The Gravitational Double Copy and Spin

Refs

- 1909.01358 "BCJ review"
- 2203.13013 "SAGEX review"
- 2204.06547 "Snowmass White paper"

Double copy: • Ampl's in GR can be obtained from YM ampl's

• $GR \sim (YM)^2$

Classical solu's version

• Schwarzschild $\sim$ (Coulomb)^2

$\downarrow$ spin $\qquad\qquad \downarrow$ spin

• Kerr solution $\sim$ ("root-Kerr")^2

YM

$$\frac{1}{4 g_{YM}} \mathrm{Tr}\,(F_{\mu\nu})^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]$$

EoM: $D_\mu F^{\mu\nu} = 0$

<u>Weak Coupling</u>

$$A^a_\mu = c^a \varepsilon_\mu e^{ip\cdot x}$$

<u>Polarizations</u> $\varepsilon\cdot p = 0$, $\varepsilon\cdot\varepsilon = 0$

$\varepsilon^\mu \sim \varepsilon^\mu + p^\mu$ (gauge inv.)

GR

$(\kappa^2 = 32\pi G_N)$

$$\frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$h_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu e^{ip\cdot x}$$

Two states

ε^+_μ, ε^-_μ gluons

$\varepsilon^+_\mu \varepsilon^+_\nu$, $\varepsilon^-_\mu \varepsilon^-_\nu$ gravitons

3pt amplitudes

$$\mathcal{A}(123) = g_{YM} f^{a_1 a_2 a_3} \overbrace{\left(\eta_{\mu\nu} (p_1 - p_2)_\rho + \text{cyclic} \underset{123}{(\mu\nu\rho)} \right)}^{V_{123}} \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho$$

$$\mathcal{M}(123) = \frac{\kappa}{2} \left[\left(\eta_{\mu\nu} (p_1 - p_2)_\rho + \text{cyclic} \underset{123}{(\mu\nu\rho)} \right) \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho \right]^2$$

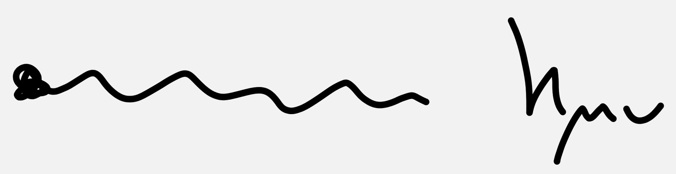
$\xrightarrow{\text{helicity basis}}$

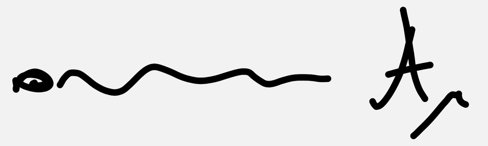
$$A(1^- 2^- 3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \quad \mathcal{M}(1^{--}, 2^{--}, 3^{++}) = \left(\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \right)^2$$

