Calculations in QCD

$\mathcal{N}=4$ SUSY

$\mathcal{N}=8$ SUGRA

Computing Grav. Waves

Finding new principles for QFT

Analytic structure of pert. QFT

Key ideas

* Use only information from physical on-shell states

* Use properties of scatt. ampls to compute them

  - Factorization

  - Unitarity

  - Existence of P.I. $\to$ Feynman Integrals

Elvang - Huang (1308.1697)

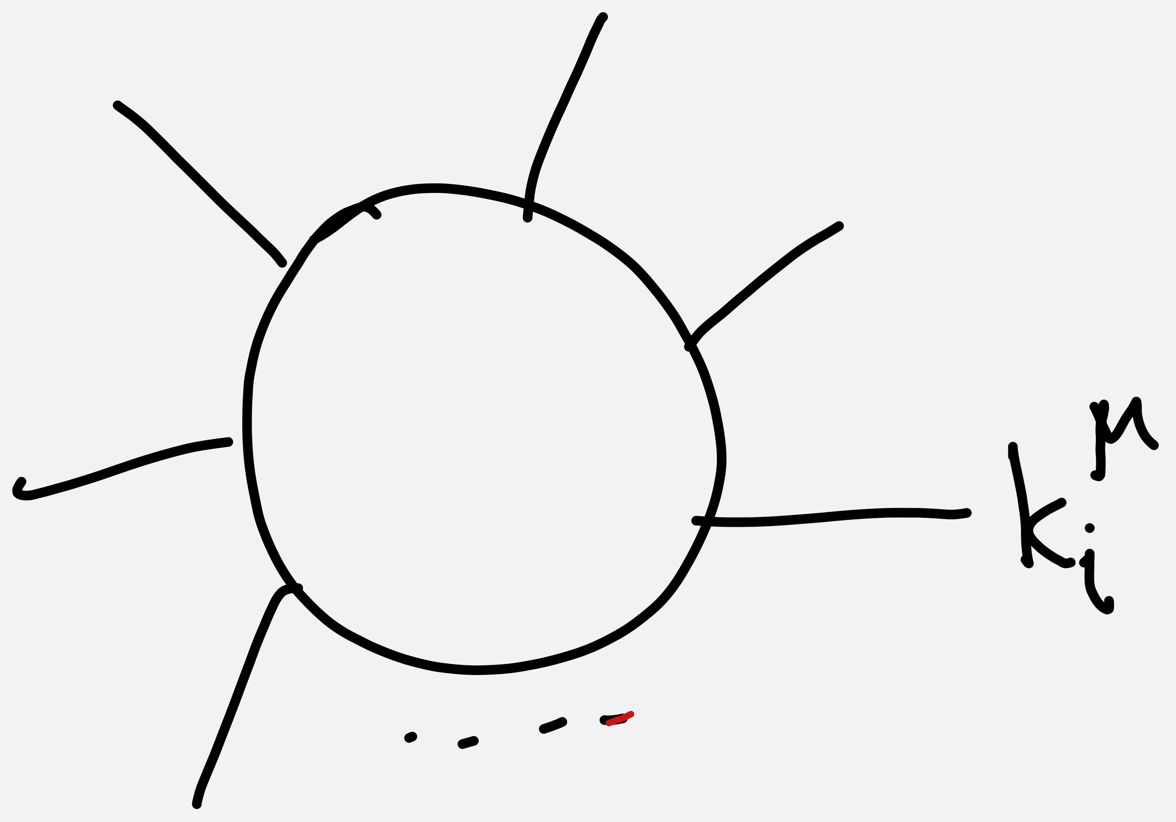
Henn - Plefka

Del Duca

Yang - Mills

Parke - Taylor ampl.

$$A_n(1^+ \ldots m_1^- \ldots m_2^- \ldots n^+) = \frac{\langle m_1 m_2 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1) n \rangle \langle n 1 \rangle}$$



$$0 \to k_1 + k_2 + \dots + k_n$$

$$k_1 + k_2 \to k_3 + \dots + k_n$$

$$\sum_i k_i^\mu = 0$$

$$\sqrt{k^\mu}$$

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma}) \qquad \sigma^0 = \mathbb{1} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$1..2$

$$k_{\alpha\dot{\alpha}} \equiv k \cdot \sigma = \begin{pmatrix} k^0 - k^3 & -k^1 + ik^2 \\ -k^1 - ik^2 & k^0 + k^3 \end{pmatrix}.$$

$$\det(\underline{k}) = k^2$$

$$k_{\alpha\dot{\alpha}} = \underbrace{\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}}_{SL(2,\mathbb{C})}$$

