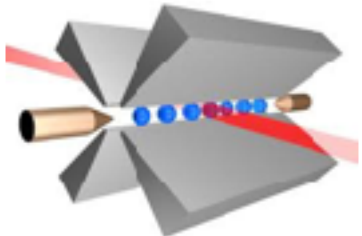
Quantum Connections in Sweden - 11, Summer School,
Högberga Gård 11.-24 June 2023

Programmable Quantum Simulators with Atoms and Ions

Peter Zoller

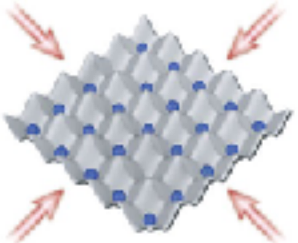
Engineered Quantum Many-Body Systems with Atoms and Ions

Trapped ions



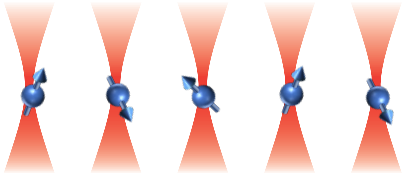
© UIBK, Duke, Honeywell …

Optical Lattices



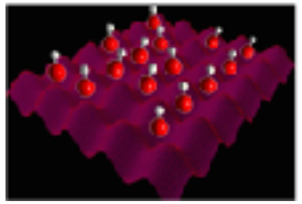
© MPQ, …

Rydberg Arrays



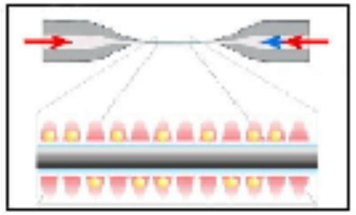
© Harvard, Paris …

Polar Molecules



© JILA

CQED & Photonic



© TU Wien

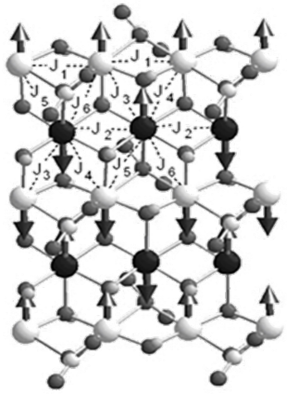
… as Atomic NISQ Devices

noisy: no error correction

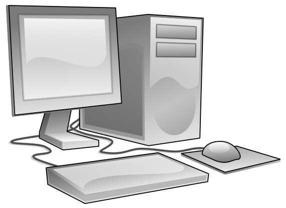
350 universität innsbruck
IQI
PASQuanS
FWF Der Wissenschaftsfonds.
AFOSR MURI (JILA)
Q-SEnSE
Simons Collaboration on Ultra-Quantum Matter
AQT

Quantum Simulation

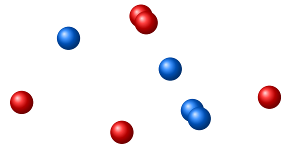
problem: `solving' a quantum many-body problem



$$\hat{H} = \sum_{i,j} \left[J_1 \hat{S}_i^z \hat{S}_j^z + J_2 \left(\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right) \right]$$



Classical Computer



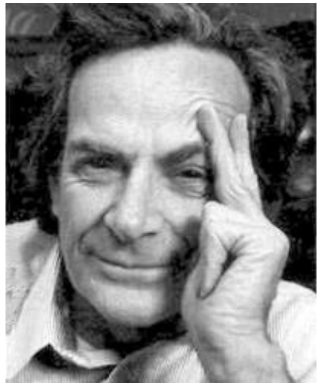
Quantum Simulator

International Journal of Theoretical Physics, Vol. 21, Nos. 6/7, 1982

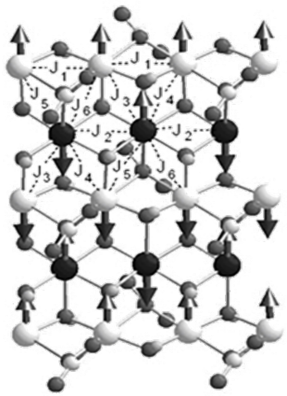
Simulating Physics with Computers

Richard P. Feynman

a thing which is usually described by local differential equations. But the physical world is quantum mechanical, and therefore the proper problem is the simulation of quantum physics—which is what I really want to talk about, but I'll come to that later. So what kind of simulation do I mean?