Diagrammatic double copy

$\epsilon_\mu e^{ip\cdot x}$

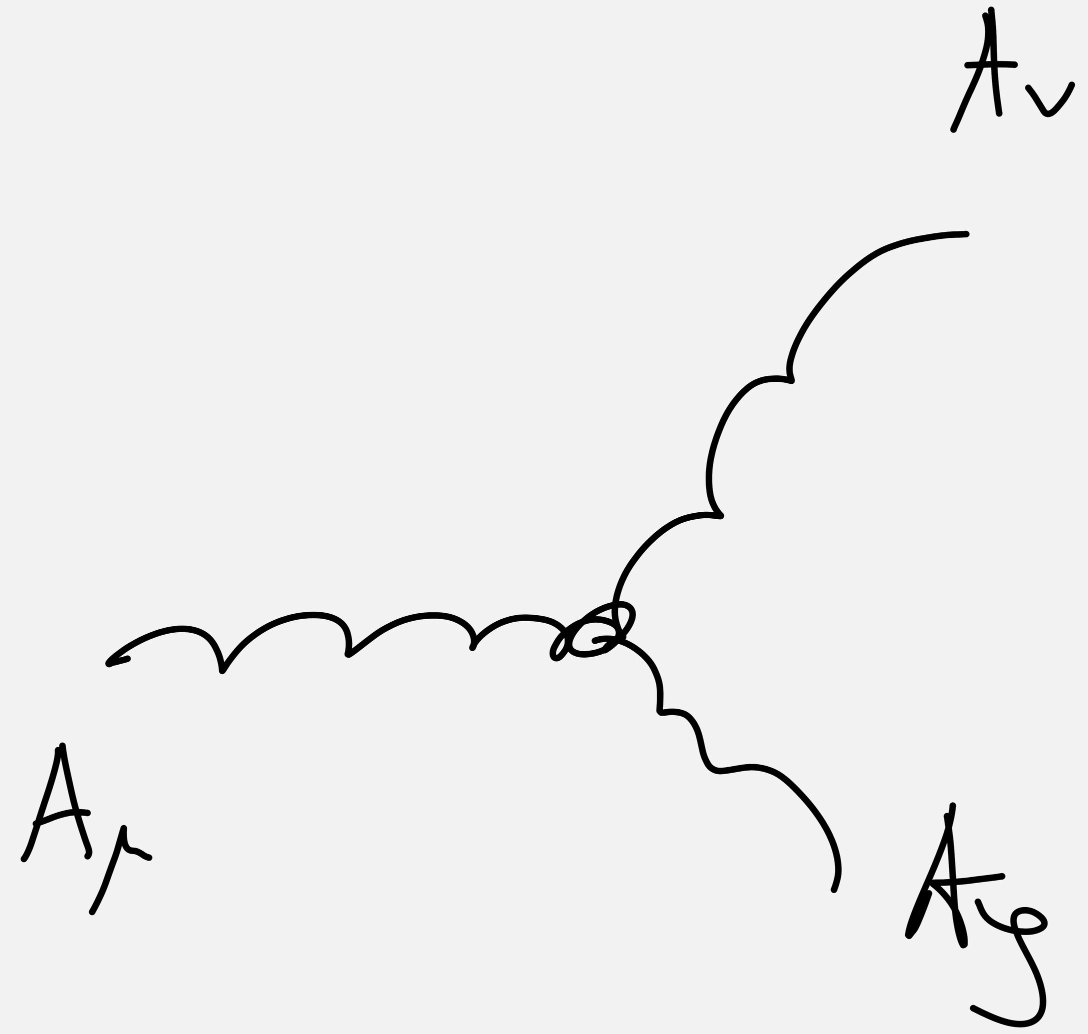
$\Longleftrightarrow$



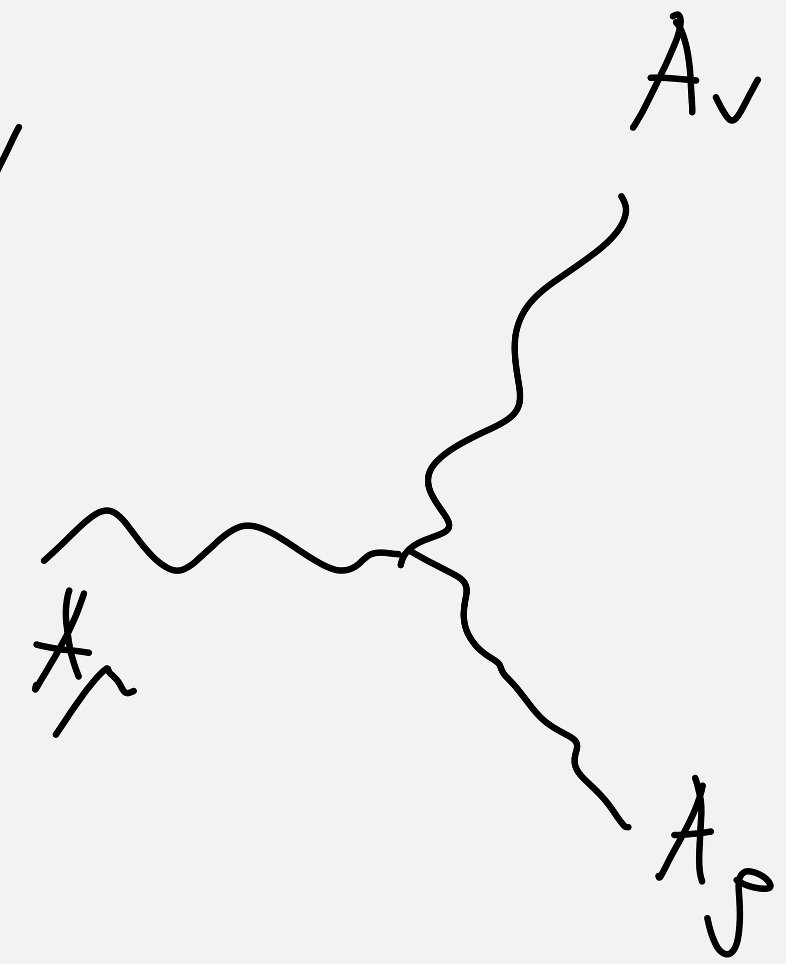
$= \Big($ 

$\Big)^2$

$\epsilon_\mu \epsilon_\nu e^{ip\cdot x}$

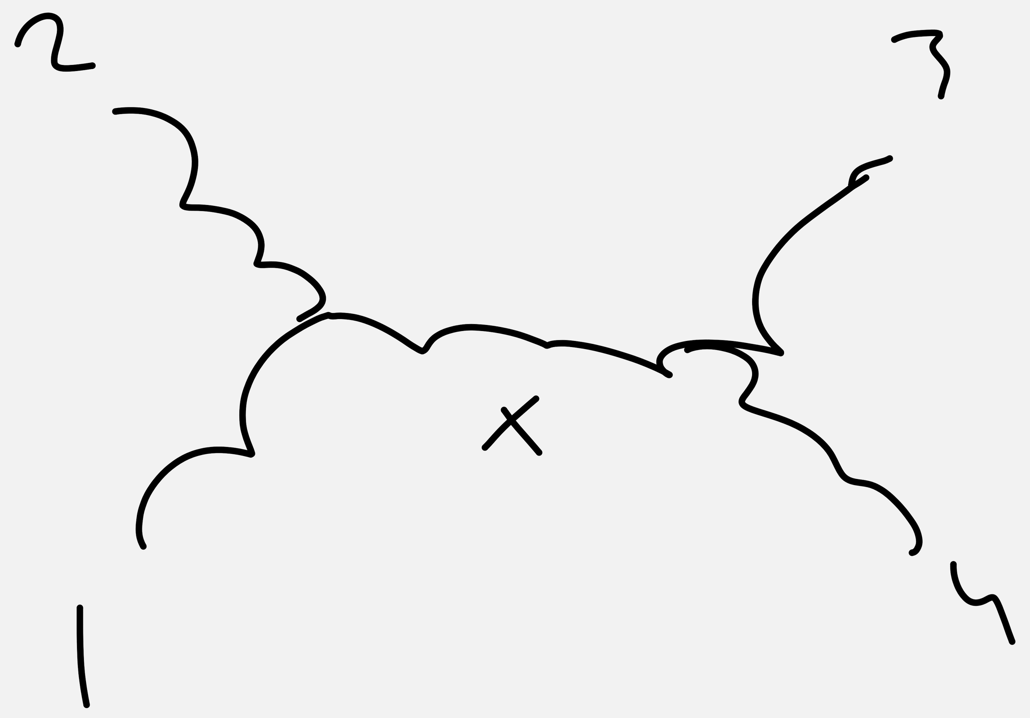
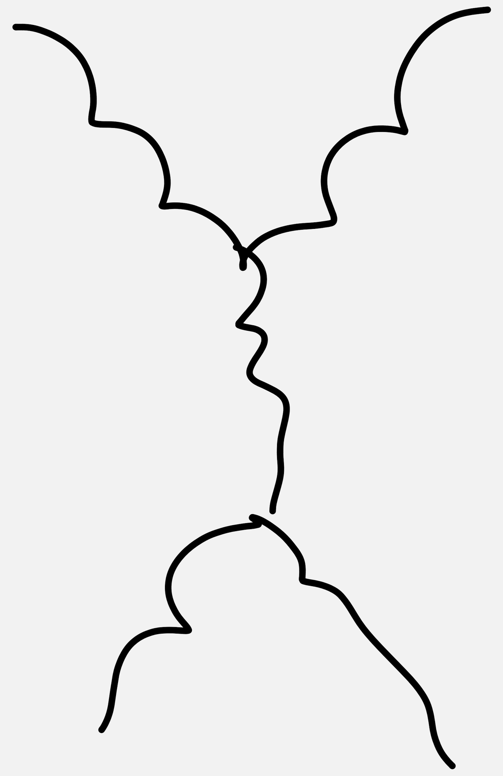
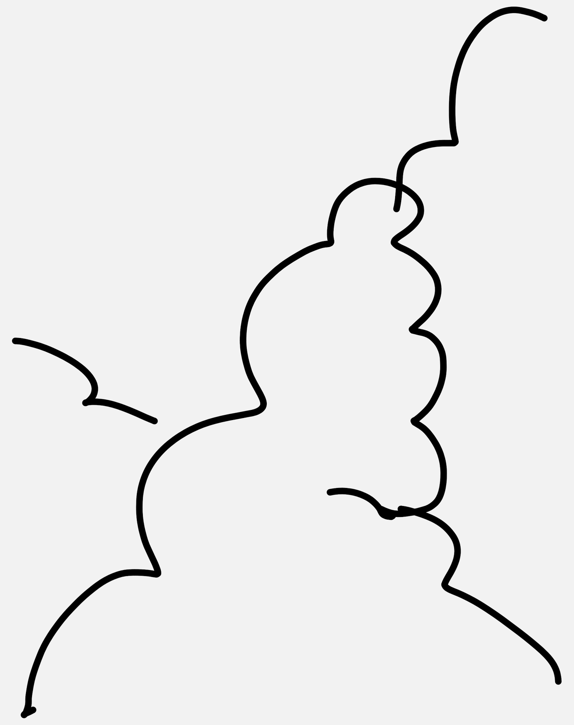
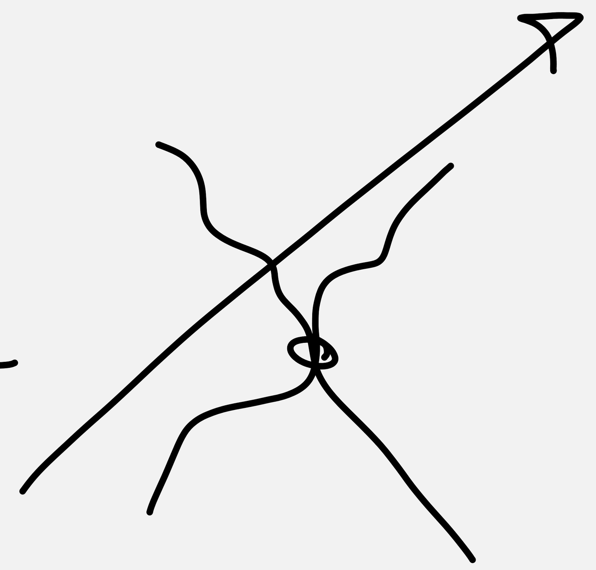


$\Longleftrightarrow$

$= \Bigg($ 

$\Bigg)^2$

4pt Tree amplitudes

$A_4^{YM} =$  $+$  $+$  $+$ 

0 absorbed into n_s, n_u, n_t

$$= \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

$$n_s = V_{12x}^{\mu} V_{\mu x34} + s\left[(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot\varepsilon_4) - (\varepsilon_1\cdot\varepsilon_4)(\varepsilon_2\cdot\varepsilon_3)\right]$$

