remains.

At one point in spacetime

If expressions and objects are G-invariant, it suffices to do all calculations above a single point.

- The Spencer differential reduces to

$$\partial : \mathrm{Hom}(V, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^2 V, V)$$
$$(\partial\kappa)(u \wedge v) = \kappa(u)v - \kappa(v)u,$$

  a (linear) map of representations!

- The subspace $W \subset V$ is invariant and creates a subbundle $F \subset TM$.

Classification

Theorem

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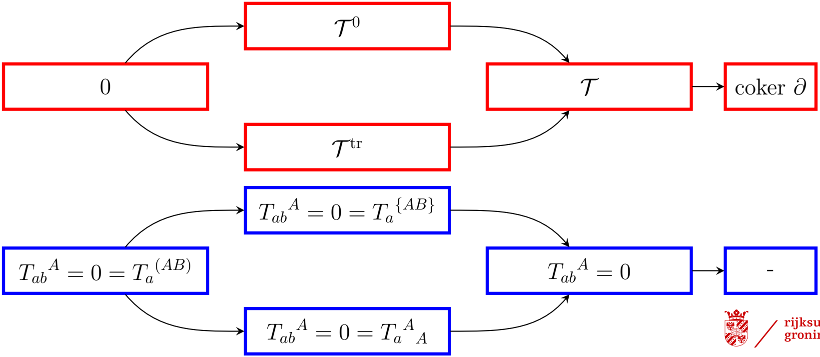
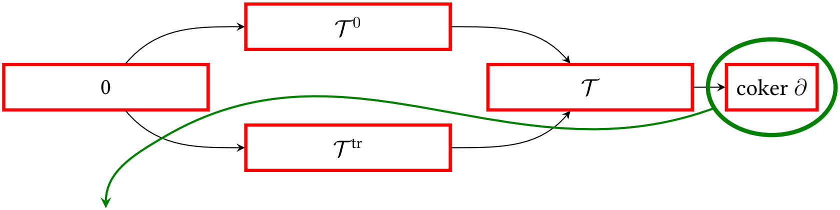
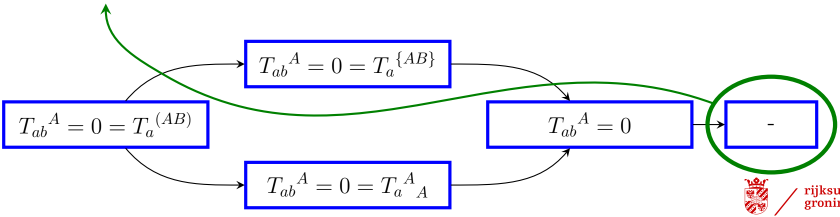


Classification



Generic torsion: adapted connections preserve F



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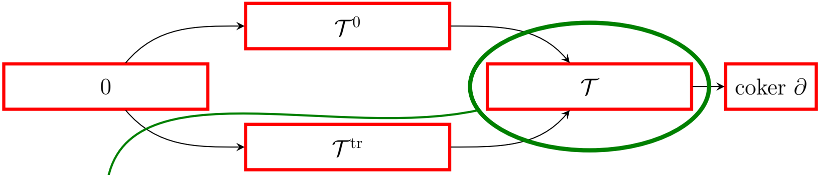
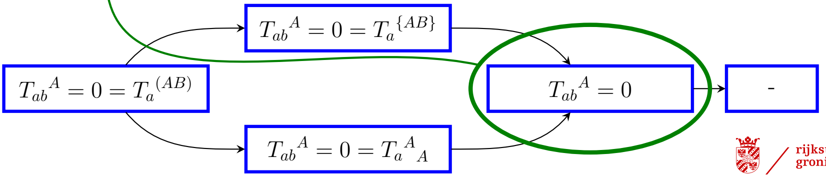


Twistless torsion: if and only if F is involutive $\rightsquigarrow$ sliced spacetime



