Quantum Field Theory on the Lattice

Z. Fodor

Two sets of lattice field theory talks

Michael Creutz: three talks
Zoltan Fodor: four talks

"computational details ... might be better for Zoltan to cover, i.e. things like hybrid monte carlo, the hadron spectrum ... g-2" and QCD thermodynamics.

- Scalar theory, Higgs bound & Monte Carlo
- QCD and hadron spectrum (Wilson)
- QCD thermodynamics (staggered & overlap)

(Krishna: "unless Zoltan is reporting some miracles")

- g-2 of the muon (staggered)

Outline

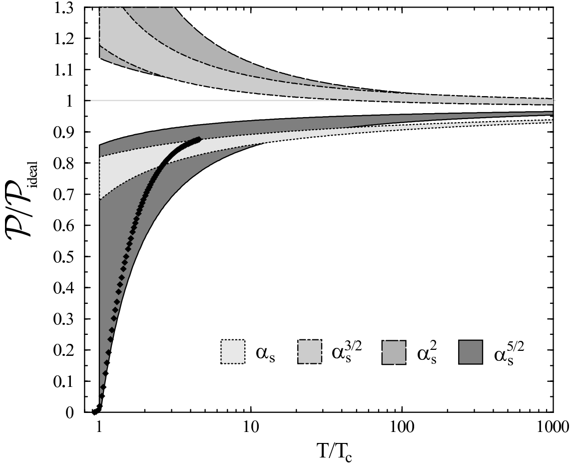
1. Nature of the transition: SU(3) & QCD
2. Transition temperature
3. Equation of state
4. Cumulants
5. Topology at high T

FLAG review of lattice results Colangelo et al. Eur.Phys.J. C71 (2011) 1695

Collaboration	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	$m_{ud,\overline{\rm MS}(2\rm GeV)}$	$m_{s,\overline{\rm MS}(2\rm GeV)}$
PACS-CS 10	P	★	■	■	★	*a*	2.78(27)	86.7(2.3)
MILC 10A	C	•	★	★	•	–	3.19(4)(5)(16)	–
HPQCD 10	A	•	★	★	★	–	3.39(6)*	92.2(1.3)
BMW 10AB	P	★	★	★	★	*b*	3.469(47)(48)	95.5(1.1)(1.5)
RBC/UKQCD	P	•	•	★	★	*c*	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum et al. 10	P	•	■	•	★	–	3.44(12)(22)	97.6(2.9)(5.5)

QCD: need for a systematic non-perturbative method

pressure at high temperatures converges at $T=10^{300}$ MeV



Finite temperature QCD

Quantum system partition function: Hamiltonian H at temperature T:

$$Z = \mathrm{Tr}\left[e^{-H/T}\right] = \int [\mathrm{d}\varphi]\, \langle\varphi|\, e^{-H/T}\, |\varphi\rangle$$

Path integral representastion with

$$\begin{aligned} Z_{\mathrm{QCD}} &= \int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\, e^{-S_{\mathrm{E}}(U,\psi,\overline{\psi})} \\ &= \int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\, \exp\left[\int_0^{1/T} \mathrm{d}x_4 \int \mathrm{d}^3\mathbf{x}\ \mathcal{L}_{\mathrm{E}}(U,\psi,\overline{\psi})\right] \end{aligned}$$

Commuting bosonic & anticommuting (Grassmann) fermionic fields
Boundary condition in the imaginary time (temperature) direction:
Gluons: periodic whereas Quarks: antiperiodic.

Temperature: T=1/$N_t a$, therefore $a \to 0$ is $N_t \to \infty$

Increase of β $\overset{\text{asymptotic freedom}}{\Longrightarrow}$ decrease of a $\Longrightarrow$ increase of T.

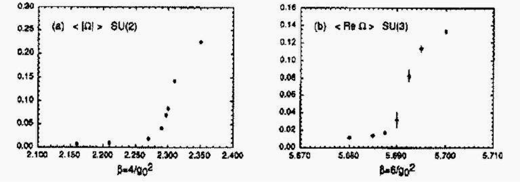
Nature of the transition: finite-size scaling theory

problem with phase transitions in Monte-Carlo studies

Monte-Carlo applications for pure gauge theories ($V = 24^3 \cdot 4$)

existence of a transition between confining and deconfining phases:

Polyakov loop exhibits rapid variation in a narrow range of β



• theoretical prediction: SU(2) second order, SU(3) first order

$\Longrightarrow$ Polyakov loop behavior: SU(2) singular power, SU(3) jump

data do not show such characteristics!

