The Anisotropic Compactification of Non-Relativistic M-Theory

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Non-Relativistic Strings and Beyond

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Based on work in progress with Z. Yan

Goal

Discuss a few results on M-theory with Ziqi Yan regarding:

1. Nonrelativistic limit.
2. Anisotropic geometry.
3. Relations between D3- and M5-branes using the anisotropic torus.

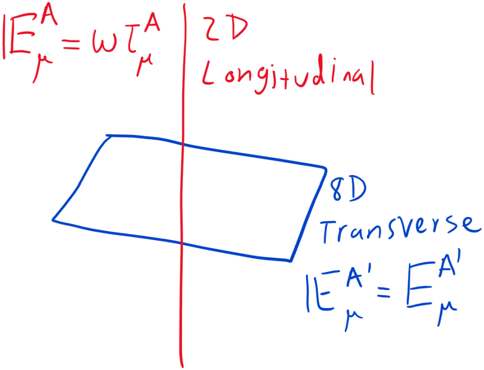
There is a U-duality relating NR M-theory and Matrix theory & DLCQ of M-theory motivating this work.

Features of nonrelativistic string theory

1. Self-contained corner of string/M-theory.
2. String excitations have a Galilean-invariant dispersion relation.
3. Propagates on a codimension-two geometry called the string Newton-Cartan geometry.
4. Low energy effective theory is Newtonian gravity.

Geometric view of nonrelativistic string theory

$$\mathbb{G}_{\mu\nu} = \mathbb{E}_\mu{}^{\mathsf{A}}\mathbb{E}_\nu{}^{\mathsf{B}}\eta_{\mathsf{AB}} + \mathbb{E}_\mu{}^{A'}\mathbb{E}_\nu{}^{B'}\delta_{A'B'}$$



$A = 0, 1$ and $A' = 2, \cdots, 9$. NR limit defined by $\omega \to \infty$. NR M-theory has a 3d longitudinal sector.

Bosonic background fields

1. Kalb-Ramond

$$\mathbb{B} = -\omega^2 \tau^{\mathsf{A}} \wedge \tau^{\mathsf{B}} \epsilon_{\mathsf{AB}} + B^{(2)}$$

2. Ramond-Ramond potentials

$$\mathbb{C}^{(q)} = \omega^2 \tau^{\mathsf{A}} \wedge \tau^{\mathsf{B}} \wedge C^{(q-2)} \epsilon_{\mathsf{AB}} + C^{(q)}$$

3. Dilaton

$$\Phi = \phi + \ln \omega^2$$

4. $U(1)$ field strength

$$\mathbb{F} = F$$

Example: nonrelativistic Dp-brane action

Textbook definition of a relativistic Dp-brane action

$$\mathbb{S}_{\mathrm{D}p} = -\int d^{p+1}\sigma \; e^{-\Phi}\sqrt{-\det\left(\mathbb{G}_{\mu\nu} + \mathbb{F}_{\mu\nu}\right)} - \int \sum_{q=0}^{p+1} \mathbb{C}^{(q)} \wedge e^{\mathbb{F}}\Big|_{p+1}$$

Plugging the expansions from the previous slide and take $\omega \to \infty$ limit to see the divergences in DBI and WZ cancel out individually (Gomis, Yan, Yu 2020; SE, Sun, Yan 2021)

$$S_{\mathrm{D}p} = -\int d^{p+1}\sigma \; e^{-\phi}\sqrt{-\det\begin{pmatrix} 0 & \tau_\nu \\ \bar{\tau}_\mu & E_\mu^{A'} E_\nu^{A'} + F_{\mu\nu} \end{pmatrix}} - \int \sum_{q=0}^{p+1} C^{(q)} \wedge e^{F}\Big|_{p+1}$$

with $E_{\mu\nu} = E_\mu^{A'} E_\nu^{A'}$, $\tau_\mu = \tau_\mu{}^0 + \tau_\mu{}^1$ and $\bar{\tau}_\mu = \tau_\mu{}^0 - \tau_\mu{}^1$.

