

# The Gravitational Double Copy and Spin

Double copy: • Amplitudes in GR can be obtained from YM amplitudes

$$\bullet \text{ GR} \sim (\text{YM})^2$$

Classical solutions version

$$\bullet \text{ Schwarzschild} \sim (\text{Coulomb})^2$$

↓ spin

$$\bullet \text{ Kerr solution} \sim (\text{"root-Kerr"})$$

Refs

• 1907.01358

"BCJ review"

• 2203.13013

"SAGrEX review"

• 2204.06547

"Stromass White paper"

YM

$$\frac{1}{4g_{\text{YM}}^2} \text{Tr} (F_{\mu\nu})^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

$$EOM: D_\mu F^{\mu\nu} = 0$$

Weak coupling

$$A_\mu^a = c^a \epsilon_\mu e^{ip \cdot x}$$

Polarizations

$$\epsilon \cdot p = 0, \quad \epsilon \cdot \epsilon = 0$$

$$\epsilon^\mu \sim \epsilon^\mu + p^\mu \quad (\text{gauge inv.})$$

GR

$$(\kappa^2 = 32\pi G_N)$$

$$\frac{2}{\kappa^2} \sqrt{-g} R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$h_{\mu\nu} = \epsilon_\mu \epsilon_\nu e^{ip \cdot x}$$

Two states

$$\epsilon_\mu^+, \epsilon_\mu^- \quad \text{gluons}$$

$$\epsilon_\mu^+ \epsilon_\nu^+, \epsilon_\mu^- \epsilon_\nu^- \quad \text{gravitons}$$

3pt amplitudes

$$A(123) = g_{\text{YM}} \int^{a_1, a_2, a_3} \overbrace{\left( \eta_{\mu\nu} (p_1 - p_2)_\mu + \text{cyclic} \left( \frac{\mu\nu\rho}{123} \right) \right)}^{V_{123}} \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho$$

$$M(123) = \frac{\kappa}{2} \left[ \left( \eta_{\mu\nu} (p_1 - p_2)_\mu + \text{cyclic} \left( \frac{\mu\nu\rho}{123} \right) \right) \epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \right]^2$$

helicity basis

$\implies$

$$A(1^-, 2^-, 3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \quad M(1^-, 2^-, 3^+) = \left( \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \right)^2$$