The nature of the SU(3) & QCD transitions

finite size scaling study of the Polyakov/chiral susceptibilities

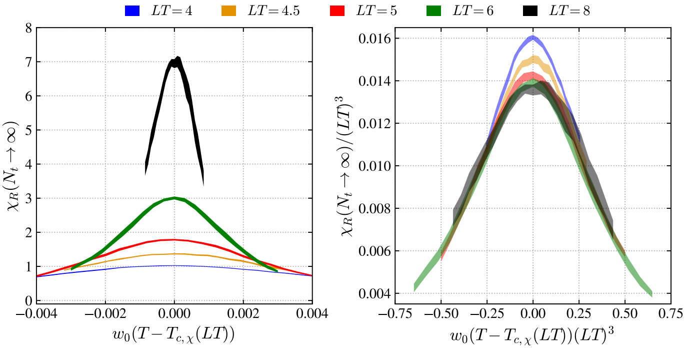
$\chi_P = N_s^3 \left(\langle |P|^2 \rangle - \langle |P| \rangle^2 \right) \qquad \chi = (T/V) \partial^2 \log Z / \partial m^2$

phase transition: finite V analyticity V$\to \infty$ increasingly singular
(e.g. first order phase transition: height $\propto$ V, width $\propto$ 1/V)
for an analytic cross-over χ does not grow with V

two steps (three-five volumes, four-five lattice spacings):
a. fix V and determine χ through a continuum extrapolation
b. using the continuum extrapolated χ_{max}: finite size scaling

Volume dependence of the susceptibility: SU(3)

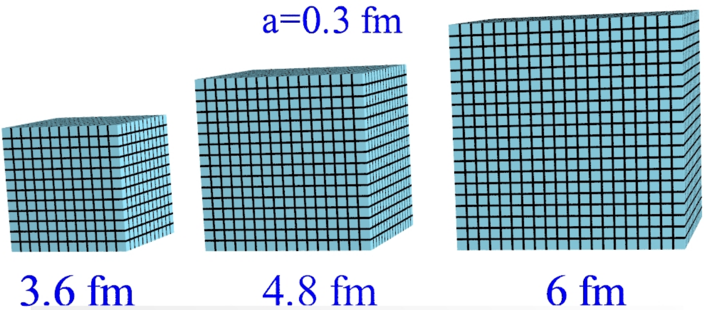
continuum extrapolated renormalized Polyakov loop susceptibilities narrower and higher: rescale it with the volume:

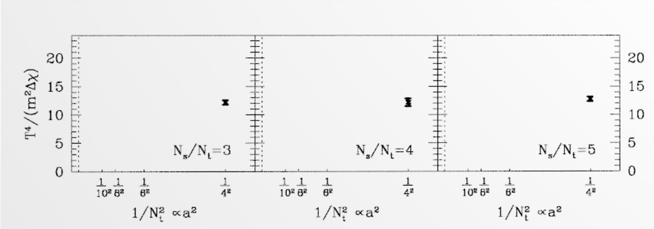


w_0 scale parameter to make it dimensionless

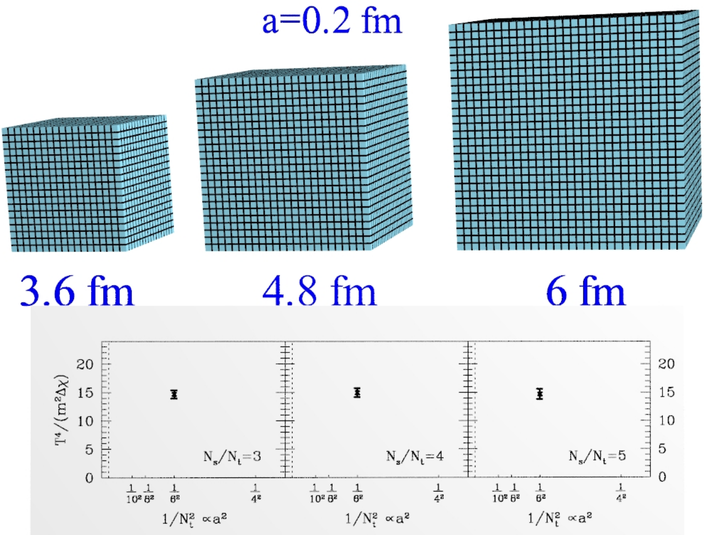
1/V and V behavior ⇒ first order phase transition