M-theory on a torus and SL$(2,\mathbb{Z})$ duality

Toroidal dimensional-reduction

$$\mathbb{G}_{\mathbb{IJ}} = \begin{pmatrix} \mathbb{G}_{\mu\nu} & 0 \\ 0 & \mathbb{G}^{\mathsf{T}^2}_{mn} \end{pmatrix}, \quad \mu,\nu = 0,1,\cdots,8, \quad m,n = 9,10$$

$\mathbb{G}_{\mu\nu}$ 10d superstring metric and torus metric is

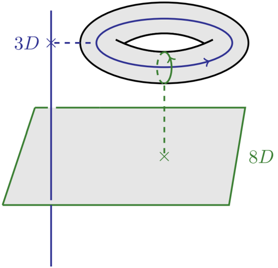
$$\mathbb{G}^{\mathsf{T}^2}_{mn} d\hat{x}^m d\hat{x}^n = \frac{A}{\hat{\tau}_2} |d\hat{x}^{10} - \hat{\tau} d\hat{x}^9|^2$$

Torus has isometry group SL$(2,\mathbb{Z})$ with $\alpha,\beta,\gamma,\delta \in \mathbb{Z}$

$$\hat{x}'^m = \Lambda^m{}_n \hat{x}^n, \quad \hat{\tau}' = \frac{\alpha\hat{\tau}+\beta}{\gamma\hat{\tau}+\delta}, \quad \det\Lambda^m{}_n = \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$$

Zweibein formalism

Convenient to work in this formalism because we can easily obtain IIB quantities from projecting fields in M-theory e.g. $\hat{\mathcal{C}}^{(3)}$.



$$\mathbb{G}^{\mathsf{T}^2}_{mn} = \mathbb{V}_m \mathbb{V}_n + \mathbb{E}_m \mathbb{E}_n,$$

SL$(2,\mathbb{Z})$ transformation to $\mathbb{V} = \mathbb{V}_m d\hat{x}^m$ and $\mathbb{E} = \mathbb{E}_m d\hat{x}^m$:

$$\begin{pmatrix} \mathbb{V}' \\ \mathbb{E}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbb{V} \\ \mathbb{E} \end{pmatrix}$$

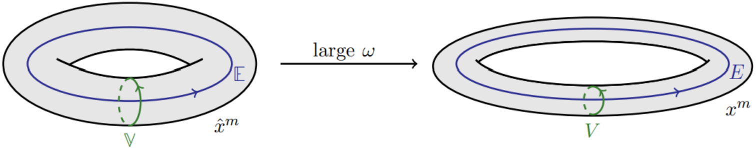
$$\theta = \tan^{-1}\hat{\kappa} + \frac{\pi}{2}\left[\mathrm{sgn}(\gamma\hat{\tau}_1 + \delta) - 1\right], \quad \hat{\kappa} = \frac{\gamma\hat{\tau}_2}{\gamma\hat{\tau}_1 + \delta}$$

The anisotropic torus

$$\mathbb{V} = \omega^{\frac{2}{3}} V, \quad \mathbb{E} = \omega^{-\frac{1}{3}} E$$

and IIB from M-theory

$$\begin{pmatrix} \mathcal{B}^{(2)} \\ \mathcal{C}^{(2)} \end{pmatrix} = \begin{pmatrix} V^m \mathcal{C}^{(3)}_m \\ E^m \mathcal{C}^{(3)}_m \end{pmatrix}$$



SL$(2,\mathbb{Z})$ transformations become Galilean boosts!

$$\begin{pmatrix} V' \\ E' \end{pmatrix} = \lim_{\omega\to\infty} \frac{\mathrm{sgn}(\gamma\tau_1+\delta)}{\sqrt{1+\kappa^2/\omega^2}} \begin{pmatrix} 1 & -\frac{\kappa}{\omega^2} \\ \kappa & 1 \end{pmatrix} \begin{pmatrix} V \\ E \end{pmatrix}, \quad \kappa = \frac{\gamma\tau_2}{\gamma\tau_1+\delta}$$

$$= \mathrm{sgn}(\gamma\tau_1+\delta) \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix} \begin{pmatrix} V \\ E \end{pmatrix}$$