# Diagrammatic double copy

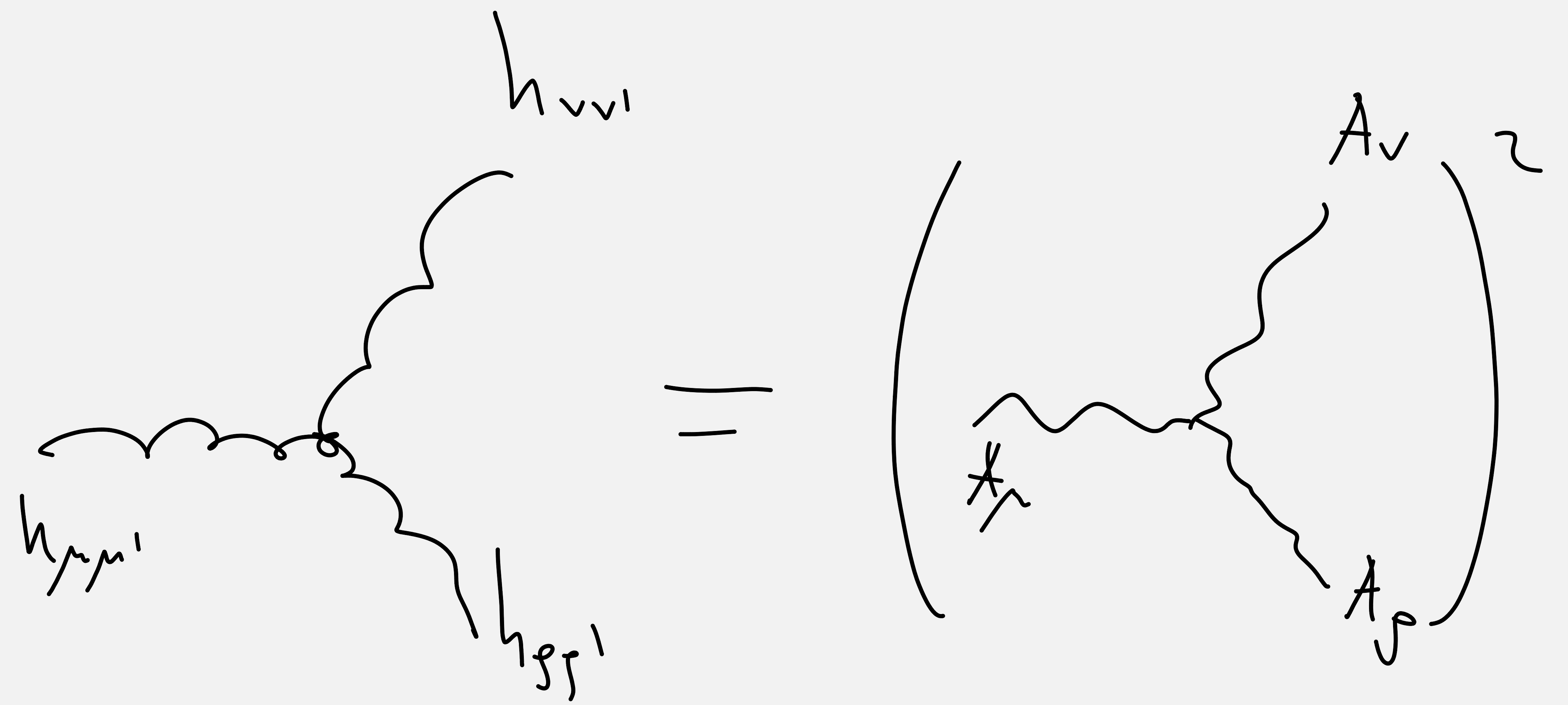
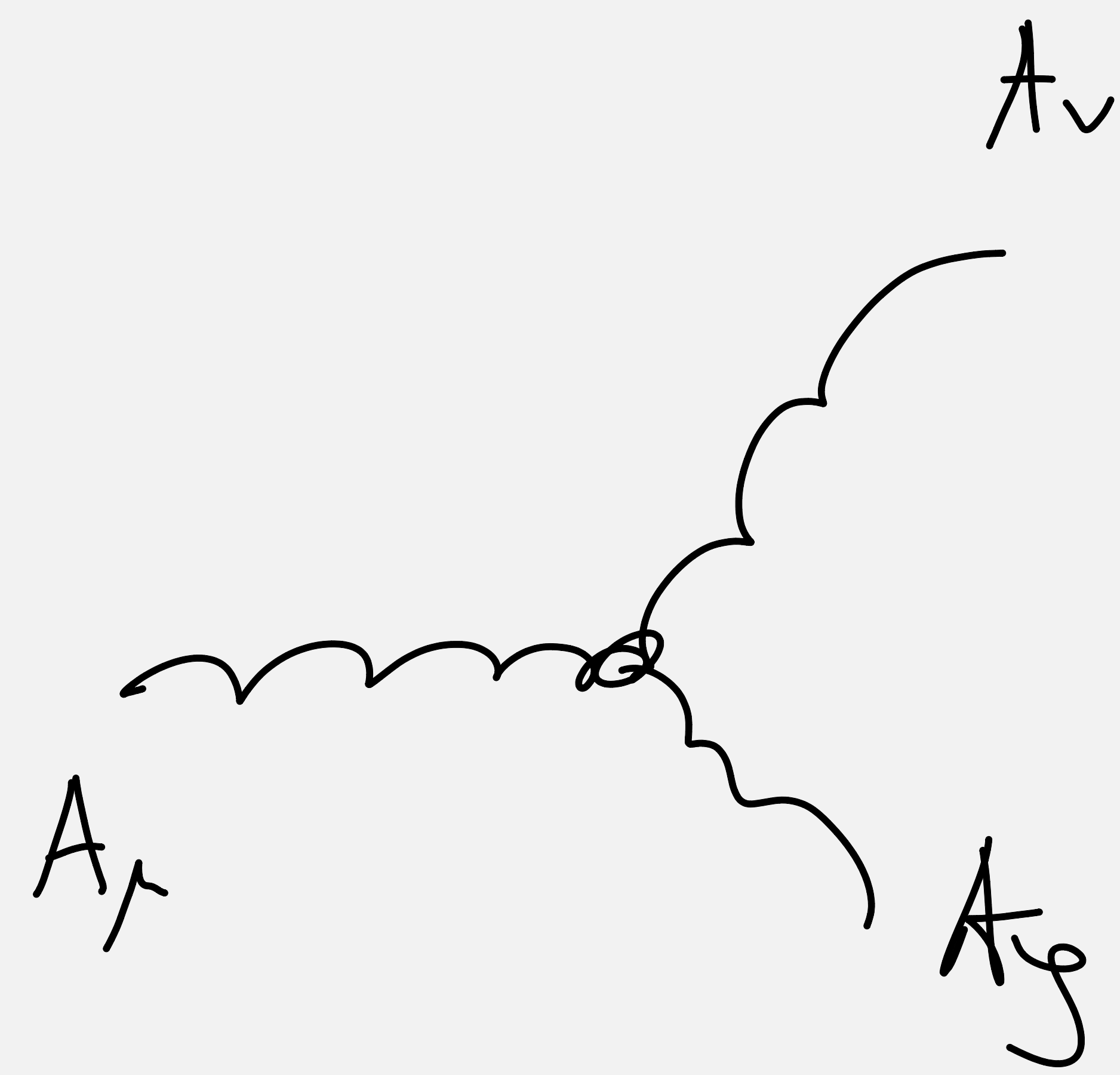
$$\text{wavy line } A_\mu$$