Approaching the continuum limit: QCD

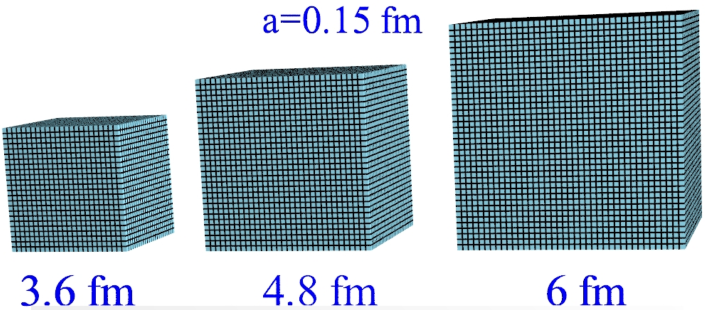


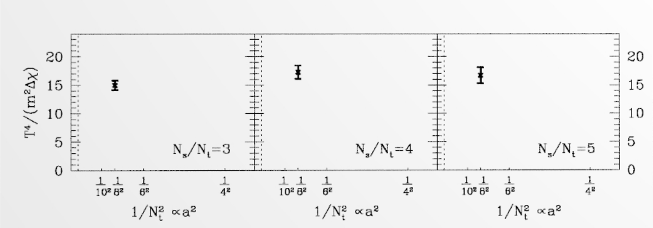


Approaching the continuum limit: QCD

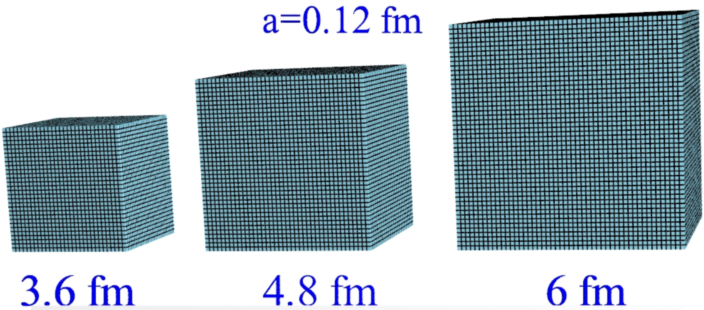


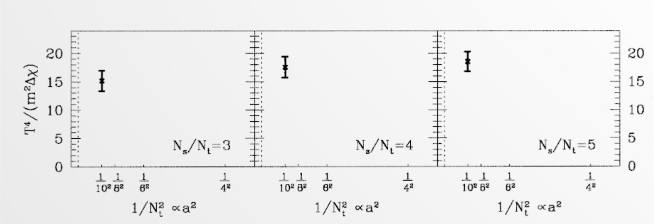
Approaching the continuum limit: QCD



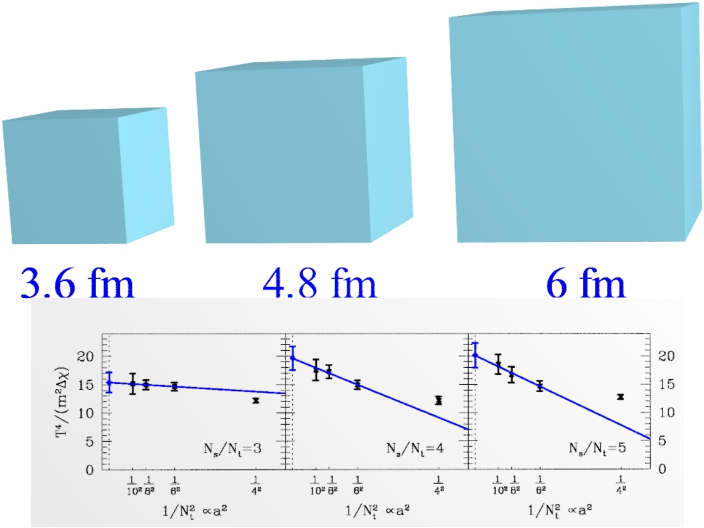


Approaching the continuum limit: QCD



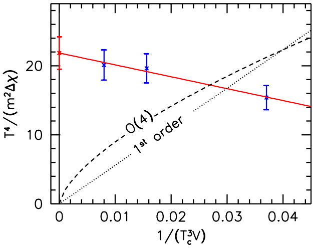


Approaching the continuum limit: QCD



The nature of the QCD transition: analytic

- finite size scaling analysis with continuum extrapolated $T^4/m^2\Delta\chi$



the result is consistent with an approximately constant behavior
for a factor of 5 difference within the volume range
chance probability for 1/V is 10^{-19} for O(4) is $7 \cdot 10^{-13}$
continuum result with physical quark masses in staggered QCD:

the QCD transition is a cross-over

Literature: discrepancies between T_c

Bielefeld-Brookhaven-Riken-Columbia Collaboration:

M. Cheng et.al, Phys. Rev. D74 (2006) 054507

T_c from $\chi_{\bar{\psi}\psi}$ and Polyakov loop, from both quantities:

T_c=192(7)(4) MeV

Bielefeld-Brookhaven-Riken-Columbia merged with MILC: 'hotQCD'

Wuppertal-Budapest group: WB

Y. Aoki, Z. Fodor, S.D. Katz, K.K. Szabo, Phys. Lett. B. 643 (2006) 46

chiral susceptibility: T_c=151(3)(3) MeV

'chiral T_c': $\approx$40 MeV difference both groups give continuum

extrapolated results with physical m_π

freeze out: 172 MeV $\rightarrow$ dramatic differences in physics:

need for strongly interacting hadronic matter

Discretization errors in the transition region

we always have discretization errors: nothing wrong with it as long as

a. result: close enough to the continuum value (error subdominant)
b. we are in the scaling regime (a^2 in staggered)