$\varepsilon^{\alpha\beta}$ $\varepsilon^{\dot\alpha\dot\beta}$

$\varepsilon^{\alpha\beta}\lambda_\alpha\lambda'_\beta$ $\varepsilon^{\dot\alpha\dot\beta}\tilde\lambda_{\dot\alpha}\tilde\lambda'_{\dot\beta}$

$|j^+\rangle \equiv \lambda_j^\alpha \equiv |j\rangle$

$|j^-) \equiv \tilde\lambda_{j\dot\alpha} \equiv |j]$

$\langle j| \equiv \langle j^-| \leftrightarrow \varepsilon_{\alpha\beta}\lambda_j^\beta$

$[j| \equiv \langle j^+| \leftrightarrow \varepsilon^{\dot\alpha\dot\beta}\tilde\lambda_{j\dot\beta}$

$\langle ij\rangle \equiv \langle i^-|j^+\rangle = \varepsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta}$

$[ij] = \langle i^+|j^-) = \varepsilon^{\dot\alpha\dot\beta}\tilde\lambda_{j\dot\alpha}\tilde\lambda_{i\dot\beta}$

Explicit rep

$$k_\pm = k^0 \pm k^3$$

$$e^{\pm i\phi_k} = \frac{k^1 \pm i k^2}{\sqrt{k_+ k_-}}$$

$$\langle ij\rangle = e^{i\phi_j}\sqrt{k_{i+}k_{j-}} - e^{i\phi_i}\sqrt{k_{j+}k_{i-}}$$

$$k^0_{i,j} > 0 \qquad [ij] = \langle ji\rangle^*$$

$$[ij] = \operatorname{sign}(k_i^0 k_j^0)\langle ji\rangle^*$$

Exercise Show that this is a phase. $\langle ij\rangle[ji] = 2k_i\cdot k_j$

$$\lambda_\alpha = \begin{pmatrix} -\sqrt{k_+} \\ e^{i\phi_k}\sqrt{k_-} \end{pmatrix}, \quad \tilde{\lambda}_{\dot\alpha} = \begin{pmatrix} -\sqrt{k_+} \\ e^{-i\phi_k}\sqrt{k_-} \end{pmatrix}$$

Antisymmetric $\langle ij\rangle = -\langle ji\rangle$

$\langle ii\rangle = 0 = [ii]$

"Gordon" $2k_j^\mu = (\sigma^\mu)^{\alpha\dot\alpha}\lambda_{j\alpha}\tilde\lambda_{j\dot\alpha} = \langle j|\sigma^\mu|j]$

$\equiv \langle j|\mu|j]$

Fierz identity $\langle i|\mu|j]\langle q|\mu|r] = 2\langle iq\rangle[rj]$

Any two-dim vect $\leftarrow$ 2 basis vector

$\lambda_q = c_i\lambda_i + c_j\lambda_j$

Schouten identity

$$\langle ij\rangle\langle rq\rangle = \langle iq\rangle\langle rj\rangle + \langle ir\rangle\langle jq\rangle$$

Exercise $\mathrm{tr}(\not{k}_1 \cdots \not{k}_6)$

(a) using scalar products

(b) using spinor products

(c) How many ops does each require?

(d) how does that scale w/ more momenta?

For real momenta, $\tilde{\lambda}_j = \mathrm{sign}(k_j^0)\, \lambda_j^*$

$$\lambda \mapsto \tau \lambda$$

$$k_j \longrightarrow |\tau|^2 k_j$$

τ is a phase

helicity $-\frac{1}{2} \cdot \lambda$, $+\frac{1}{2}\ \tilde{\lambda}$

Amplitudes

3-pt amplitude

$$k_3^2 = (k_1+k_2)^2 = 2k_1 \cdot k_2 = 0.$$

$$\langle 12 \rangle = 0 = \langle 23 \rangle = \langle 31 \rangle$$

$$[12] = 0 = [23] = [31]$$

$$A(1,2,3) = 0$$

$$\tilde{\lambda} \neq \pm \lambda^*$$

$$k_i \cdot k_j = 0 \Rightarrow \langle ij \rangle = 0 \text{ or } [ij] = 0$$

but not necessarily both

$$[12] = 0 = [23] = [31]$$

$$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$$

$$\langle 12 \rangle, \langle 23 \rangle, \langle 31 \rangle$$

$$A_3 = \text{const}\ \langle 12\rangle^{e_{12}} \langle 23\rangle^{e_{23}} \langle 31\rangle^{e_{31}}$$