Quantum Simulation



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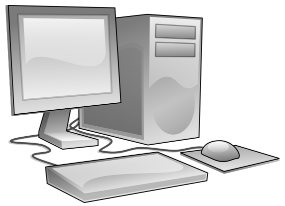
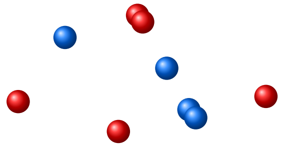


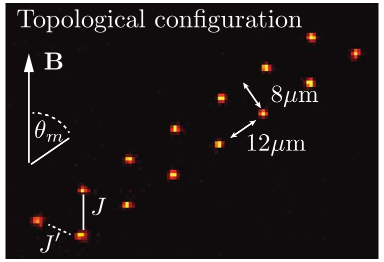
Classical Computer



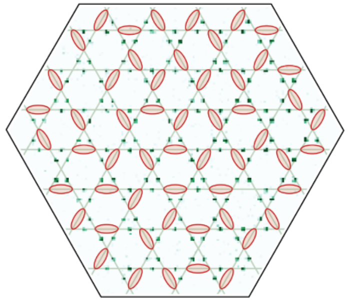
Quantum Simulator

- Programmable analog quantum simulators (AMO)

Rydberg Atoms



S. Léséleuc
et al., Science 2019

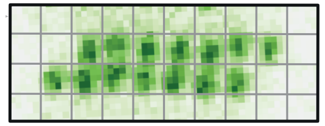


G. Semeghini
et al., Science 2021

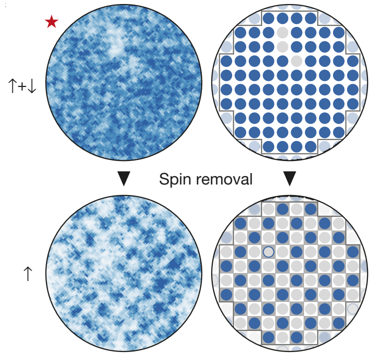
Innsbruck, NIST, JQI etc.

Hubbard models

Tilted-edge ladder

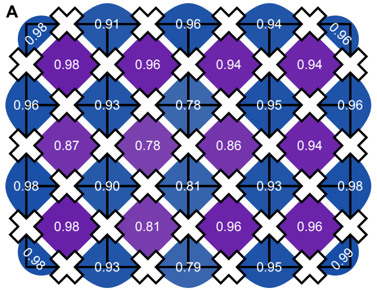


Sompet et.al.
arXiv: 2103.10421



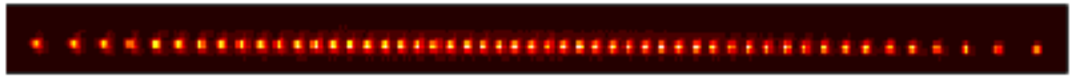
Mazurenko et.al.
Nature 2017

Superconducting Circuits



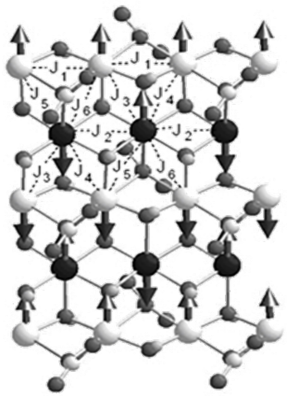
Satzinger et.al (2021)

Google



Ions (51 Ions Innsbruck)

Quantum Simulation



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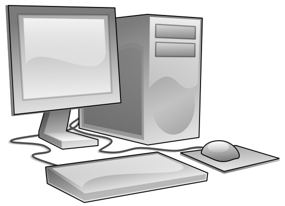
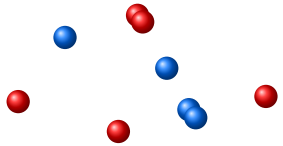