$$n_s + n_t + n_u = 0$$

$s = (p_1+p_2)^2$

$+$

$t = (p_1+p_4)^2$

$+$

$u = (p_1+p_3)^2$

$=$

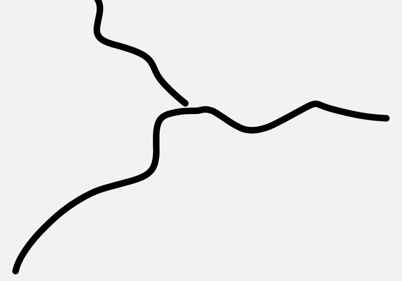
0

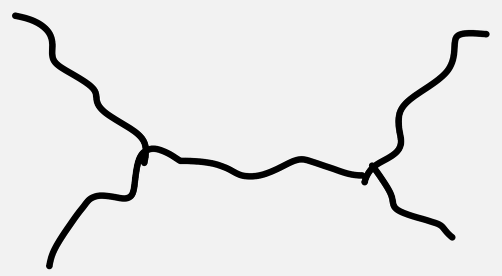
$c_s = f^{a_1 a_2 x} f^{x a_3 a_4}$

$c_t =$ perm

$c_u =$ perm

GR

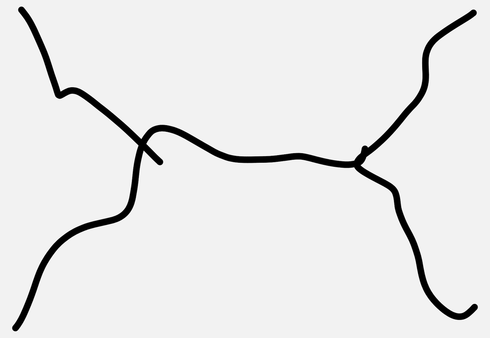
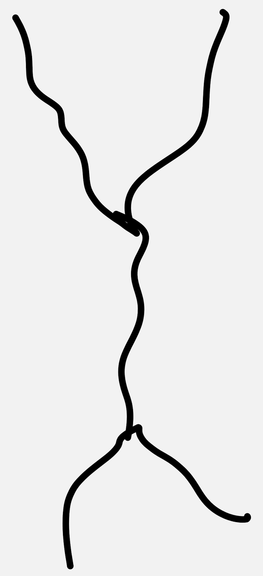
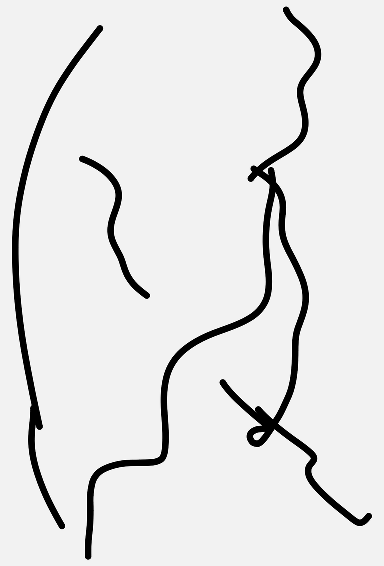
 ~ 100 ,  ~ 3

 $= 100^2 \times 3 = 30000$

SU(N_c) Lie algebra

$$f^{a_1 a_2 x} f^{x a_3 a_4} + \text{cyclic}(123) = c_s + c_t + c_u = 0$$

Kinematic Lie algebra ((S → Volume-pres Diffeos))

$$n_s + n_t + n_u = 0 =$$ $n\Big($  $\Big) + n\Big($  $\Big) + n\Big($  $\Big)$

Gravitational double copy

$$\mathcal{M}_4^{GR} = A_4^{YM}\Big|_{c_i \to n_i} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

Why is this GR?

Claim: tree-level GR uniquely fixed by

1) Unitarity $\Longrightarrow$ factorization & physical states h^{++}, h^{--}

2) 2-derivative interactions

3) Diffeo inv.

Check Diffeo inv.

$$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta A_\mu = \partial_\mu \xi$$

gauge parameters

Consider only variation of ε_4, $\delta \varepsilon_4 = p_4$, $\delta \varepsilon_1, \varepsilon_2, \varepsilon_3 = 0$

Lie algebra

$$\delta A_4^{YM} = \frac{c_s\, \delta n_s}{s} + \frac{c_t\, \delta n_t}{t} + \frac{c_u\, \delta n_u}{u} = \alpha\,(c_s + c_t + c_u) = 0$$

$$\left(\text{HW:} \quad \frac{\delta n_s}{s} = \frac{\delta n_u}{u} = \frac{\delta n_t}{t} = \alpha(\varepsilon_i, p_i)\right)$$

Kinematic algebra

$$\delta M_4^{GR} = 2\left(\frac{n_s\, \delta n_s}{s} + \frac{n_t\, \delta n_t}{t} + \frac{n_u\, \delta n_u}{u}\right) = \alpha\,(n_s + n_t + n_u) = 0$$

Gravitational double copy @ n points

color factors from f^{abc}'s

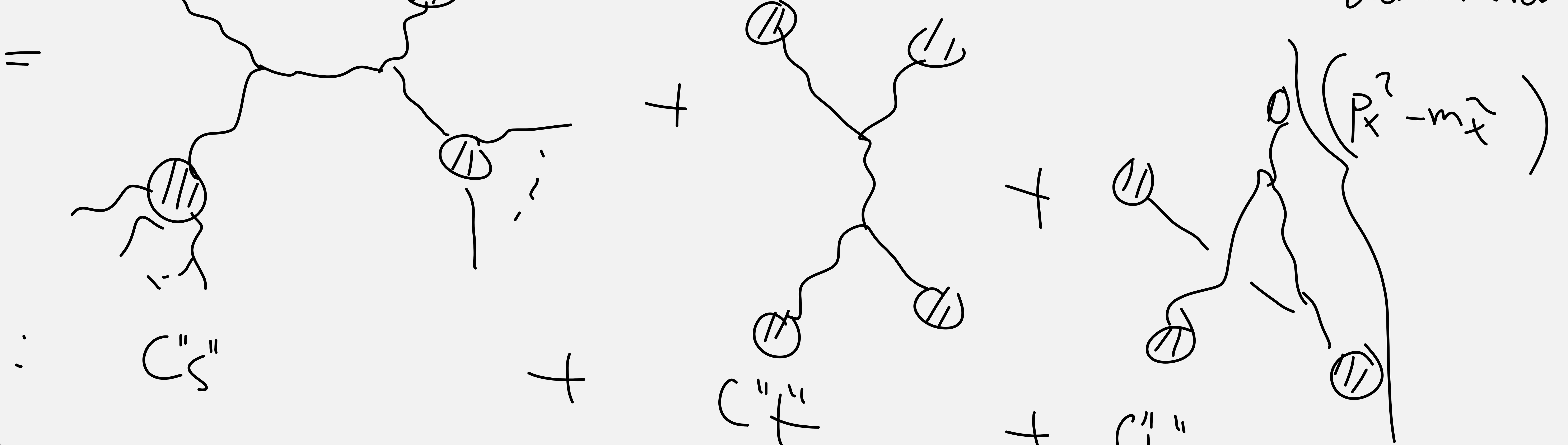
kinematic numerators

Feynman propagator denominator

n-point YM : $\mathcal{A}_n^{YM} = \sum_{i \in \{\text{cubic diagrams}\}} \frac{c_i n_i}{D_i}$