$$- - +$$

$$h_1 = -\frac{1}{2}(e_{12} + e_{31})$$

$$h_2 = -\frac{1}{2}(e_{23} + e_{12})$$

$$h_3 = -\frac{1}{2}(e_{31} + e_{23})$$

$$e_{12} = h_3 - h_1 - h_2$$

$$e_{23} = h_1 - h_2 - h_3$$

$$e_{31} = h_2 - h_1 - h_3$$

$$e_{12} = 3, \quad e_{23} = e_{31} = -1.$$

$$e_{12} = 6, \quad e_{23} = e_{31} = -2$$

$$A_3 = \text{const}\ \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

$$M_3 = \text{const}\left(\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\right)$$

$$M_3 = A_3^2$$

$$\text{Gravity} \sim (\text{Yang-Mills})^2$$

$$(++-) \qquad e_{12} = -3\,, \quad e_{23} = e_{31} = 1.$$

$$\cancel{A_3 = \text{const}\,\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle^3}}$$

$$A_3 = \text{const}\,\frac{[12]^3}{[23][31]}$$

Lecture II

$$A_3 = ct \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \qquad (- - +)$$

M_3

$ct \sim \frac{1}{M}$

$(- - -)$

$$A_3 = \text{const} \, \langle 12\rangle\langle 23\rangle\langle 31\rangle$$

$\mathrm{Tr}(G^3) \qquad G_{\mu\nu}$

$$ct = g f^{a_1 a_2 a_3}$$

$$A_4(1\,2\,3\,4) \sim A_3(1\,2\,x)\, f^{a_1 a_2 a_x}\, \delta_{a_x a_y}\, f^{a_y a_3 a_4} A_3(y\,3\,4)$$

$$f^{abc} = -\frac{i}{\sqrt{2}} \mathrm{Tr}\left([T^a, T^b]\, T^c\right) \qquad \mathrm{Tr}(T^a T^b) = \delta^{ab}$$

$$U(N) = SU(N) \times U(1)$$

$$(T^a)_{i_1}^{\ \bar{\imath}_1} (T^a)_{i_2}^{\ \bar{\imath}_2} = \delta_{i_1}^{\ \bar{\imath}_2}\, \delta_{i_2}^{\ \bar{\imath}_1}$$

$$f^{a_1 a_2 a_x} f^{a_x a_3 a_4} = -\frac{1}{2} \mathrm{Tr}\left([T^{a_1}, T^{a_2}] T^{a_x}\right) \mathrm{Tr}\left(T^{a_x} [T^{a_3}, T^{a_4}]\right)$$

$$= -\frac{1}{2} \mathrm{Tr}\left([T^{a_1}, T^{a_2}][T^{a_3}, T^{a_4}]\right)$$

$\longrightarrow \sum$ single traces.

$$A_n(\{k_i, h_i, a_i\}) = \sum_{\rho \in S_n / \mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) A_n\left(\rho(1)^{h_{\rho(1)}} \cdots \rho(n)^{h_{\rho(n)}}\right).$$

$A_n(1,\dots,n)$ sym under Z_n

$= (-1)^n A_n(n \dots 1)$

$a_n \in U(1)$

$\mathrm{Tr}(T^{a_1} \dots T^{a_{n-1}})$:

$A_n(1 \dots n) + A_n(1 \dots n-2, n, n-1) + \dots + A_n(1\, n\, 2 \dots n-1) = 0$