Polynomial realization of SL$(2,\mathbb{Z})$ in NR IIB

Other fields under the $\omega \to \infty$ limit projected from the three-form's $\mathbb{C}^{(3)}_m$ components in M-theory

$$\begin{pmatrix} \mathcal{B}^{(2)} \\ \mathcal{C}^{(2)} \end{pmatrix} = \begin{pmatrix} V^m \mathbb{C}^{(3)}_m \\ E^m \mathbb{C}^{(3)}_m \end{pmatrix} = \begin{pmatrix} e^{-\frac{\phi}{2}} B^{(2)} \\ e^{\frac{\phi}{2}} \left(C^{(2)} + C^{(0)} B^{(2)} \right) \end{pmatrix}$$

and the SL$(2,\mathbb{Z})$ transformations of these 2-forms are

$$\mathcal{B}'^{(2)} = \mathcal{B}^{(2)} - \kappa \mathcal{C}^{(2)} + \frac{1}{2}\kappa^2 \tau^A \wedge \tau^B \epsilon_{AB}, \quad \mathcal{C}'^{(2)} = \mathcal{C}^{(2)} - \kappa \tau^A \wedge \tau^B \epsilon_{AB}$$

Also the RR 4-form $C^{(4)}$ in IIB from the 6-form in M-theory $\mathcal{C}^{(4)} = \frac{1}{2}\mathbb{C}^{(6)}_{mn} V^m E^n$:

$$\mathcal{C}'^{(4)} = \mathcal{C}^{(4)} - \frac{1}{2}\kappa \mathcal{B}^{(2)} \wedge \tau^A \wedge \tau^B \epsilon_{AB} + \frac{1}{4}\kappa^2 \mathcal{C}^{(2)} \wedge \tau^A \wedge \tau^B \epsilon_{AB}$$

where

$$\mathcal{C}^{(4)} = C^{(4)} + \frac{1}{2} B^{(2)} \wedge C^{(2)}$$

M5-brane

The M5-brane has an extensive history filled with important applications. Very incomplete timeline:

1. Perry-Schwarz (1996): flat relativistic M5, but loss of general covariance.
2. Schwarz (1997): curved relativistic M5, but loss of general covariance.
3. Pasti-Sorokin-Tonin (1997): curved relativistic M5 with general covariance!
4. Aganagic-Park-Popescu-Schwarz (1997): Supersymmetrization of PST.
5. Cherkis-Schwarz (1998): PST action on K3 realizes the heterotic string.
6. Berman (1998): PST action on T^2 realizes D3-brane with manifestly $\mathrm{SL}(2,\mathbb{Z})$.
7. Heydeman-Schwarz-Wen (2017-2020): M5-brane amplitudes.
8. SE, Z. Yan (2023): Nonrelativistic PST action & relation to manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant D3-brane action!

Relativistic PST action

PST (Pasti, Sorokin, and Tonin) Action for a single M5-brane:

$$\mathbb{S}_{\mathsf{M5}} = -\int d^6\sigma \sqrt{-\det\left(\mathbb{G}_{\mathsf{MN}} + i\widetilde{\mathbb{H}}_{\mathsf{MN}}\right)} + \mathbb{S}_{\mathsf{WZ}}$$

where

$$\mathbb{N}_{\mathsf{M}} = \frac{v_{\mathsf{M}}}{\sqrt{\mathbb{G}^{\mathsf{NL}} v_{\mathsf{N}} v_{\mathsf{L}}}}, \quad \widetilde{\mathbb{H}}^{\mathsf{MN}} = \frac{1}{6\sqrt{-\mathbb{G}}}\epsilon^{\mathsf{MNOPQL}}\mathbb{H}_{\mathsf{OPQ}},\ \mathbb{H}^{(3)} = \Theta^{(3)} - \mathbb{A}^{(3)}$$

and

$$\Theta^{(3)} = d\theta^{(2)}, \quad \Theta^{(3)} = \star\Theta^{(3)}.$$

1. Fundamental d.o.f. is $\theta^{(2)}$ for M5 unlike D-branes' d.o.f. being a 1-form A_μ.
2. General covariance is guaranteed by the purely auxiliary vector $\mathbb{N}_{\mathsf{M}}$.