$$\epsilon_\mu e^{ip \cdot x}$$

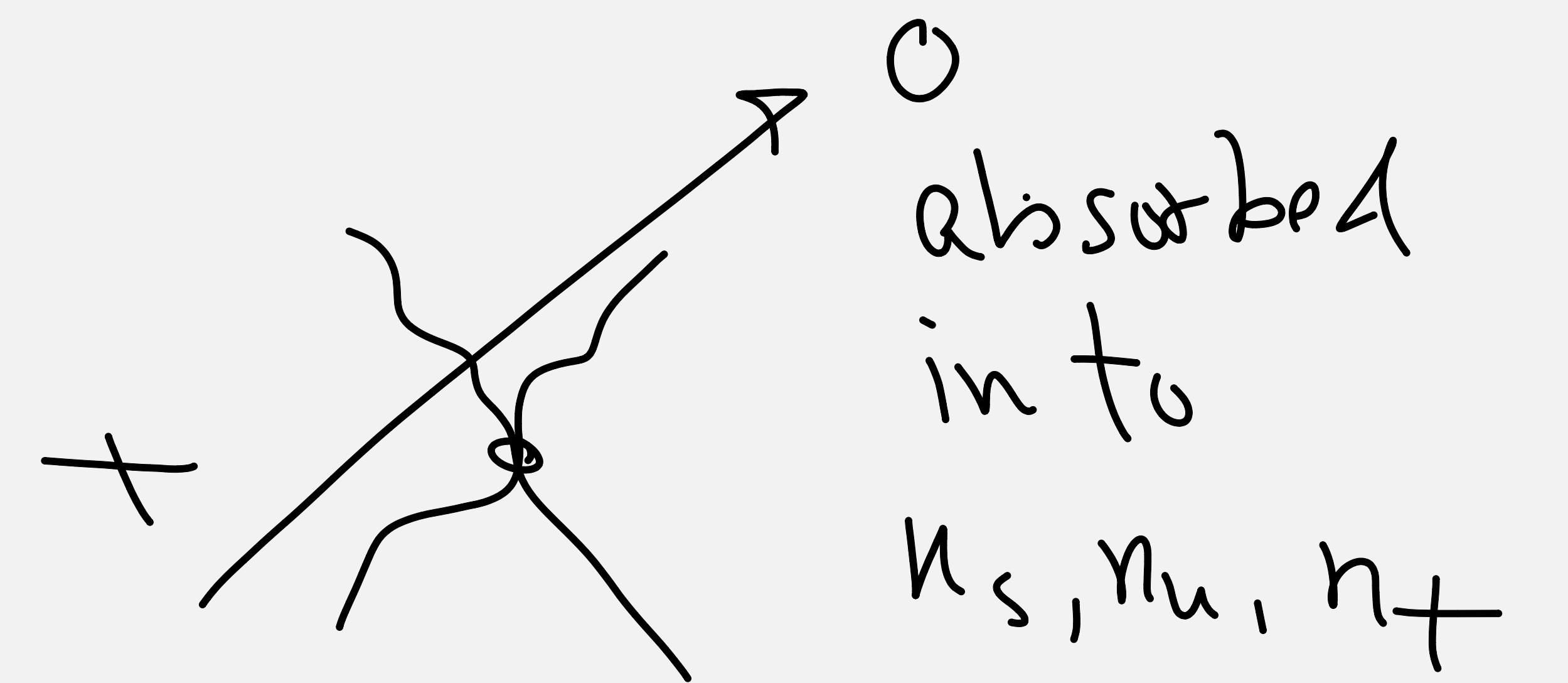
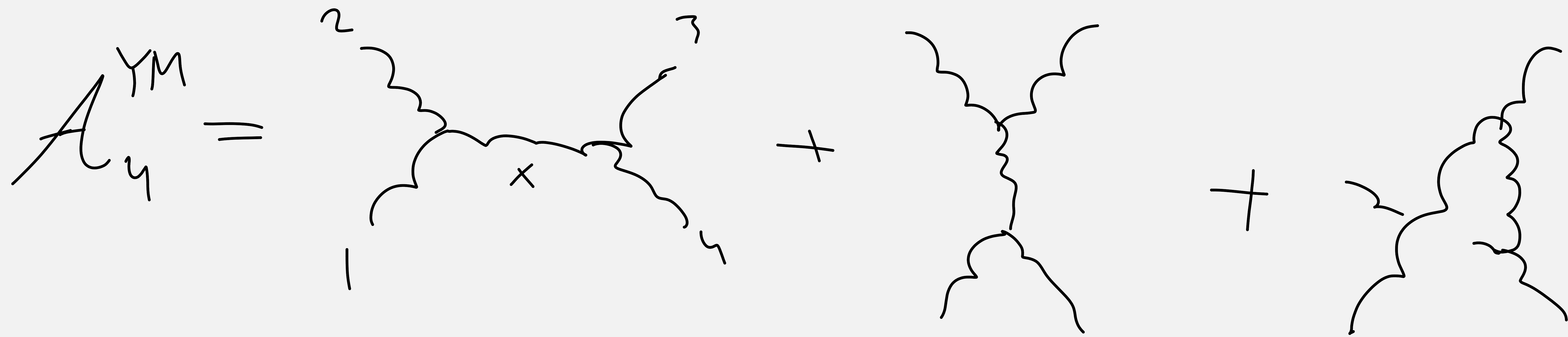


$$\text{wavy line } h_{\mu\nu} = (\text{wavy line } A_\mu)^2$$

$$\epsilon_\mu \epsilon_\nu e^{ip \cdot x}$$



# 4pt Tree amplitudes



$$= \frac{n_s C_s}{s} + \frac{n_t C_t}{t} + \frac{n_u C_u}{u}$$

$$n_s = V_{12}^{\mu} V_{\mu 34} + s \left[ (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) \right]$$

$$n_s + n_t + n_u = 0$$

$$\begin{aligned}
 s &= (p_1 + p_2)^2 \\
 t &= (p_1 + p_4)^2 \\
 u &= (p_1 + p_3)^2 \\
 &= 0
 \end{aligned}$$

$$C_s = f^{a_1 a_2 \mu} f^{\mu a_3 a_4}$$

$$C_t = \text{perm}$$

$$C_u = \text{perm}$$

$SU(N_c)$  Lie algebra

$\text{GR} \sim 100$ ,  $\sim 3$   
 $\text{Diagram} = 100^2 \times 3 = 30000$

$f^{a_1 a_2 \times} f^{x a_3 a_4} + \text{cyclic}(123) = C_S + C_t + C_u = 0$

Kinematic Lie algebra

( $S \rightarrow$  Volume-pres Diffeos)

$n_S + n_t + n_u = 0 \equiv n \left( \text{Diagram 1} \right) + n \left( \text{Diagram 2} \right) + n \left( \text{Diagram 3} \right)$

# Gravitational double copy

$$M_4^{\text{GR}} = A_4^{\text{YM}} \Big|_{c_i \rightarrow n_i} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

Why is this GR?

Claim: tree-level GR uniquely fixed by

- 1) Unitarity  $\Rightarrow$  factorization & physical states  $h^{++}, h^{--}$
- 2) 2-derivative interactions
- 3) Diffeo inv.