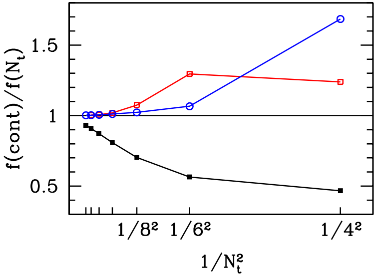
various types of discretization errors $\Rightarrow$ we improve on them (costs)

we are speaking about the transition temperature region
interplay between hadronic and quark-gluon plasma physics
smooth cross-over: one of them takes over the other around T_c

both regimes (low T and high T) are equally important
improving for one: T$\gg T_c$, doesn't mean improving for the other: T$<T_c$

Examples for improvements, consequences

how fast can we reach the continuum pressure at T=∞?



p4 action is essentially designed for this quantity T$\gg T_c$

asqtad designed mostly for T=0 physics (but good at high T, too)

stout-smeared one-link converges slower but in the a^2 scaling regime (e.g. extrapolation from N_t=8,10 provides a result within about 1%)

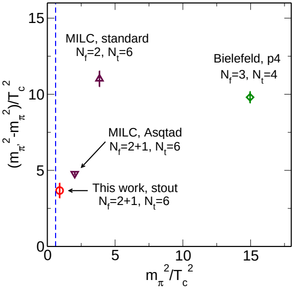
one can improve on the action (expensive) or observable level

Chiral symmetry/pions

Wuppertal-Budapest: JHEP 0601 (2006) 089. [hep-lat/0510084]

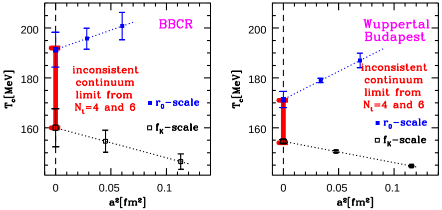
transition temperature for remnant of the chiral transition:
balance between the f's of the chirally broken & symmetric sectors
chiral symmetry breaking: 3 pions are the pseudo-Goldstone bosons

staggered QCD: 1 pseudo-Goldstone instead of 3 (taste violation)
staggered lattice artefact ⇒ splitting disappears in the continuum limit
WB: stout-smeared improvement is designed to reduce this artefact



Consequences of the non-scaling behaviour

for large 'a' no proper a^2 scaling (e.g. due to large m_π splitting)
how do we monitor it, how to be sure being in the scaling regime?
dimensionless combinations in the a$\to$0 limit:
$T_c r_0$ or T_c/f_K for the remnant of the chiral transition

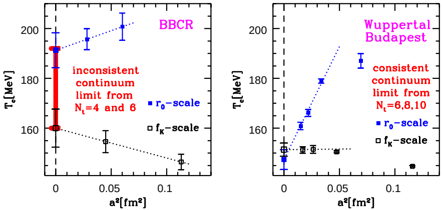


N_t=4,6: inconsistent continuum limit
N_t=6,8,10: consistent continuum limit (stout-link improvement)

independently which quantity is taken one obtains the same T_c
signal: extrapolation is safe, we are in the a^2 scaling regime

Consequences of the non-scaling behaviour

no proper a^2 scaling for large 'a' (e.g. due to large m_π splitting)
how do we monitor it, how to be sure being in the scaling regime?
dimensionless combinations in the a$\rightarrow$0 limit:
$T_c r_0$ or T_c/f_K for the remnant of the chiral transition



N_t=4,6: inconsistent continuum limit
N_t=6,8,10: consistent continuum limit (stout-link improvement)

independently which quantity is taken one obtains the same T_c
signal: extrapolation is safe, we are in the a^2 scaling regime

Equation of state

Integral method: J. Engels et al., Phys. Lett. B252 (1990) 625
on the lattice the dimensionless pressure is given by

$$p^{\rm lat}(\beta, m_q) = (N_t N_s^3)^{-1} \log \mathcal{Z}(\beta, m_q)$$

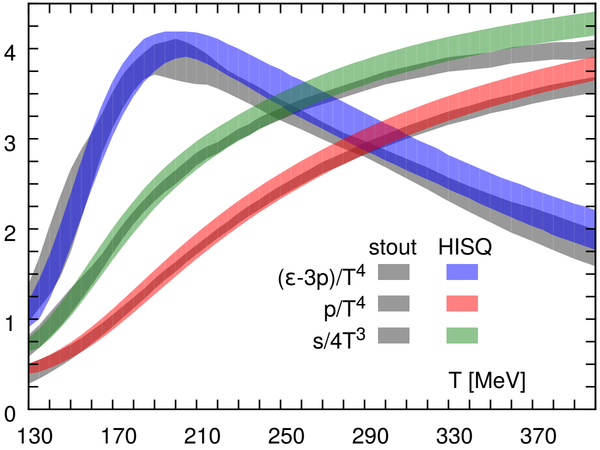
not accessible using conventional algorithms, only its derivatives

$$p^{\rm lat}(\beta, m_q) - p^{\rm lat}(\beta^0, m_q^0) = (N_t N_s^3)^{-1} \int_{(\beta^0, m_q^0)}^{(\beta, m_q)} \left(d\beta \frac{\partial \log \mathcal{Z}}{\partial \beta} + dm_q \frac{\partial \log \mathcal{Z}}{\partial m_q} \right),$$

first term: gauge action & second term: chiral condensate

the pressure has to be renormalized: subtraction at T=0 (or T>0)