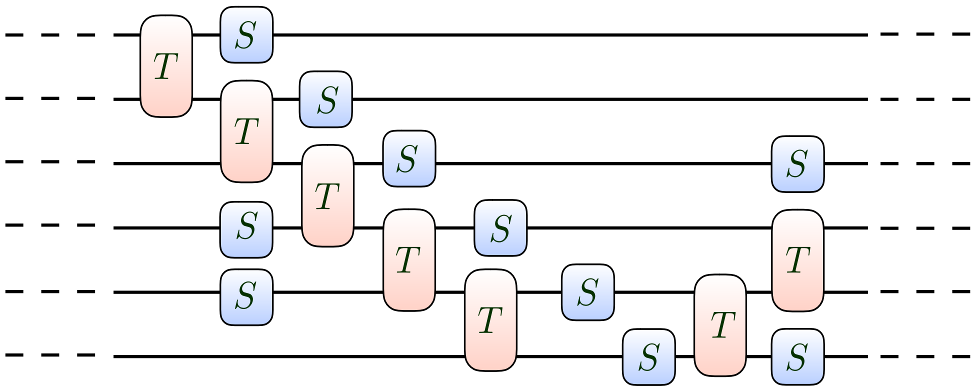
Classical Computer

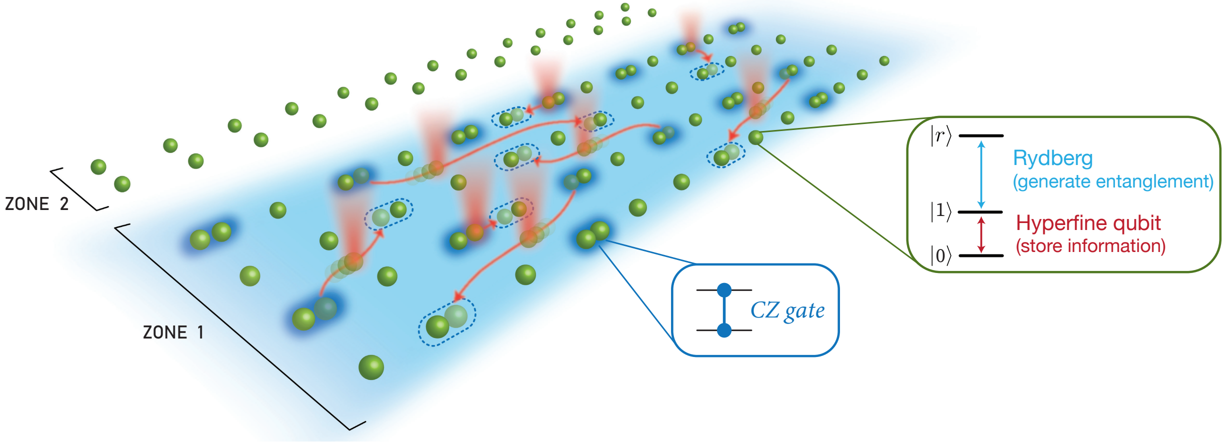


Quantum Simulator

- Digital quantum simulation

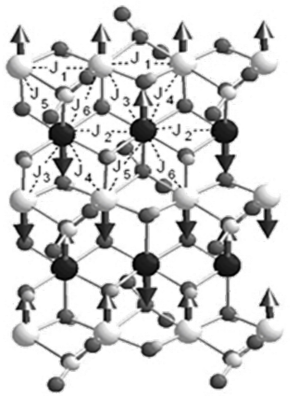
$$\begin{aligned} |\Psi(T)\rangle &= e^{-iHT}\,|\Psi_0\rangle \\ &\simeq U_N(\Delta t)\cdots U_2(\Delta t)U_1(\Delta t)\,|\Psi_0\rangle \end{aligned}$$