Check Diffeo inv.

$$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta A_\mu = \partial_\mu \xi$$

↑  
gauge parameters

Consider only variation of  $\xi_4$ ,  $\delta \xi_4 = p_4$ ,  $\delta \xi_1, \xi_2, \xi_3 = 0$

$$\delta A_4^{YM} = \frac{c_s \delta n_s}{s} + \frac{c_t \delta n_t}{t} + \frac{c_u \delta n_u}{u} = \alpha (c_s + c_t + c_u) \stackrel{\text{Lie algebra}}{=} 0$$

$$\delta M_4^{GR} = 2 \left( \frac{n_s \delta n_s}{s} + \frac{n_t \delta n_t}{t} + \frac{n_u \delta n_u}{u} \right) = \alpha (n_s + n_t + n_u) \stackrel{\text{Kinematic algebra}}{=} 0$$

(HW:  $\frac{\delta n_s}{s} = \frac{\delta n_u}{u} = \frac{\delta n_t}{t} = \alpha (\xi_i, p_i)$ )



# Graviton double copy @ m points

color factors sum  $f^{abc}$

m-point YM :  $A_m^{YM} = \sum_{i \in \{\text{cubic diagrams}\}} \dots$

Consider all 3-term Jacobi Id

Color-Lie alg. :  $C^s$

Kinematic Lie alg  $n_s$

m-pt GR :  $M_n^{GR} = A_m^{YM} |_{C_i \rightarrow n_i} = \sum_i \frac{n_i^2}{D_i}$

$C_i n_i$  ← kinematic numerators

$D_i$  ← Feynman propagator denominator  $(p_x^2 - m_x^2)$

$C_4^s = 0$

$n_u = 0$

## Comments

- 1) Note numbers  $n_i$  are gauge dependent  
hence non-trivial task to find  $N_{12}^{11} + N_{21}^{11} + N_{11}^{11} = 0$
- 2) Double copy generalizes to loop level  
numbers unknown (in general)  $\Leftrightarrow$  kinematical algebra unknown
- 3) When including matter  $\Rightarrow$  Web of theories that double copy

$$A = \sum_i \frac{n_i c_i}{D_i}, \quad M = \sum_i \frac{n_i^2}{D_i}$$

$$c_i \leftrightarrow n_i$$

Color - kinematics duality

$U(1)$  decoupling Id

$$A(1243) = A(1324) - A(1234)$$

$$A_4 = \left. \frac{c_s n_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right| c_t = -c_s - c_u$$

$$= c_s \left( \frac{n_s}{s} - \frac{n_t}{t} \right) + c_u \left( \frac{n_u}{u} - \frac{n_t}{t} \right)$$

$$= c_s A(1234) + c_u A(1324)$$

$$= \sum_{p \in S_4/Z_4} \text{Tr}(T^{a_{p(1)}} \dots T^{a_{p(4)}}) A(p(1)p(2)p(3)p(4))$$

$$s+t+u=0$$

$$\begin{pmatrix} A(1234) \\ A(1324) \end{pmatrix} = \begin{pmatrix} \frac{1}{s} + \frac{1}{t} & \frac{1}{t} \\ \frac{1}{t} & \frac{1}{u} + \frac{1}{t} \end{pmatrix} \begin{pmatrix} n_s \\ n_u \end{pmatrix}$$

not invertible

$$\Rightarrow A(1324) = \frac{t}{u} A(1234)$$

BCD relations

$\rightarrow (m-3)!$  basis

Add Matter

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4g^2} \text{Tr}(F_{\mu\nu})^2 + \bar{q} (i\not{D} - m_q) q \quad \leftarrow \text{fund rep.}$$

$$SU(N_c) \times SU(N_f)$$

Claim: Color-kinematics duality & double copy works @ tree level

but what is the double copy in this case?

asymptotic states  
weak coupling

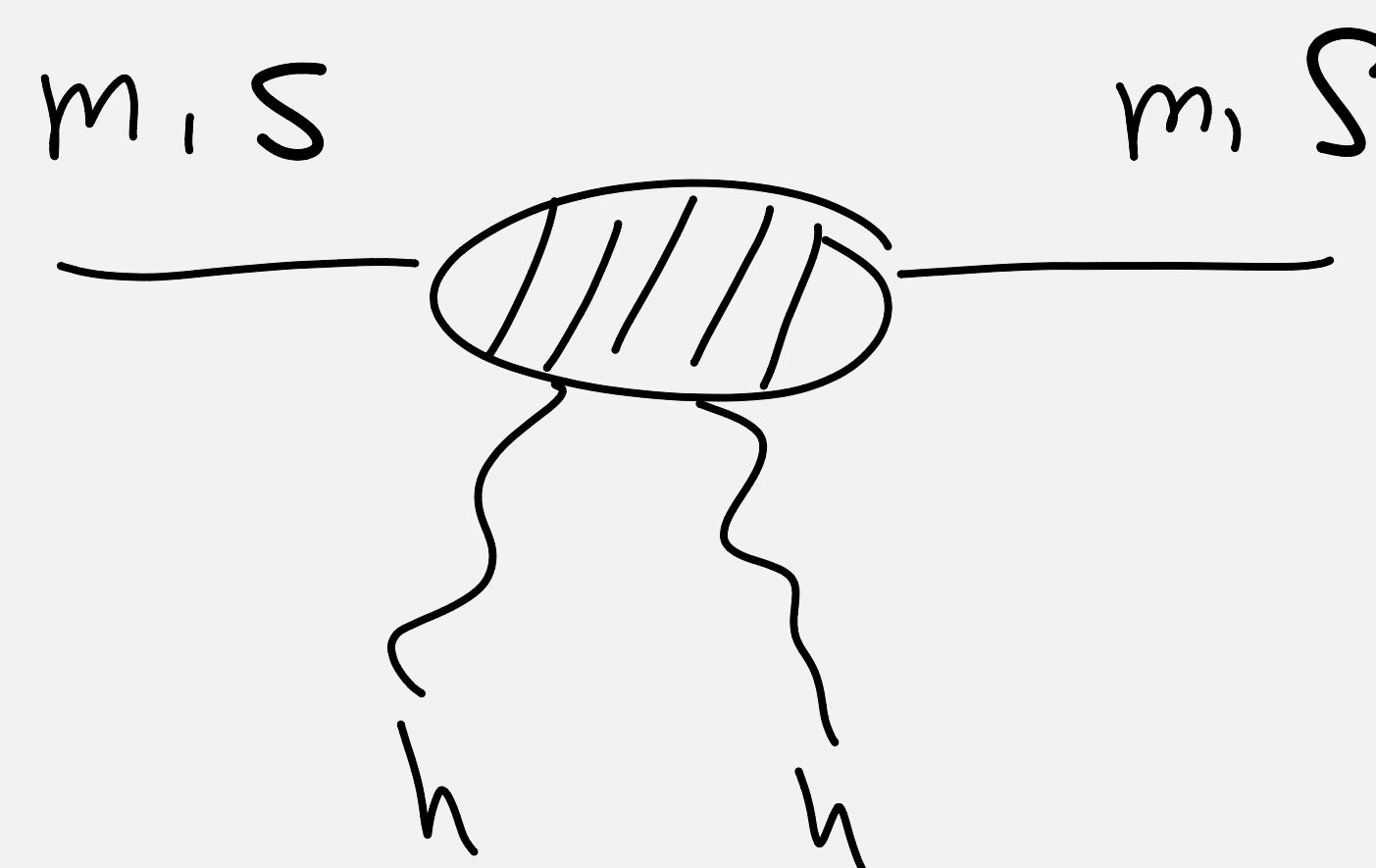
$$\left\{ \begin{array}{l} A_\mu^2 \otimes A_\nu^2 \sim h_{\mu\nu}^2, B_{\mu\nu}^1, \phi^1 \leftarrow \text{dilaton} \\ \Downarrow \\ \text{d} \ln B_{\mu\nu} \leftrightarrow \text{d} \ln \phi \leftarrow \text{axion} \\ \frac{q}{2} \otimes \frac{\bar{q}}{2} \sim \frac{V_\mu}{3} \text{ (Proca field)}, \psi \text{ (complex)} \\ 2 \times 2 \qquad \qquad \qquad 3 \qquad \qquad \qquad + \qquad \qquad \qquad 1 \end{array} \right.$$