Equation of state



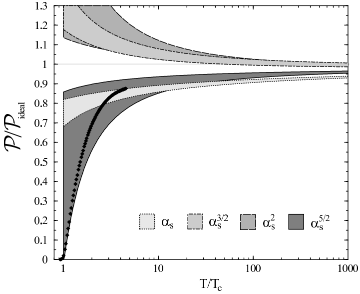
Energy density, pressure, entropy density

Big-Bang and heavy ion: Brookhaven, CERN)

Equation of state: difficulties at high temperatures

lattice results for the EoS extend upto a few times T_c

perturbative series "converges" only at asymptotically high T



applicability ranges of perturbation theory and lattice don't overlap
it was believed to be "impossible" to extend the range for lattice QCD

The standard technique is the integral method

$\bar{p}$=T/V·log(Z), but Z is difficult
$\Rightarrow$ $\bar{p}$ integral of (∂log(Z)/$\partial\beta$,∂log(Z)/∂m)

substract the T=0 term, the pressure is given by: $p(T)$=$\bar{p}(T)$-$\bar{p}(T=0)$

back of an envelope estimate:

$T_c\approx$150–200 MeV, m_π=135 MeV
try to reach T=20·T_c for N_t=8 (a=0.0075 fm)

$\Rightarrow N_s > 4/m_\pi \approx 6/T_c$ = 6·20/T = 6·20·$N_t \approx$ 1000

$\Rightarrow$ completely out of reach

Practical solution for the problem

a. substract successively:

$p(T)=\bar{p}(T)-\bar{p}(T=0)= [\bar{p}(T)-\bar{p}(T/2)]+[\bar{p}(T/2)-\bar{p}(T/4)]+...$

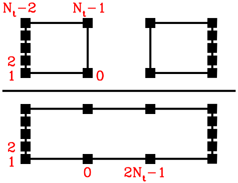
$\Longrightarrow$ for substractions at most twice as large lattices are needed (physical reason: there are no new UV divergencies at finite T)

b. instead of the integral method calculate:

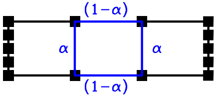
$\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

and introduce an interpolating partition function $Z(\alpha)$

$$\frac{Z^2(N_t)}{Z(2N_t)} =$$



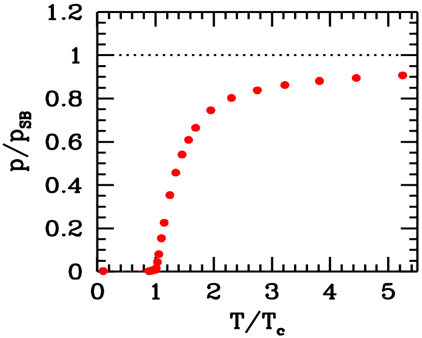
$$\overline{Z}(\alpha) =$$



define $\bar{Z}(\alpha)=\int \mathcal{D}U \exp[-\alpha S_{1b}-(1-\alpha)S_{2b}] \Rightarrow Z^2(N_t)=\bar{Z}(0), \quad Z(2N_t)=\bar{Z}\ (1)$

one gets directly for $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

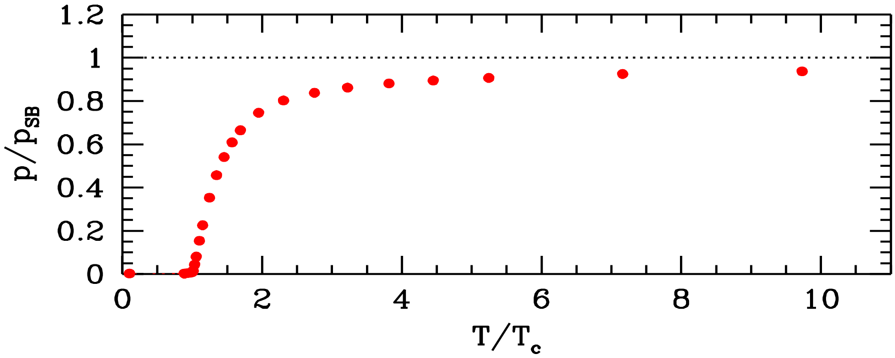
$T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1\langle S_{1b}-S_{2b}\rangle_\alpha\cdot d\alpha$



define $\bar{Z}(\alpha)=\int \mathcal{D}U \exp[-\alpha S_{1b}-(1-\alpha)S_{2b}] \Rightarrow Z^2(N_t)=\bar{Z}(0), \quad Z(2N_t)=\bar{Z}(1)$