D. Bluvstein et.al. Nature 604 451 (2022)

Quantum Simulation



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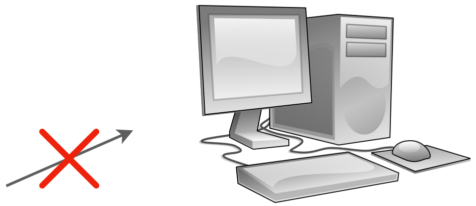
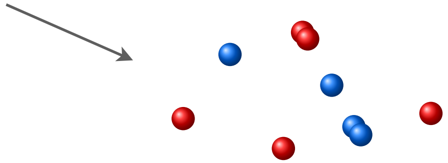


Classical Computer



Quantum Simulator

- Digital quantum simulation

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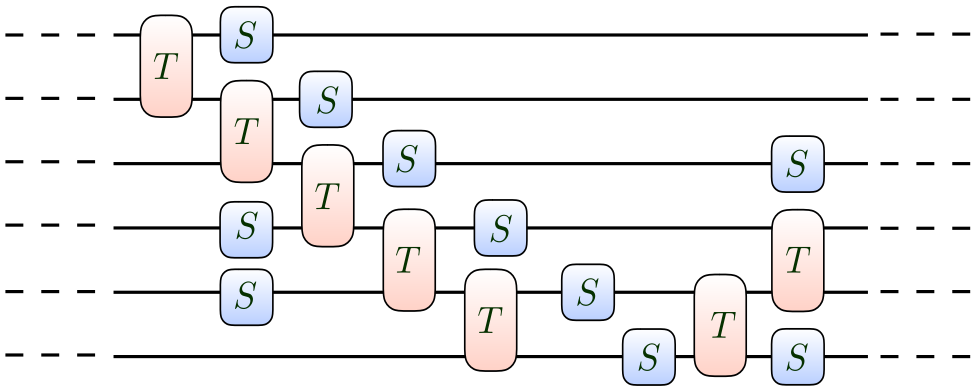


Article Nature | Vol 618 | 15 June 2023

Evidence for the utility of quantum computing before fault tolerance

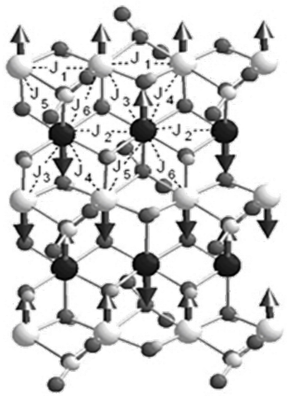
Youngseok Kim[1,6 ✉], Andrew Eddins[2,6 ✉], Sajant Anand[3], Ken Xuan Wei[1], Ewout van den Berg[1], Sami Rosenblatt[1], Hasan Nayfeh[1], Yantao Wu[3,4], Michael Zaletel[3,5], Kristan Temme[1] & Abhinav Kandala[1 ✉]

[1]IBM Quantum, IBM Thomas J. Watson Research Center, Yorktown Heights, NY, USA.

out of reach for current processors. Here we report experiments on a noisy 127-qubit processor and demonstrate the measurement of accurate expectation values for circuit volumes at a scale beyond brute-force classical computation. We argue that this

$H = -J \sum_{\langle i,j \rangle} Z_i Z_j + h \sum_i X_i,$ (2D) NISQ & error mitigation (zero noise extrapolation)