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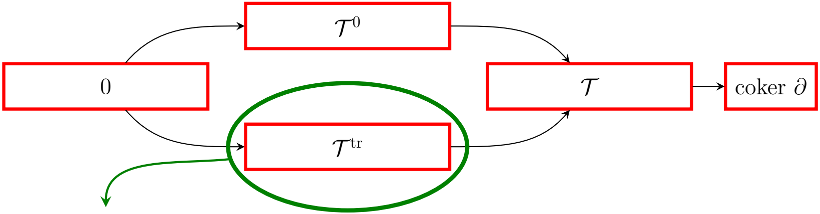
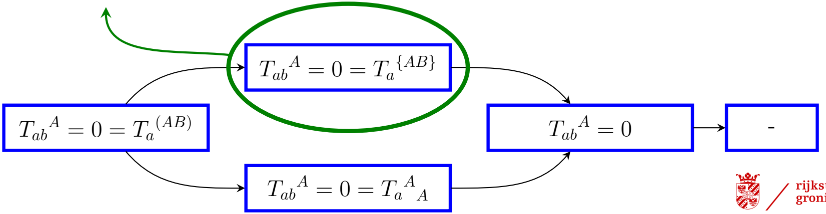


Twistless torsion, trace: for $X \in \Gamma(F)$, we have $\mathcal{L}_X \eta \propto \eta$



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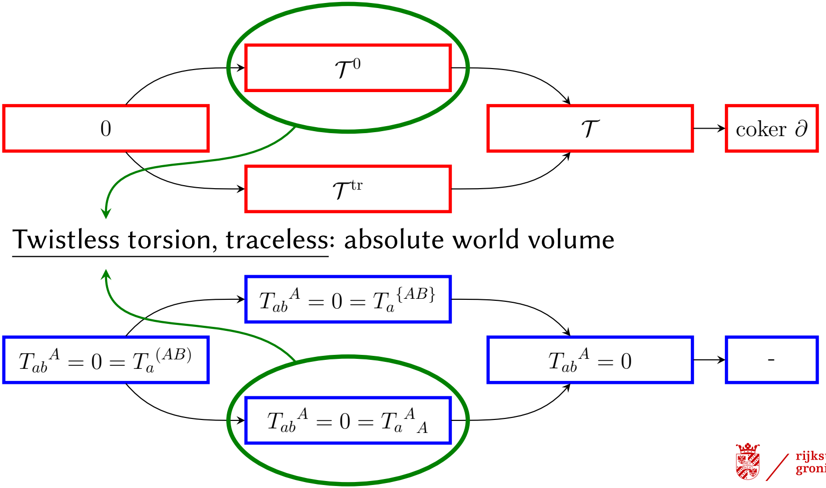

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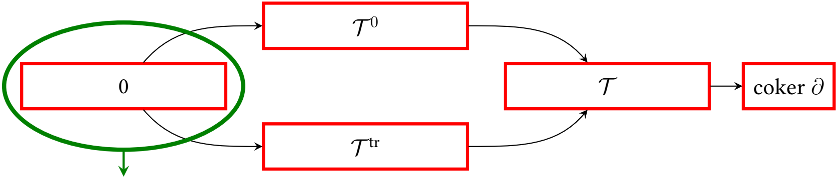
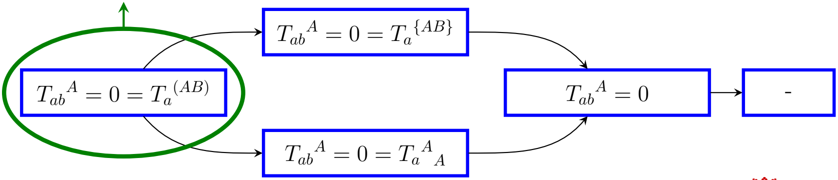


No intrinsic torsion: for $X \in \Gamma(F)$, we have $\mathcal{L}_X \eta = 0$



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Discussion

- Newton-Cartan geometry
- Limits of General Relativity
- Supersymmetry
- Different symmetry groups (e.g. Aristotelian)
- Irreducibility

References

Thank you!

Eric Bergshoeff, Kevin van Helden, Johannes Lahnsteiner, Luca Romano, and Jan Rosseel. 2023. *Generalized Newton-Cartan geometries for particles and strings*, Classical and Quantum Gravity **40**, DOI 10.1088/1361-6382/acbe8c.

José Figueroa-O'Farrill. 2020. *On the intrinsic torsion of spacetime structures*, available at arXiv:2009.01948.

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