one gets directly for $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

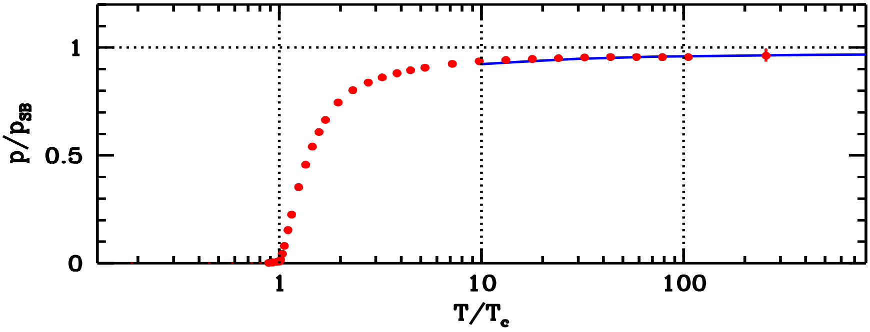
$T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1\langle S_{1b}-S_{2b}\rangle_\alpha\cdot d\alpha$



define $\bar{Z}(\alpha)=\int\mathcal{D}U\exp[-\alpha S_{1b}-(1-\alpha)S_{2b}] \Rightarrow Z^2(N_t)=\bar{Z}(0), \quad Z(2N_t)=\bar{Z}\ (1)$

one gets directly for $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

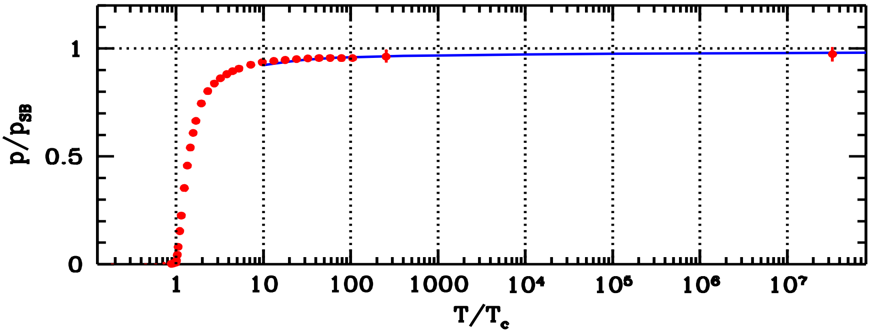
$T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1\langle S_{1b}-S_{2b}\rangle\alpha\cdot d\alpha$



define $\bar{Z}(\alpha)=\int\mathcal{D}U\exp[-\alpha S_{1b}-(1-\alpha)S_{2b}] \Rightarrow Z^2(N_t)=\bar{Z}(0),\quad Z(2N_t)=\bar{Z}\,(1)$

one gets directly for $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

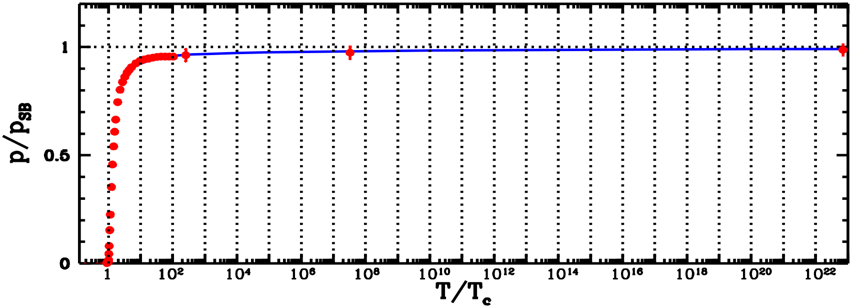
$T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1\langle S_{1b}-S_{2b}\rangle\alpha\cdot d\alpha$



long awaited link between lattice thermodynamics and pert. theory define $\bar{Z}(\alpha)=\int\mathcal{D}U\exp[-\alpha S_{1b}-(1-\alpha)S_{2b}] \Rightarrow Z^2(N_t)=\bar{Z}(0), \quad Z(2N_t)=\bar{Z}\,(1)$

one gets directly for $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

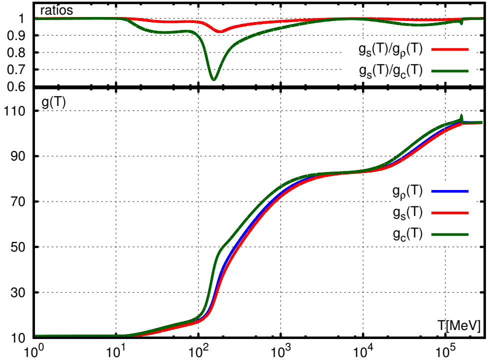
$T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1\langle S_{1b}-S_{2b}\rangle_\alpha\cdot d\alpha$