Quantum Simulation



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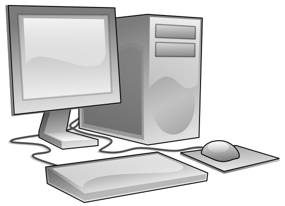
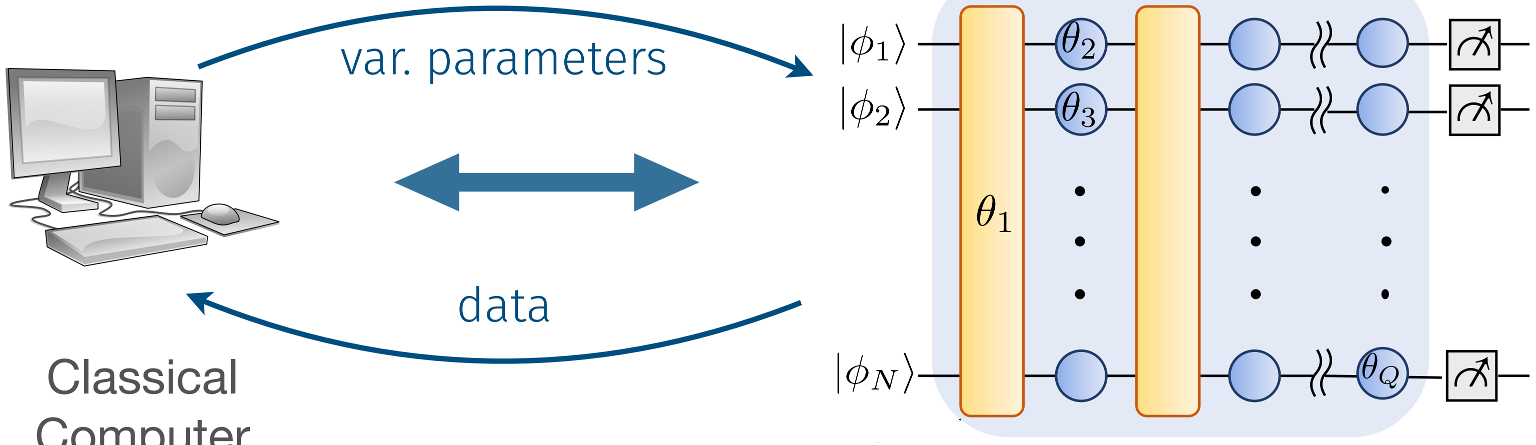


Classical Computer

- Variational quantum algorithms

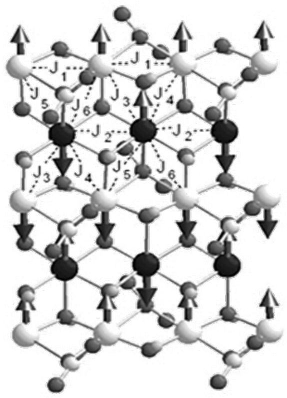


Classical Computer

programmable Quantum Simulator

- lattice models in cond mat & HEP
- quantum chemistry
- classical optimization
- quantum metrology

Quantum Simulation



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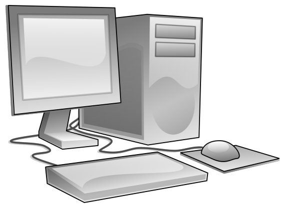
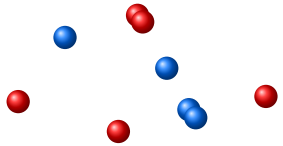


Classical Computer



Quantum Simulator

While we make progress in building larger and more powerful quantum devices, …

Perspective

Practical quantum advantage in quantum simulation

Nature | Vol 607 | 28 July 2022

Andrew J. Daley[1✉], Immanuel Bloch[2,3,4], Christian Kokail[5,6], Stuart Flannigan[1], Natalie Pearson[1], Matthias Troyer[7] & Peter Zoller[5,6]

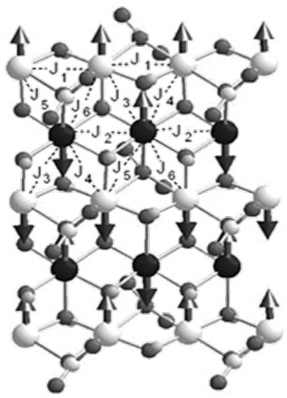
Today's challenges:

- scalability vs. controllability vs. decoherence
- quantitative predictions in q-simulation
- verification of quantum devices