Consider all 3-term Jacobi Id

$\{ \; 0 =$



Color-Lie alg. : $c_{"s"} + c_{"t"} + c_{"u"} = 0$

Kinematic Lie alg $n_s + n_t + n_u = 0$

n-pt GR : $M_n^{GR} = \mathcal{A}_n^{YM}\Big|_{c_i \to n_i} = \sum_i \frac{n_i^2}{D_i}$

Comments

1) Note numerators n_i are gauge dependent

hence non-trivial task to find $n_s + n_t + n_u = 0$

2) Double copy generalizes to loop level

numerators unknown (in general) $\Longleftrightarrow$ kinematical algebra unknown

3) When including matter $\Longrightarrow$ Web of theories that double copy

$$A = \sum_i \frac{n_i c_i}{D_i} \,, \qquad M = \sum_i \frac{n_i^2}{D_i} \,,$$

$$\boxed{\begin{array}{l} U(1) \text{ decoupling Id} \\ A(1243) = A(1324) - A(1234) \end{array}}$$

$$c_i \longleftrightarrow n_i$$

Color - Kinematics duality

$$A_4 = \left.\frac{c_s n_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right|_{c_t = -c_s - c_u}$$

$$= c_s\left(\frac{n_s}{s} - \frac{n_t}{t}\right) + c_u\left(\frac{n_u}{u} - \frac{n_t}{t}\right)$$

$$= c_s A(1234) + c_u A(1324)$$

$$= \sum_{\rho \in S_4/\mathbb{Z}_4} \mathrm{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(4)}}\right) A(\rho(1)\rho(2)\rho(3)\rho(4))$$

$s+t+u=0$

$$\begin{pmatrix} A(1234) \\ A(1324) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{s}+\frac{1}{t} & \frac{1}{t} \\ \frac{1}{t} & \frac{1}{u}+\frac{1}{t} \end{pmatrix}}_{\text{not invertible}} \begin{pmatrix} n_s \\ n_u \end{pmatrix}$$

$$\Rightarrow \quad A(1324) = \frac{t}{u} A(1234)$$

BCJ relations

$\rightarrow (n-3)!$ basis

Add Matter

$$\mathcal{L}_{QCD} = -\frac{1}{4g^2}\,\mathrm{Tr}(F_{\mu\nu})^2 + \bar{q}\,(i\not{D} - m_q)\,q$$

fund rep.

$$SU(N_c) \times SU(N_f)$$

Claim : Color-kinematics duality & double copy works @ tree level

but what is the double copy in this case?

asymptotic states / weak coupling :

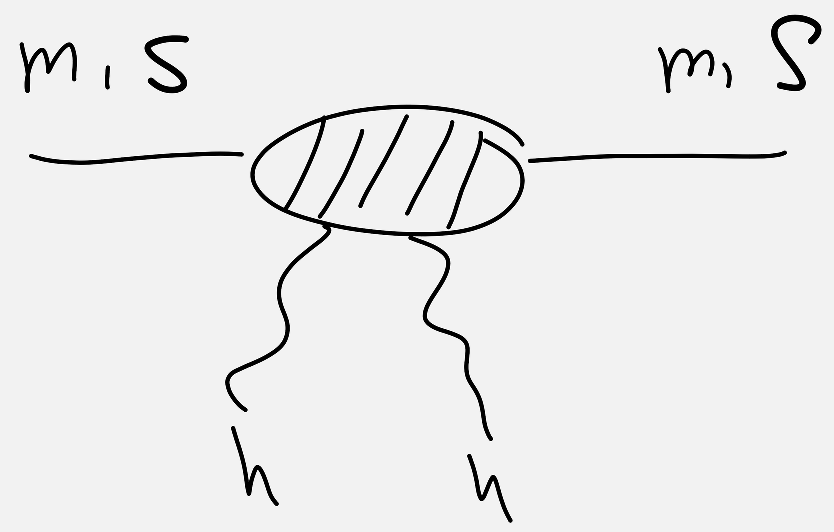
$$\begin{cases} \overset{2}{A_\mu} \otimes \overset{2}{A_\nu} \sim \overset{2}{h_{\mu\nu}},\ \overset{1}{B_{\mu\nu}},\ \overset{1}{\phi} \leftarrow \text{dilaton} \\ \qquad\qquad \epsilon^{\mu\nu\rho\sigma}\partial_{[\nu} B_{\rho\sigma]} \longleftrightarrow \partial^\mu a \leftarrow \text{axion} \\ q \otimes \bar{q} \sim V_\mu \text{ (Proca field)},\ \varphi \text{ (complex)} \\ 2 \times 2 \qquad\ \ 3 \quad + \quad 1 \end{cases}$$

What can one do with this gravitational th. ?

1) Set $m_f = 0$, $N_f = -1$ $\Longrightarrow$ cancelation of ϕ, a with φ (ghost) similar to Faddeev–Popov

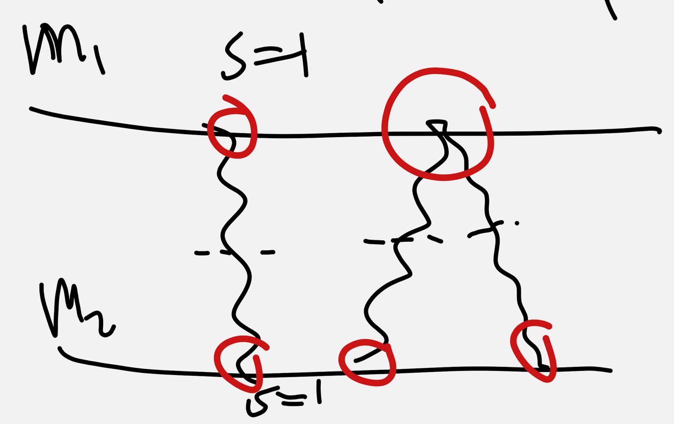
2) Consider classical scattering of Proca field