U(1) decoupling identity

$$\boxed{0 \to \{k_i\}} \qquad \sum_i k_i = 0$$

$$k_1 + k_2 \to k_3 + \cdots + k_n \qquad k_1 + k_2 = \sum_{i=3}^{n} k_i$$

$$k_1 + k_2 + k_3 \longrightarrow k_4 + \cdots + k_n$$

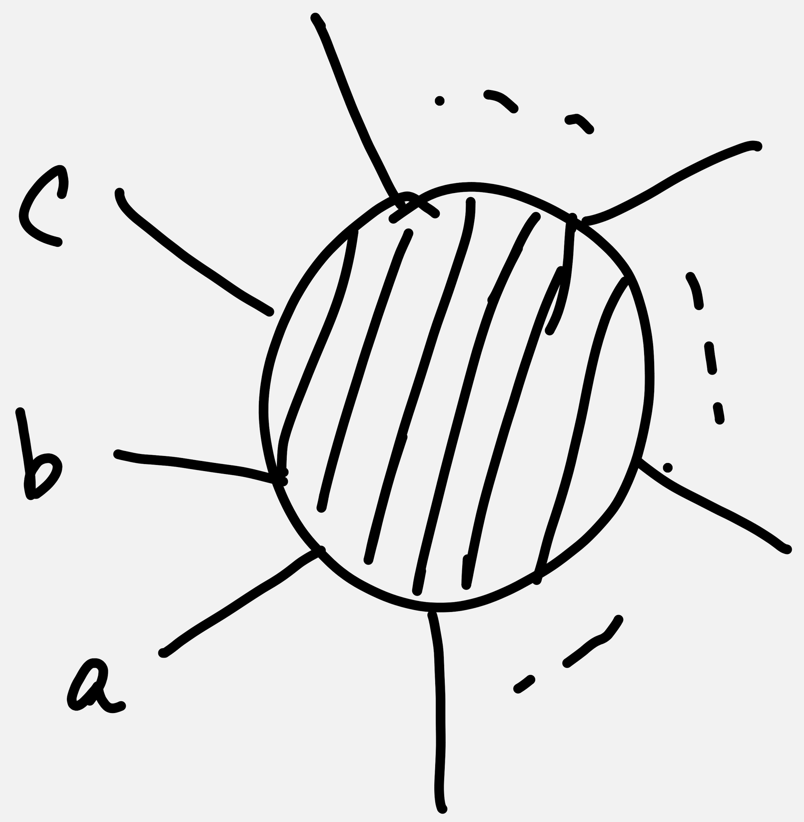
$$p^2 = (k_1+k_2)^2 \to m_X^2$$

$$A(1+2 \to 3+\cdots+n) \longrightarrow A_L(1+2 \to X)\,\frac{i}{p^2-m_X^2}\,A_R(X \to 3+\cdots+n)$$

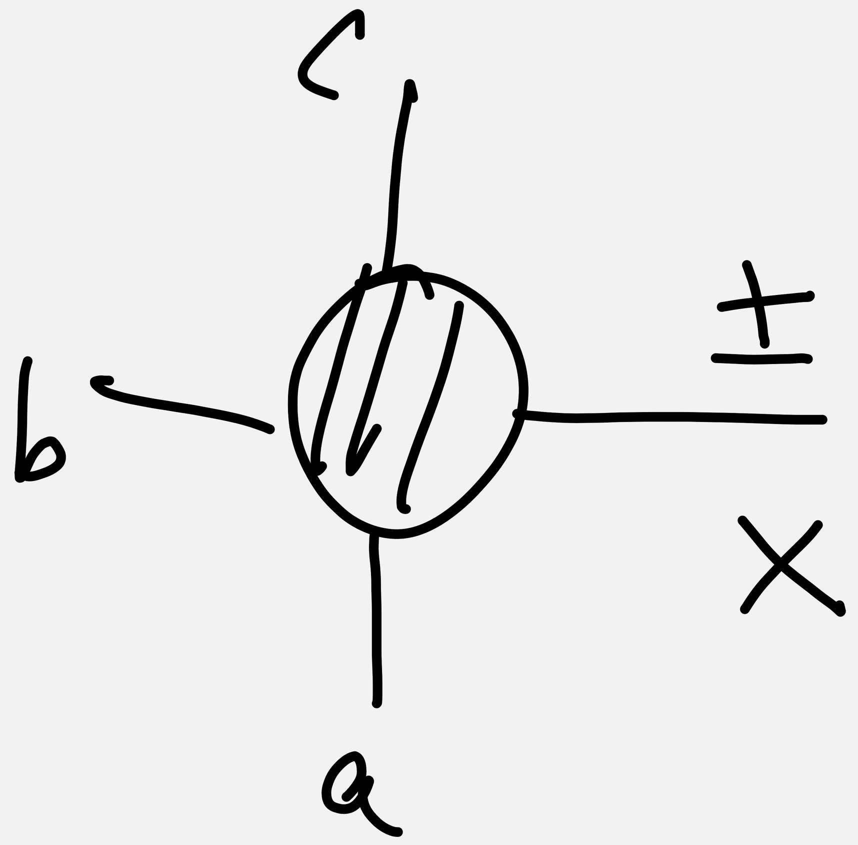
$$p^2 = (k_1+\cdots+k_r)^2 \to m_X^2$$

$$A(1+\cdots+r \to (r+1)+\cdots+n) \longrightarrow A_L(1+\cdots+r \to X)\,\frac{i}{p^2-m_X^2}\,A_R(X \to (r+1)+\cdots+n)$$