… in regime of quantum advantage

Quantum Simulation

problem: `solving’ a quantum many-body problem



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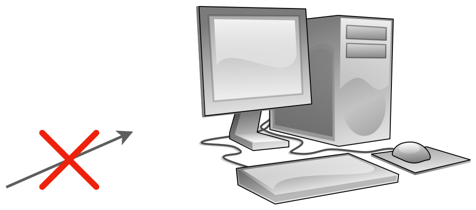
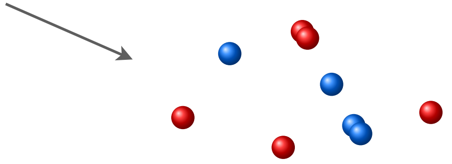
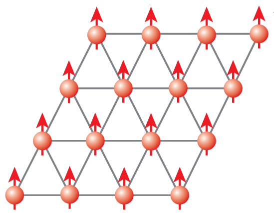
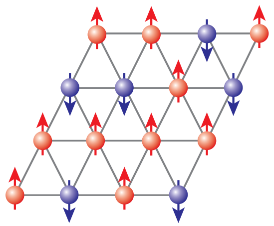
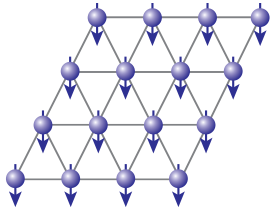


Classical Computer



Quantum Simulator

large scale entanglement

$|\Psi\rangle = c_1|$  $\rangle + c_2|$  $\rangle + \cdots + c_{2^N}|$  $\rangle$

AND AND AND

Challenge: develop tools to quantify entanglement

Lecture 1:

Introduction & Background Material

- Programmable Quantum Simulators - Atomic Platforms
- Characterizing Entanglement in Many-Body Systems

How to measure Entanglement

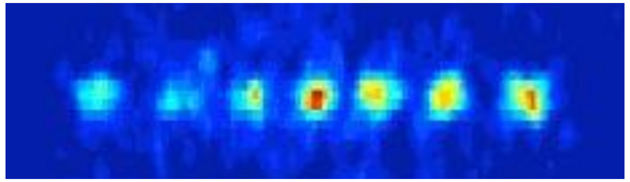
- Renyi Entanglement Entropy
- …

- quantum state tomography
- copies - quantum protocol
- randomized measurements

The randomized measurement toolbox,
A Elben, ST Flammia, HY Huang, R Kueng, J Preskill, B Vermersch & PZ,
Nature Review Physics (2022)

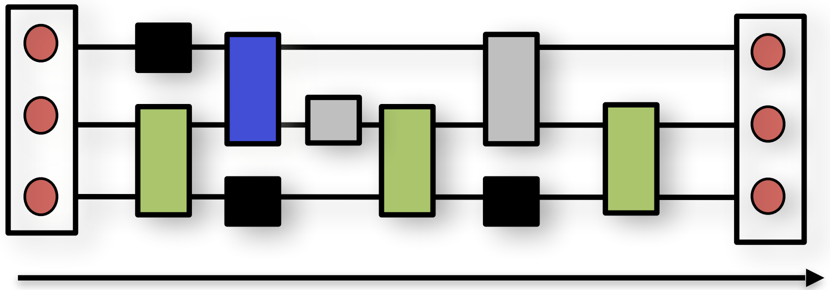
Quantum Computing [Digital]

trapped ions



exp.: Innsbruck, Maryland, NIST, …

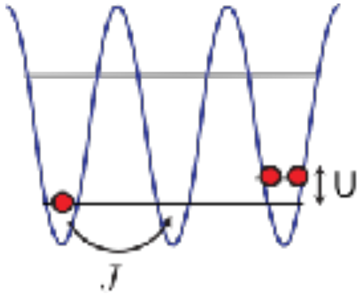
quantum logic network model

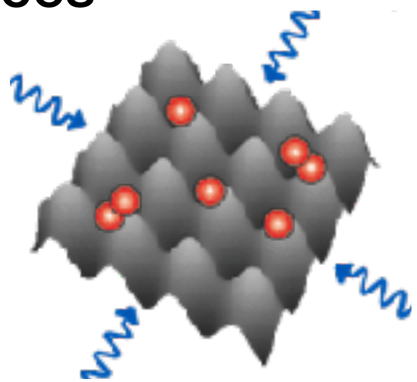


… demonstrating quantum algorithms

Quantum Simulation [Analog]