What can one do with this gravitational th. ?

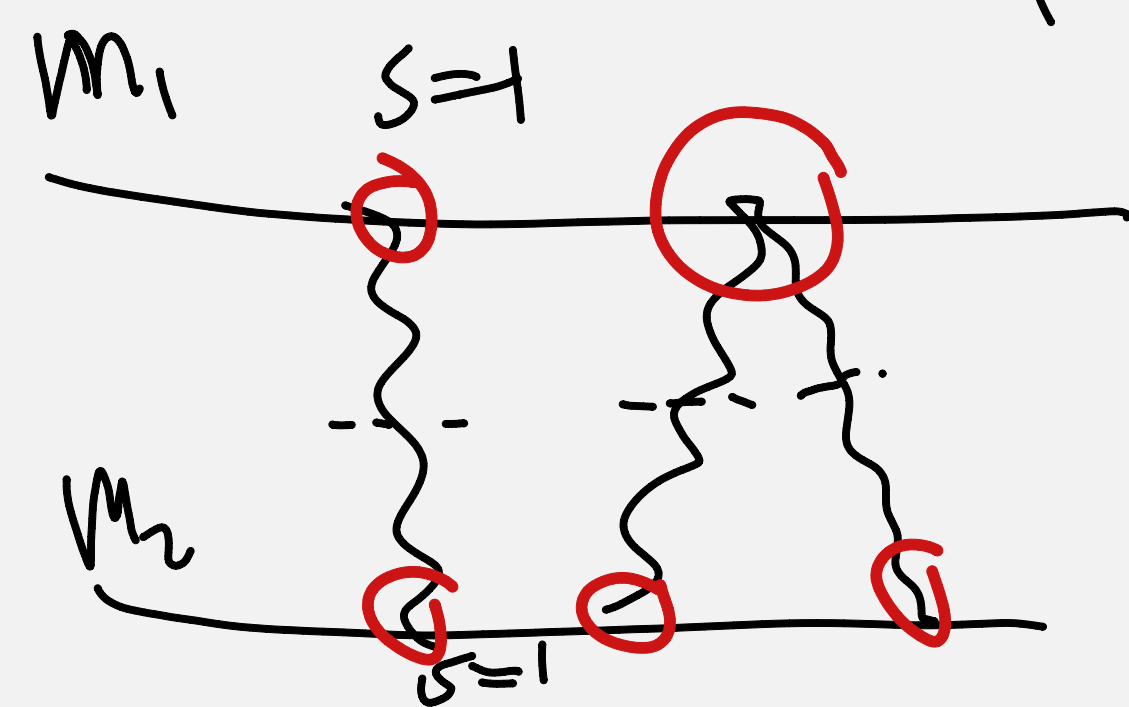
1) Set  $m_f = 0$ ,  $N_f = -1 \Rightarrow$  cancellation of  $\phi, a$  with  $\psi$  (glab) similar to Faddler-Popov