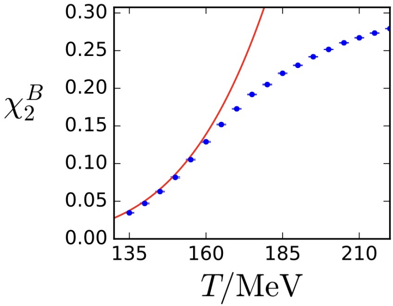
The equation of state

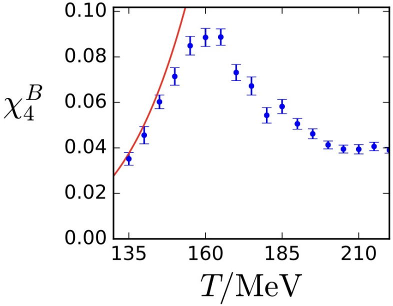
Effective number of degrees of freedom including all SM particles

$$\rho = \frac{\pi^2}{30} g_\rho T^4 \qquad s = \frac{2\pi^2}{45} g_s T^3 \qquad c = \frac{2\pi^2}{15} g_c T^3$$



Cumulants





Dozens of other (cross-)fluctuations (B,Q,S), up to eight order

About costs: quenched case from T=0 (or T_c) to $4T_c$

Cost of the conventional algorithm at relative error $\delta\chi_t$

$$\textit{costs} \propto \frac{1}{(\delta\chi_t)^2 \chi_t(T)}$$

relative cost $(4T_c)/(1T_c)$ (our highest T was $4T_c$: not enough)

from measured $\chi_t(T)$	$4^{7.1} \approx 2 \times 10^4$
from measured $\delta\chi_t$	$10^5 - 10^6$

- quenched $\chi_t(T = 0)$ calculated $\sim$ 20 years before
- Moores law leads to a factor of $\sim 10^5$ in 24 years

  $\Rightarrow$ Was just possible to do (dynamical case is probably hard)

About costs: dynamical QCD

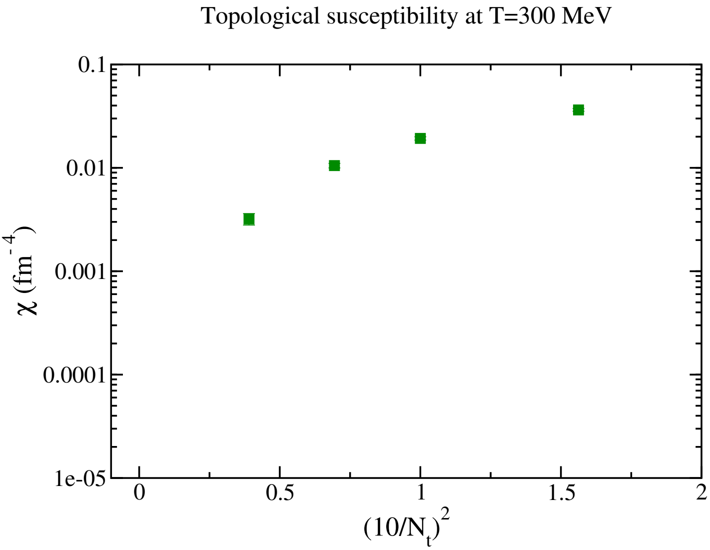
Dynamic relative cost $\$(7T_c)/\$(1T_c)$ ($7T_c \sim 1200MeV$)

from estimated $\chi_t(T)$	$7^{7-8} \approx 10^6 - 10^7$
increasing τ_{int} with T	$10^7 - 10^9$

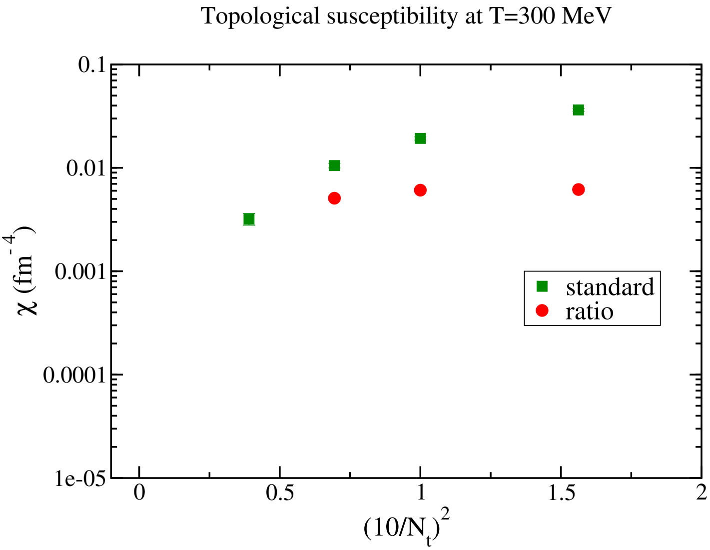
- dynamic $\chi_t(T = 0)$ in 2010, Moore cycles of ~ 30

$\Rightarrow$ conventional dynamical study not possible (needs 35 years)