Compton scattering

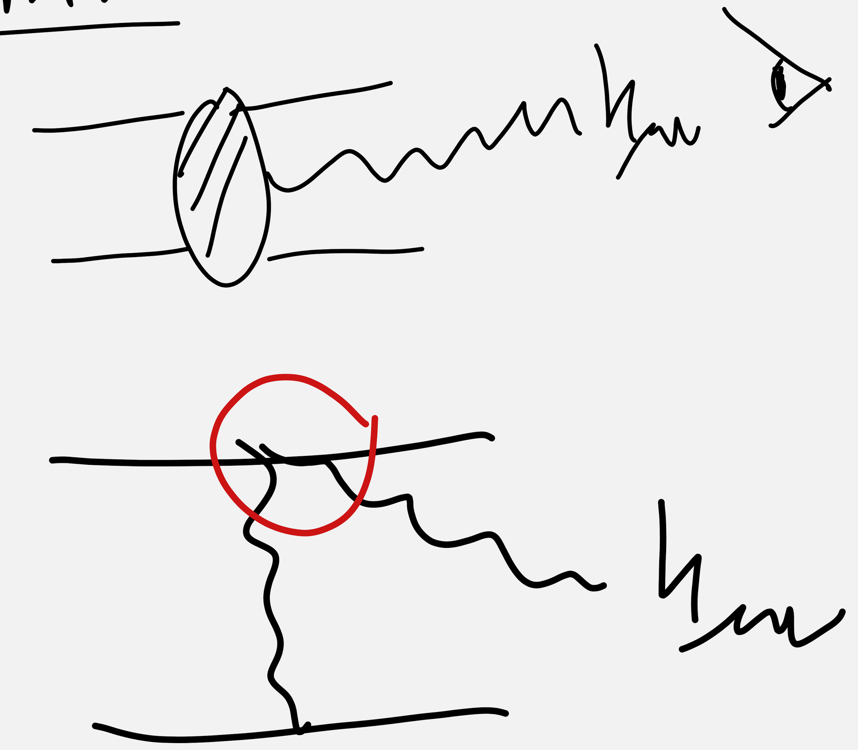


Match Kerr BH up to spin quadrupole moment !

Conservative $2 \to 2$



Radiation



Classical double copy '14 [Monteiro, O'Connell, White]

Kerr – Schild metric ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi \quad , \text{ where } k_\mu \text{ null w.r.t. } g^{\mu\nu}, \eta^{\mu\nu}$$

$$\Longrightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} - k^\mu k^\nu \phi \quad , \quad k^\mu = \eta^{\mu\nu} k_\nu$$

$\Longrightarrow$ Einstein eqns $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ become linear in $h_{\mu\nu} = k_\mu k_\nu \phi$

$\Longrightarrow$ $A_\mu = k_\mu \phi$ solve $\partial_\mu F^{\mu\nu} = 0$ Maxwell's eqns