2) Consider classical scattering of Proca field

Compton scattering

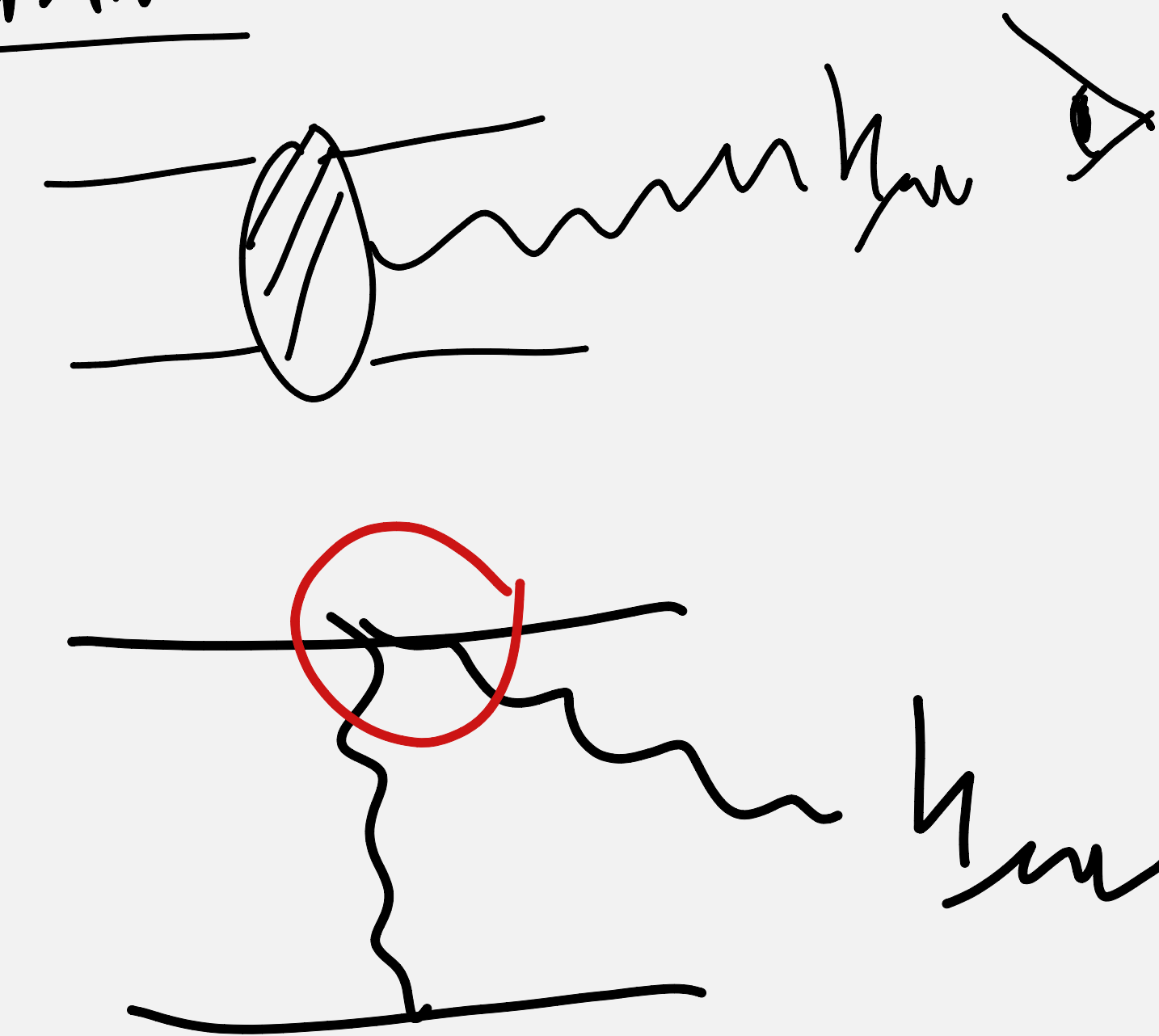


Conservable  $2 \rightarrow 2$



Match Kerr BH up to  $\sqrt{\text{spin}}$  quadrupole moment!

Radiation





Classical double copy

'14 [Monteiro, O'Connell, White]

Kerr - Schild metric ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + k_{\mu} k_{\nu} \phi, \text{ where } k_{\mu} \text{ null w.r.t. } g^{\mu\nu}, \eta^{\mu\nu}$$

$$\implies g^{\mu\nu} = \eta^{\mu\nu} - k^{\mu} k^{\nu} \phi, \quad k^{\mu} = \eta^{\mu\nu} k_{\nu}$$

$$\implies \text{Einstein eqns } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \text{ become linear in } h_{\mu\nu} = k_{\mu} k_{\nu} \phi$$

$$\implies A_{\mu} = k_{\mu} \phi \quad \text{solve } \partial_{\mu} F^{\mu\nu} = 0 \quad \text{Maxwell's eqns}$$



## Schwarzschild solution

$$k_{\mu\nu} = (1, \hat{r}) \quad , \quad \phi = \frac{2GM}{r}$$

## Coulomb solution

$$k_{\mu\nu} = (1, \hat{r}) \quad , \quad \phi = \frac{Q}{4\pi r} \quad , \quad A_{\mu\nu} \rightarrow A_{\mu\nu} - d_{\mu\nu} \left( \frac{Q}{4\pi} \log r \right)$$

# Kerr solution

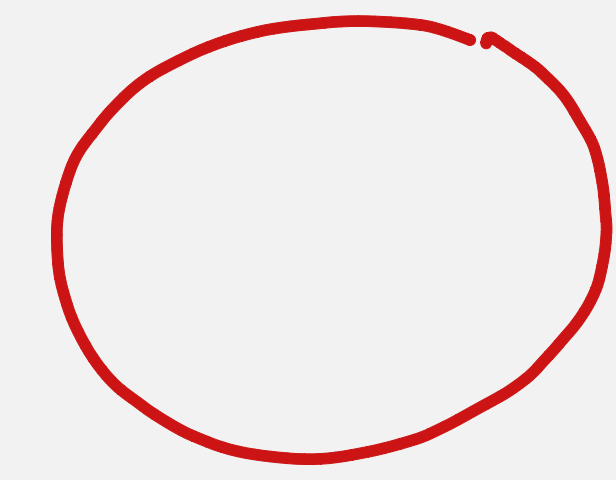
$$\phi(r) = \frac{2MG}{r + a^2 z^2 / r^3}$$

$$k_{\mu} k^{\mu} = 0, \quad k_{\mu} = \left( 1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right)$$