Nonrelativistic PST Action

$$\widetilde{\mathbb{H}}^{\mathrm{M}}{}_{\mathrm{N}} = L^{\mathrm{M}}{}_{\mathrm{N}} + \frac{1}{c^3} M^{\mathrm{M}}{}_{\mathrm{N}} + O\left(\frac{1}{c^6}\right)$$

M5-brane action

$$S_{\mathrm{M5}} = -\int d^6\sigma E \sqrt{\mathrm{Tr}(LM)\left(1 + \frac{1}{2}\,\mathrm{Tr}\left(L^2\right)\right) - \mathrm{Tr}\left(L^3 M\right)} + S_{\mathrm{WZ}}$$

with

$$L^{\mathrm{M}}{}_{\mathrm{N}} = \gamma_{\mathrm{NQ}} \widetilde{H}^{\mathrm{MQ}} + E_{\mathrm{NQ}} \widetilde{\ell}^{\mathrm{MQ}}, \quad M^{\mathrm{M}}{}_{\mathrm{N}} = -\frac{1}{2} N_u N^u L^{\mathrm{M}}{}_{\mathrm{N}} + E_{\mathrm{NQ}} \widetilde{H}^{\mathrm{MQ}}$$

and

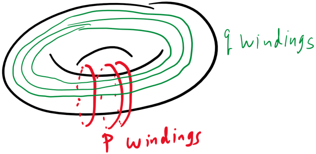
$$\tilde{H}^{\mathrm{MN}} = \frac{1}{6E} \epsilon^{\mathrm{MNQPSL}} H_{\mathrm{QPS}} N_{\mathrm{L}}, \quad \tilde{\ell}^{\mathrm{MN}} = \frac{1}{6E} \epsilon^{\mathrm{MNQPSL}} \ell_{\mathrm{QPS}} N_{\mathrm{L}},$$

$$\ell = \frac{1}{3!} \tau^u \wedge \tau^v \wedge \tau^w \epsilon_{uvw}, \quad u, v = 0, 1, 2, \quad u', v' = 3, 4, 5.$$

Dimensional reduction setup

$M, N = (\underbrace{0,1}_{\text{D3}}, \underbrace{2,3}_{T^2}, \underbrace{4,5}_{\text{D3}})$. Longitudinal are $0,1,2$ and transverse are $3,4,5$. Torus must have both long. and trans. directions to lead to NR IIB or else it would be the DLCQ of relativistic IIB. 6D PST vector is valued on T^2 to preserve covariance of D3

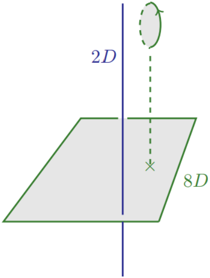
$$N_a = \begin{pmatrix} p \\ q \end{pmatrix}$$



Origins of the $\mathrm{SL}(2,\mathbb{Z})$ D3-brane! E.g. $N_a = (1,0)$ and $(0,1)$ are related via $\mathrm{SL}(2,\mathbb{Z})$ transformation (Berman 1998).

Matching M5-brane on T^2 with $\mathbf{SL}(2,\mathbb{Z})$ D3-brane

From the $\mathrm{SL}(2,\mathbb{Z})$ D3-brane side (Bergshoeff, Grosvenor, Lahnsteiner, Yan, Zorba 2022) we compactify one of the 10d target space dimensions on a circle



and dualize the scalars arising from dimensional reduction to obtain

$$S_{\text{Dual D3}} = -e^{-\phi}\int d^4\sigma \sqrt{-\det\begin{pmatrix} 0 & \tau_\nu \\ \bar{\tau}_\mu & E_\mu \bar{E}_\nu + \mathscr{F}_{\mu\nu} \end{pmatrix} + \det\begin{pmatrix} 0 & J_\nu & 0 \\ J_\mu & \tau_\mu \bar{\tau}_\nu + i \star \mathscr{F}_{\mu\nu} & E_\mu \\ 0 & \bar{E}_\nu & 0 \end{pmatrix}} + S_{\text{WZ}}$$

Conclusion

Open questions:

1. Study U-duality groups $E_{n(n)}(\mathbb{Z})$ from NR M-theory on T^n. Only considered $E_2(\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$ here. Test for Matrix theory to better understand these groups in the non-relativistic context.
2. Top-down holography. Do black holes exist in nonrelativistic string theory from stacking coincident Dp-, M2- and M5-branes?
   NR limit of ABJM theory (i.e. M2-branes in $\mathrm{AdS}_4 \times \mathrm{S}^7$, M2-branes in $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ or D2-branes in $\mathrm{AdS}_4 \times \mathbb{CP}^3$), D3-branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$, M5-branes in $\mathrm{AdS}_7 \times S^4$ ...?
3. Explore more anisotropic geometries to compactify string/M-theory objects on: T^n, K3, CY ...
4. Find the scattering amplitudes of Galilean M2- and M5-branes.

Thanks for your attention!