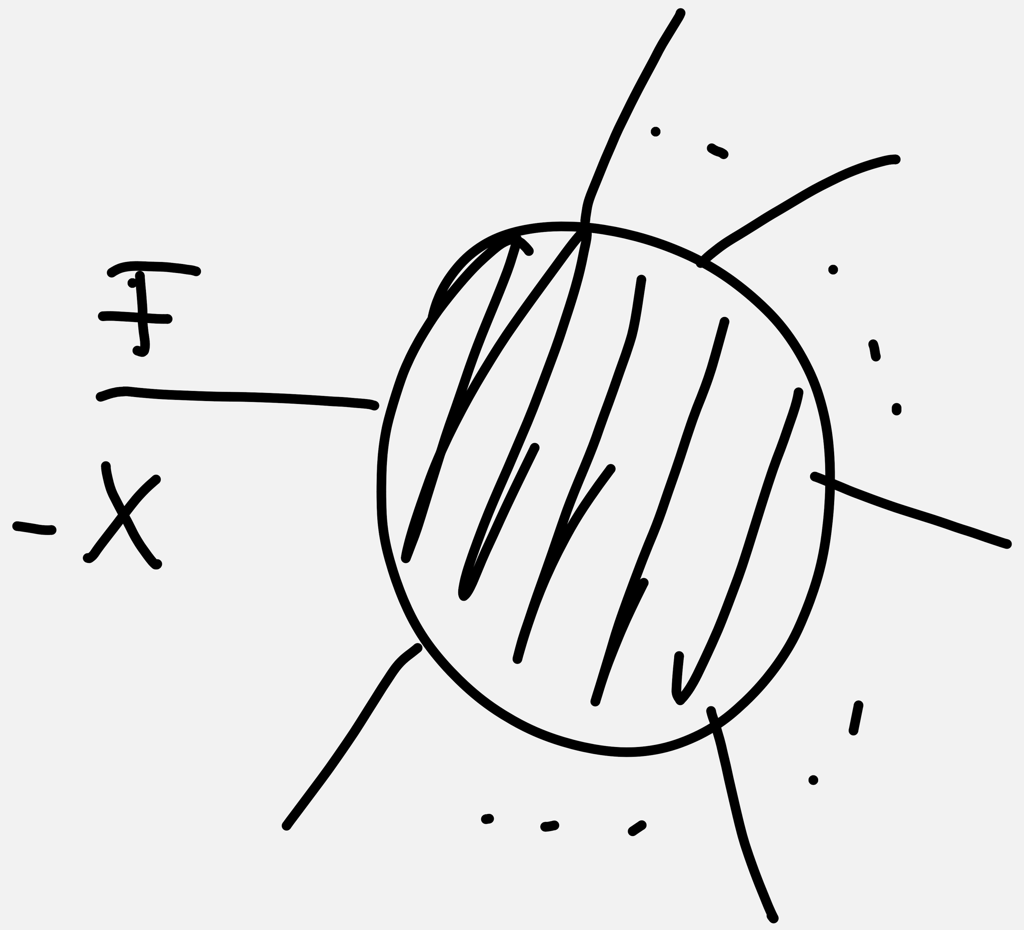
Yang-Mills $m_X = 0$



$\longrightarrow$



$$\frac{i}{(k_a+k_b+k_c)^2}$$



$$(k_a+k_b+k_c)^2 \longrightarrow 0 \qquad s_{ab},\ s_{bc},\ s_{ca} \not\longrightarrow 0.$$

$$s_{ij} = (k_i+k_j)^2 = 2\,k_i\cdot k_j$$

$A_3 = 0.$ $\qquad \frac{0}{0} \sim \frac{1}{\sqrt{s_{12}}}$

Splitting amplitudes

" $\sqrt{\text{Altarelli-Parisi}}$ "

$$\frac{E^3}{E^2} \sim E \to 0$$

$$s_{12} \sim E^2 \to 0$$

Complex momenta $\qquad A_3 \neq 0$

$$s_{12} \to 0 \qquad \langle 12\rangle \to 0 \text{ or } [12] \to 0$$

BCFW On-shell Recursion Relation

Britto Cachazo Feng & Witten

Complex parameter z

$[j, \ell\rangle$:

$|j] \to |j] - z|\ell]$

$|\ell\rangle \to |\ell\rangle + z|j\rangle$

$k_j^\mu \to k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j|\mu|\ell]$

$k_\ell^\mu \to k_\ell^\mu(z) = k_\ell^\mu + \frac{z}{2}\langle j|\mu|\ell]$

$k_j^\mu + k_\ell^\mu \to k_j^\mu(z) + k_\ell^\mu(z) = k_j^\mu + k_\ell^\mu$

$$k_j^2(z) = \underbrace{k_j^2}_{0} - z\underbrace{\langle j|j|\ell]}_{0} + \underbrace{\frac{z^2}{2^2}\langle j|\mu|\ell]^2}_{0} = 0$$

$A(x)$