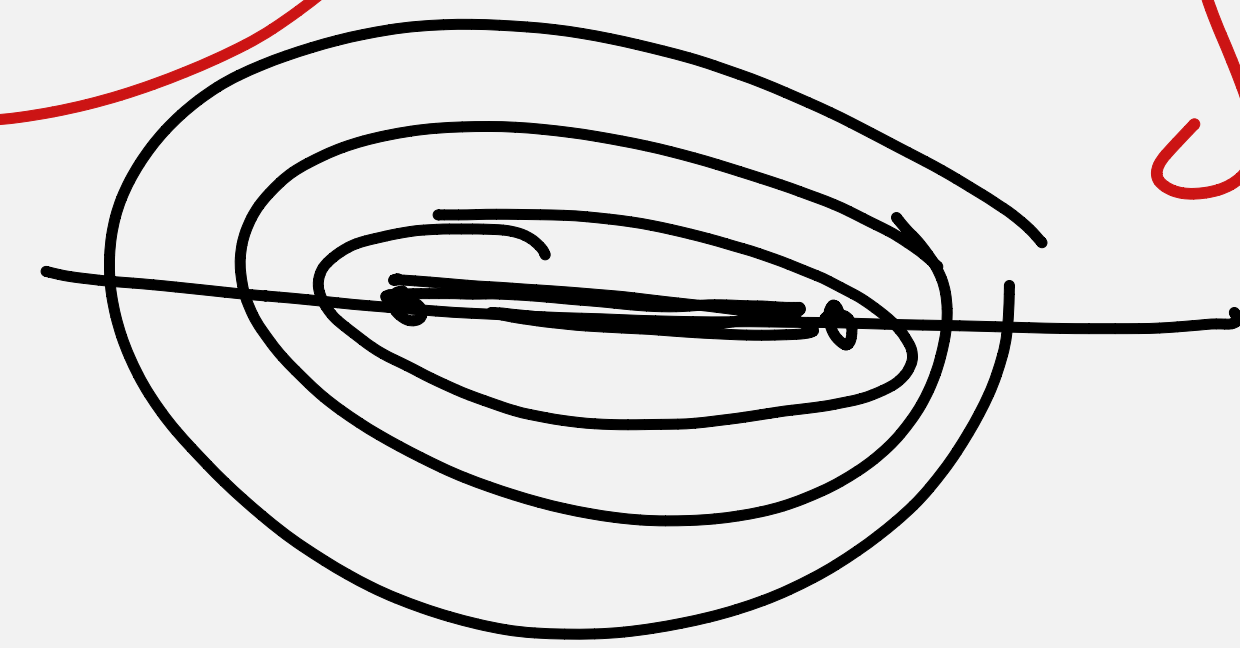
spheroidal coordinates:  $\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$

## Root-Kerr

$$A_{\mu} = \frac{qk_{\mu}}{r} \phi, \quad \text{Massive, spinning, charged disk}$$



$$\int \mathcal{M} = ma^{\mu}$$



$$\int \mathcal{F}_0 = Q$$

# Kerr BH 3pt amplitudes & Spin

'17, Arkani-Hamed, (Huang)<sup>2</sup>  
 $ma^m = S^m$



$$M(1^s, 2^s, 3^{++}) = \left[ (p_1 - p_2) \cdot \epsilon_3^+ \right]^2 \frac{\langle 12 \rangle^{2s}}{m^{2s}}$$

$$\sum_{l=0}^{2s} c_l \langle 12 \rangle^{2s-l} [12]^l$$

$$\begin{matrix} \hbar \rightarrow 0 \\ s \rightarrow \infty \end{matrix}$$

$$= \left[ (p_1 - p_2) \cdot \epsilon_3^+ \right]^2 e^{\mathcal{R}_m S^m/m}$$

[Vines '17]

[Guevara, Ochirov, Vines]

Massive Weyl spinors & momenta

$$p^\mu = k^\mu + m^2 q^\mu, \quad \text{with} \quad k^2 = q^2 = 0$$

$$\boxed{2k \cdot q = 1}$$

$$p^2 = m^2$$

$$\boxed{I=1,2 \text{ } SU(2)}$$

$$p_\mu \sigma^\mu = |k\rangle [k| + m^2 |q\rangle [q| = |p^\mp\rangle [p_\mp|$$

$$= (\lambda_k \tilde{\lambda}_k + m^2 \lambda_q \tilde{\lambda}_q)$$

$$|p^\mp\rangle = \begin{pmatrix} |k\rangle \\ m|q\rangle \end{pmatrix}, \quad [p_\mp] = \begin{pmatrix} [k] \\ m[q] \end{pmatrix}$$

$$\langle kq \rangle = [kq] = 1$$

Consider a QM wavefn  $|\psi\rangle = z_1 |\uparrow\rangle + z_2 |\downarrow\rangle$