Schwarzschild solution

$$k_\mu = (1, \hat{r}) \quad , \quad \phi = \frac{2GM}{r}$$

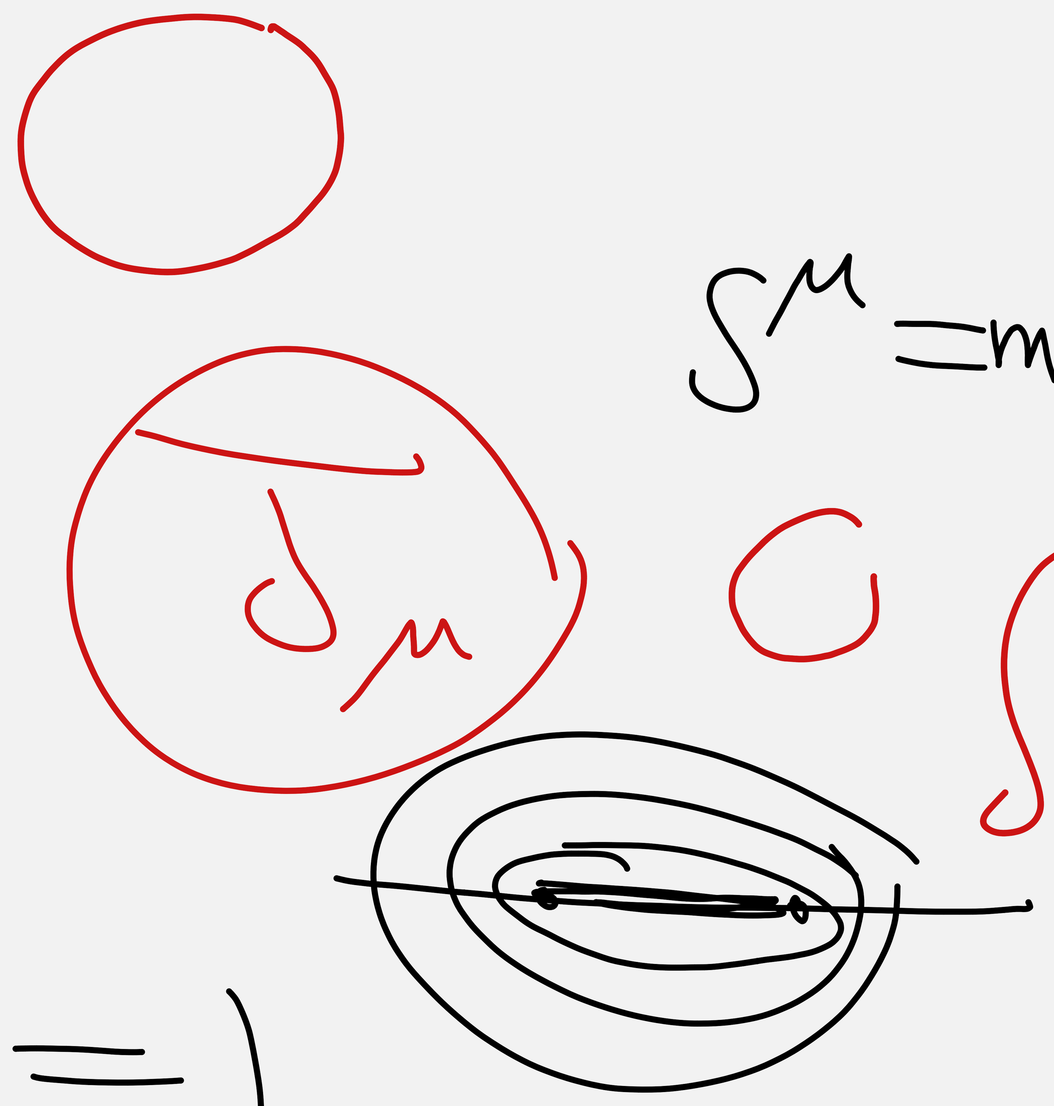
Coulomb solution

$$k_\mu = (1, \hat{r}) \quad , \quad \phi = \frac{Q}{4\pi r} \quad , \quad A_\mu \to A_\mu - \partial_\mu\left(\frac{Q}{4\pi} \log r\right)$$

Kerr solution

$$\phi(r) = \frac{2MG}{r + a^2 z^2/r^3}$$

$$k_\mu k^\mu = 0, \quad k_\mu = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right)$$



$$S^\mu = m a^\mu$$

$$\int J_0 = Q$$

spheroidal coordinates: $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$

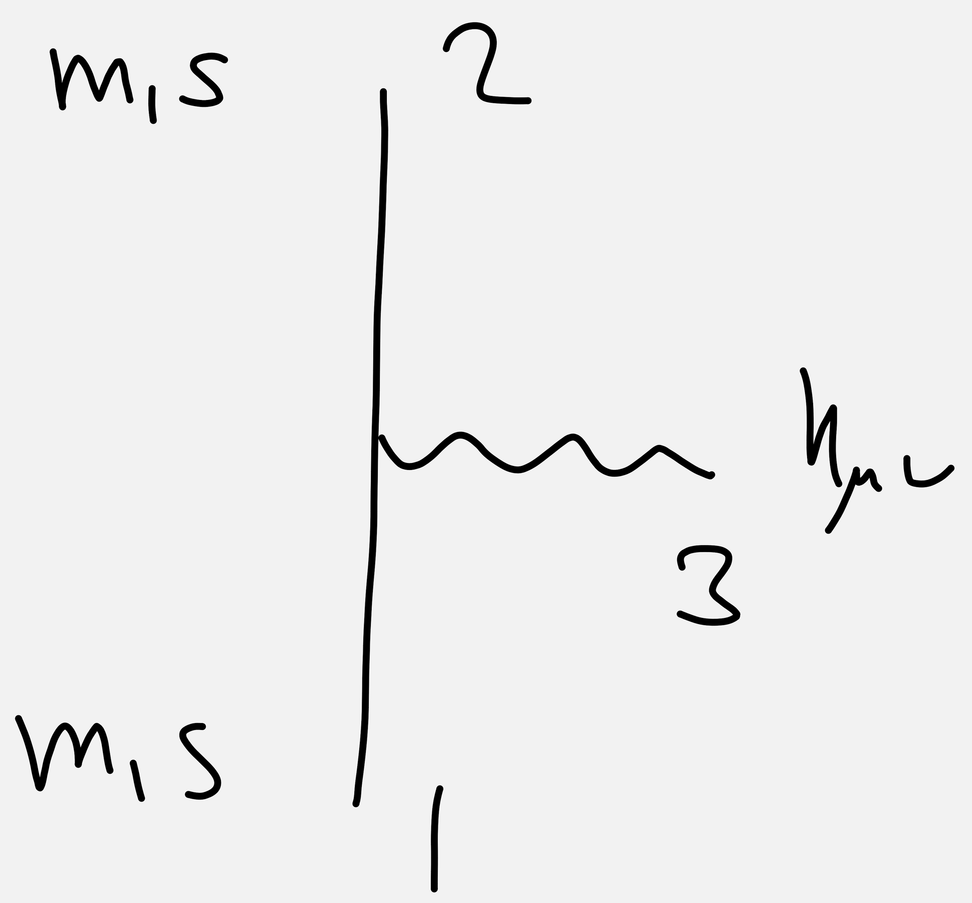
Root-Kerr

$A_\mu = \frac{Q}{4\pi} k_\mu \phi$, Massive, spinning, charged disk

Kerr BH 3pt amplitudes & spin

'17, Arkani-Hamed, (Huang)^2

$ma^\mu = S^\mu$



,

$$M(1^s, 2^s, 3^{++}) = \left[(p_1 - p_2)\cdot \varepsilon_3^+\right]^2 \frac{\langle 12\rangle^{2s}}{m^{2s}}$$

$$\sum_{k=0}^{2s} c_k \langle 12\rangle^{2s-k} [12]^k$$

[Vines '17]

$\hbar \to 0$
$s \to \infty$

$$= \left[(p_1 - p_2)\cdot \varepsilon_3^+\right]^2 e^{p_{3\mu} S^\mu / m}$$

[Guevara, Ochirov, Vines]

Massive Weyl spinors & momenta

$$p^\mu = k^\mu + m^2 q^\mu \quad , \quad \text{with} \quad k^2 = q^2 = 0$$

$$2k\cdot q = 1$$

$$p^2 = m^2$$

$I = 1,2 \quad SU(2)$

$$p_\mu \sigma^\mu = |k\rangle[k| + m^2 |q\rangle[q| = |p^I\rangle[p_I|$$

$$= (\lambda_k \tilde{\lambda}_k + m^2 \lambda_q \tilde{\lambda}_q)$$

$\langle kq\rangle = [kq] = 1$

$$|p^I\rangle = \begin{pmatrix} |k\rangle \\ m|q\rangle \end{pmatrix} \quad , \quad |p_I] = \begin{pmatrix} |k] \\ m|q] \end{pmatrix}$$

Consider a QM wavefn $|\psi\rangle = z_1 |\uparrow\rangle + z_2 |\downarrow\rangle$

z_1 = coeff.