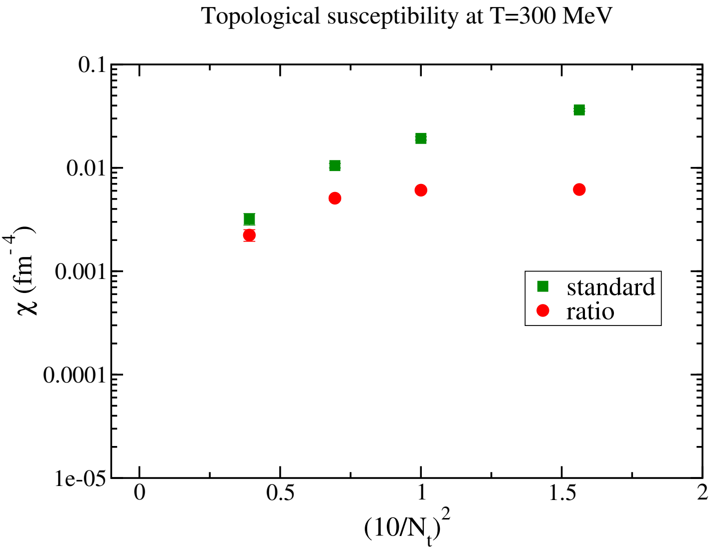
Unusually large cut-off effects: N_f=2+1+1 with 4-stout



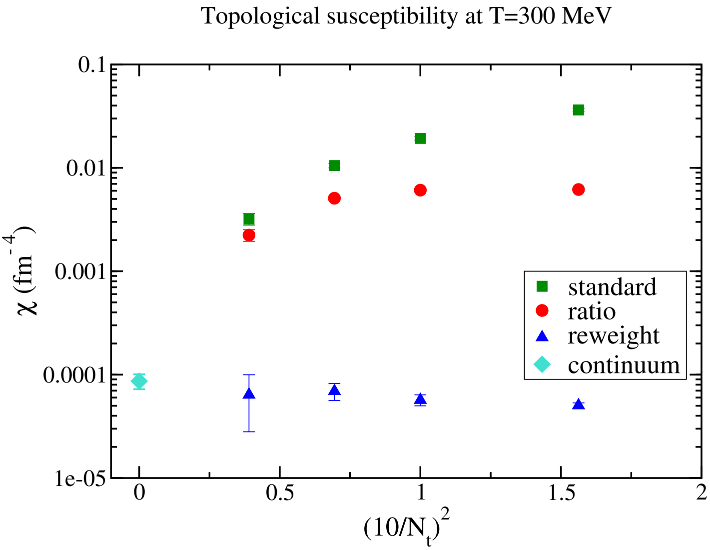
Unusually large cut-off effects: N_f=2+1+1 with 4-stout



Unusually large cut-off effects: N_f=2+1+1 with 4-stout



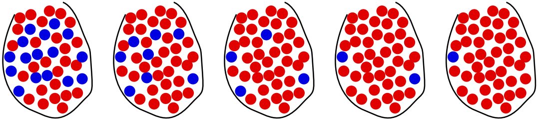
Fixed N_t: instanton's resolution doesn't change $\lambda_0 a$



Determine topological susceptibility/axion potential

Challenge

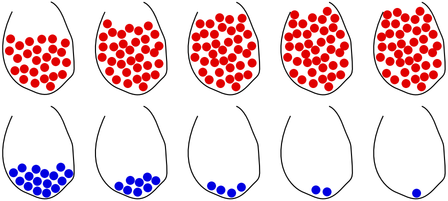
Determine the blue/red ratio by random pick!



$\longrightarrow$ getting very difficult with T $\longrightarrow$

Solution

Separate colors and determine the rate of change with T!



Fixed topological sector integral (susceptibility)

Instead of waiting for tunneling events,
we make simulations in fixed Q sectors. How to get

$$Z_1/Z_0 = ?$$

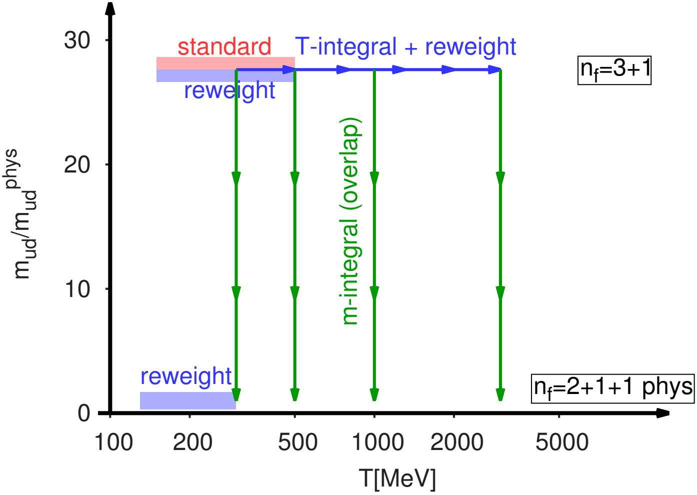
First calculate derivative of $\log Z_1/Z_0$:

$$b_1(T) \equiv \frac{d \log Z_1/Z_0}{d \log T}$$

Use fixed N_t-approach, ie. $T = (aN_t)^{-1}$ is changed by β:

$$b_1(T) = \frac{d\beta}{d \log a} \left(\langle S_g \rangle_1 - \langle S_g \rangle_0 \right)$$

Map of simulations



Topological susceptibility at the physical point

Though topological change is very rare, result up to about 3 GeV.