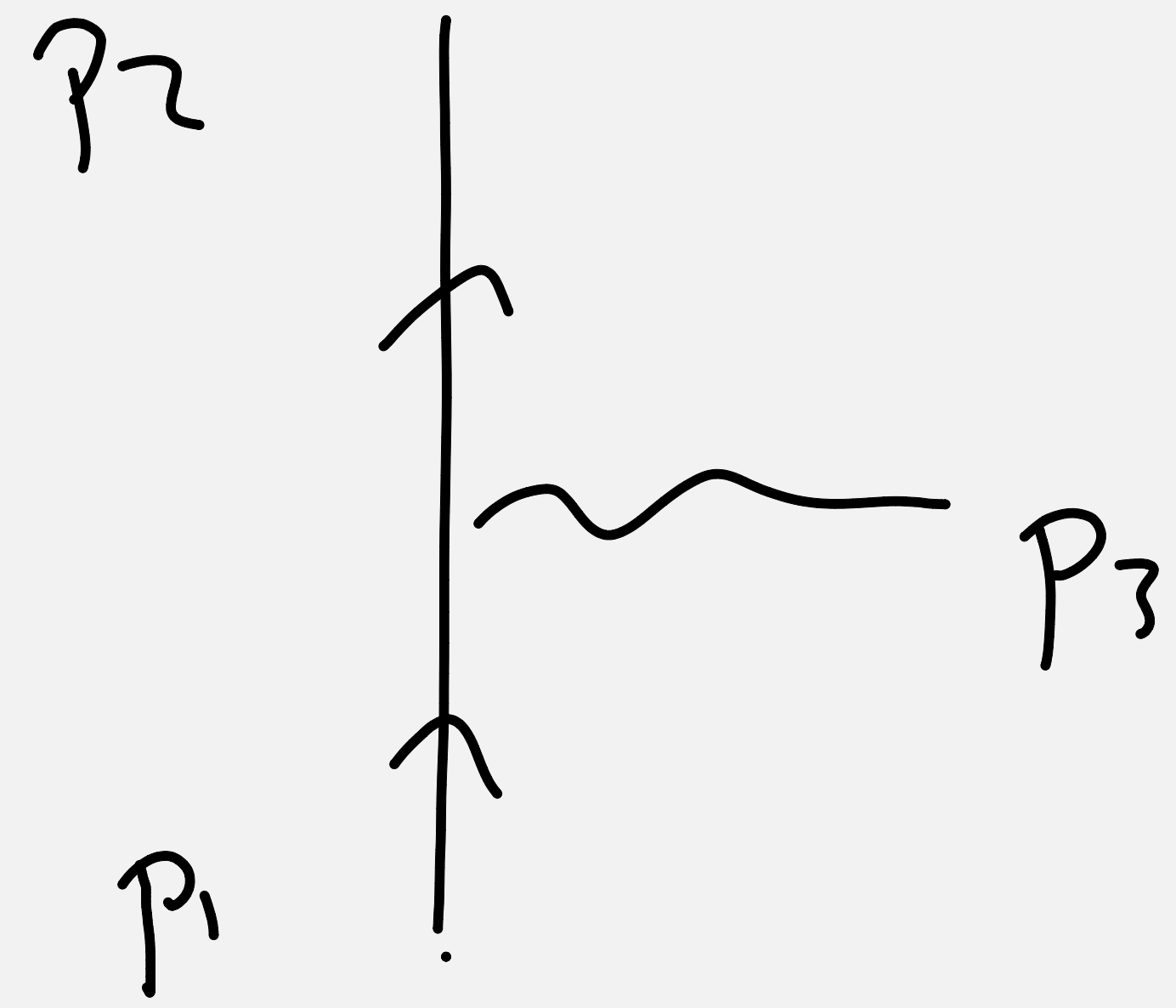
||  
 coeff.      ↑ Basis

$$|p\rangle = |p^1\rangle z_1 + |p^2\rangle z_2 = |p^I\rangle z_I$$

$$\langle 12 \rangle^{2s} = \left( -\bar{z}_I \langle p_2^I | p_1^J \rangle z_j \right)^{2s} = f(\langle \hat{S}^M \rangle)$$

where  $\hat{S}_{(s)}^M = \frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} p_\nu M_{\rho\sigma}^{(s)}$  Pauli-Lubanski pseudo-vector

what is  $\langle \hat{S}^M \rangle = \langle 1 | \hat{S}^M | 1 \rangle^{2s}$



$$p_1 \cdot p_3 = 0$$

$$p_2 \cdot p_3 = 0$$

$$p^2 = 0$$

Burst

$$|2\rangle = |1\rangle + \frac{p_3 \cdot \sigma}{2m} |1\rangle$$

$$\begin{aligned} \langle 12 | &= \langle 1 | + \frac{1}{m} \langle 1 | \sigma^\mu | 1 \rangle p_{3\mu} \\ &= 1 + \frac{1}{m} \langle \hat{S}_{(1/2)}^\mu \rangle p_{3\mu} \end{aligned}$$

$$\begin{aligned} \langle 12 \rangle^{2s} &= \left( 1 + \frac{1}{2m} \langle \hat{S}_{(1/2)}^\mu \rangle p_{3\mu} \right)^{2s} = \sum_{k=0}^{2s} \left( \frac{\langle \hat{S}_{(1/2)}^\mu p_3 \rangle}{m} \right)^k \frac{(2s)!}{(2s-k)! k!} \\ &\left[ \left( \hat{S}_{(s)}^\mu \right)^k = \frac{(2s)!}{(2s-k)!} \left( \hat{S}_{(1/2)}^\mu \right)^k + \dots \right] \end{aligned}$$

$$\langle 12 \rangle^{2s} = \left\langle \sum_{k=0}^{2s} \frac{1}{k!} \left( \frac{\hat{S}(s) \cdot P_3}{m} \right)^k \right\rangle = \left\langle e^{\frac{\hat{S}(s) \cdot P_3}{m}} \right\rangle$$

$$\begin{aligned} \hbar &\rightarrow 0 \\ s &\rightarrow \infty \end{aligned}$$

$$\langle \hat{S}^2 \rangle = \langle \hat{S} \rangle^2$$

$$e^{ix \cdot P_3} \sim h^{m\nu}(x)$$

$$= e^{\frac{\langle \hat{S}^M \rangle P_{3M}}{m}} = e^{a \cdot P_3} \quad a^1 = \frac{\langle \hat{S}^M \rangle}{m} \quad \text{classical spin}$$

Newman-Janis shift

Schwarzschild  $\rightarrow$  Kerr

$$x^M \rightarrow x^M + ia^M$$

$$A(1^s 2^s 3^+) = (p_1 - p_2) \cdot \epsilon_3^+ \left( \frac{\langle 12 \rangle}{m} \right)^{2s}$$

$$A(1^s, 2^s, 3^+) = (P_1 - P_2) \cdot \epsilon_3^+ \left( \frac{\langle 12 \rangle}{m} \right)^{2s}$$

$S=0$  charged scalar & EM

$S=1/2$  QED

$S=1$  W-boson

$S=3/2$   $N=2$  gauged super

$S=2$  ?

⋮

$$M(1^s, 2^s, 3^{++}) = \left( (P_1 - P_2) \cdot \epsilon_3 \right)^2 \left( \frac{\langle 12 \rangle}{m} \right)^{2s}$$

$S=0$  minimally coupled scalar

$S=1/2$  " " fermion

$S=1$  " " Proca

$S=3/2$  " " gravitino

$S=2$  Kaluza-Klein graviton

$S \geq 2$  ?

⋮