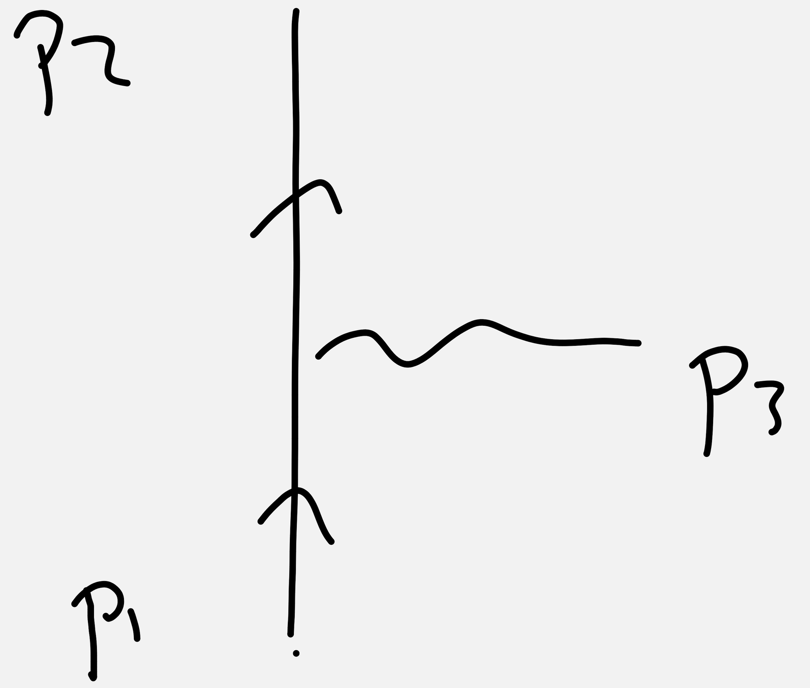
Basis

$$|p\rangle = |p^1\rangle z_1 + |p^2\rangle z_2 = |p^I\rangle z_I$$

$$\langle 12\rangle^{2s} = \left(- \bar{z}_I \langle p_2^I | p_1^J\rangle z_J\right)^{2s} = f\left(\langle \hat{S}^\mu\rangle\right)$$

where $\hat{S}^\mu_{(s)} = \frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} p_{1\nu} M^{(s)}_{\rho\sigma}$ Pauli-Lubanski pseudo-vector

what is $\langle \hat{S}^\mu\rangle = \langle \bar{1}|^{2s} \hat{S}^\mu |1\rangle^{2s}$



$p_1 \cdot p_3 = 0$

$p_2 \cdot p_3 = 0$

$p_3^2 = 0$

<u>Boost</u> $\quad |2\rangle = |\bar{1}\rangle + \dfrac{p_3 \cdot \sigma}{2m} |\bar{1}]$

$$\langle 12\rangle = \langle 1\bar{1}\rangle + \frac{1}{m}\underbrace{\langle 1|\sigma^\mu|\bar{1}]\, p_{3\mu}}$$

$$\begin{aligned} &\ \ \ \ \ \ \ \ \ \ \ \ \ \parallel \\ &= \ \ 1 + \frac{1}{m}\langle \hat{S}^\mu_{(1/2)}\rangle\, p_{\mu 3}\end{aligned}$$

$$\langle 12\rangle^{2s} = \left(1 + \frac{1}{2m}\langle \hat{S}^\mu_{(1/2)}\rangle\, p_{\mu 3}\right)^{2s} = \sum_{k=0}^{2s}\left(\frac{\langle \hat{S}_{1/2}\cdot p_3\rangle}{m}\right)^k \frac{(2s)!}{(2s-k)!\,k!}$$

$$\left[\ \left(\hat{S}^\mu_{(s)}\right)^k = \frac{(2s)!}{(2s-k)!}\left(S^\mu_{(1/2)}\right)^k + \ldots\ \right]$$

$$\langle 12\rangle^{2s} = \left\langle \sum_{k=0}^{2s} \frac{1}{k!}\left(\frac{\hat{S}_{(s)}\cdot p_3}{m}\right)^k \right\rangle = \left\langle e^{\frac{\hat{S}_{(s)}\cdot p_3}{m}} \right\rangle$$

$\hbar \to 0$
$s \to \infty$

$$\langle \hat{S}^2\rangle = \langle \hat{S}\rangle^2$$

$$e^{ix\cdot p_3} \sim h^{\mu\nu}(x)$$

$$= e^{\frac{\langle \hat{S}^\mu\rangle p_{3\mu}}{m}} = e^{a\cdot p_3} \qquad a^\mu = \frac{S^\mu}{m}$$

classical spin

<u>Newman-Janis shift</u>   Schwartzchild $\longrightarrow$ Kerr

$$x^\mu \to x^\mu + i a^\mu$$

$$A(1^s 2^s 3^+) = (p_1 - p_2)\cdot \varepsilon_3^+ \left(\frac{\langle 12\rangle}{m}\right)^{2s}$$

$$A(1^s, 2^s, 3^+) = (p_1 - p_2)\cdot \varepsilon_3^+ \left(\frac{\langle 12\rangle}{m}\right)^{2s}$$

$s=0$ charged scalar & EM

$s=1/2$ QED

$s=1$ W-boson

$s=3/2$ $\mathcal{N}=2$ gauged sugra

$s=2$?

$\vdots$

$$M(1^s 2^s 3^{++}) = \left((p_1-p_2)\cdot\varepsilon_3\right)^2 \left(\frac{\langle 12\rangle}{m}\right)^{2s}$$

$s=0$ minimally coupled scalar

$s=1/2$ " " fermion

$s=1$ " " Proca

$s=3/2$ " " gravitino

$s=2$ Kaluza-Klein graviton

$s\geq 2$?

$\vdots$