atoms in optical lattices

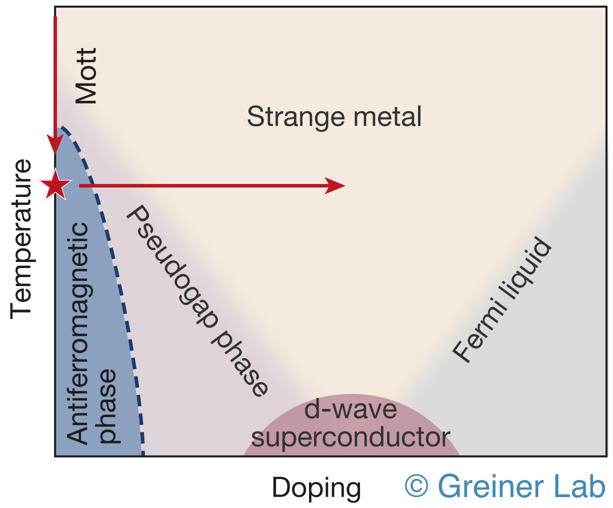




theory: Jaksch et al. 1998

exp.: Munich, ETH,, Harvard, MIT, Hamburg, UIBK, Heidelberg …

(non-)equilibrium many-body physics

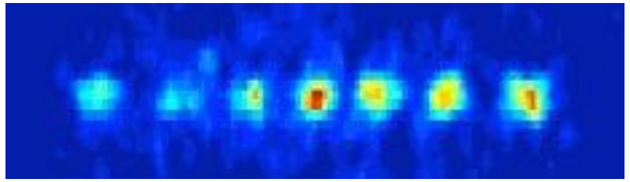


Fermi-Hubbard Model in 2D (high Tc)

… many-body quantum physics / cond mat

Quantum Computing [Digital]

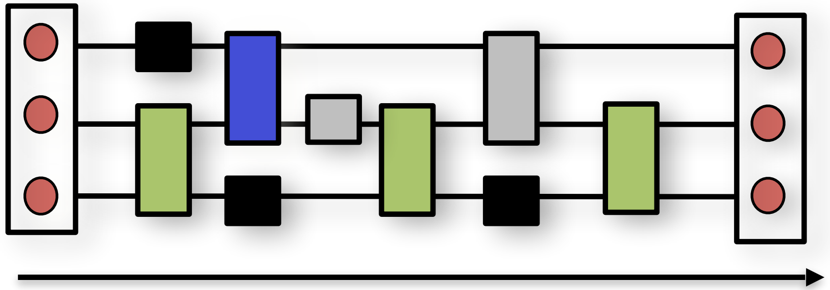
trapped ions



exp.: Innsbruck, Maryland, NIST, …

quantum logic network model

qubits quantum gates read out

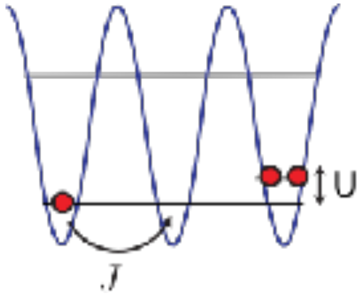


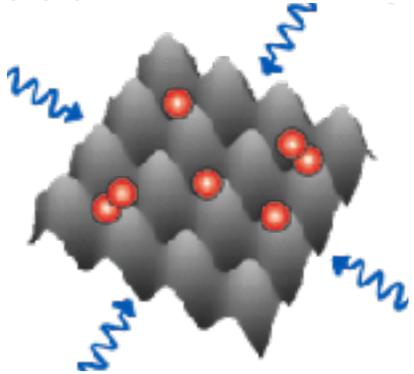
… demonst

- *fully programmable / universal*
- *small # qubits*
- *error correction*

Quantum Simulation [Analog]

atoms in optical lattices





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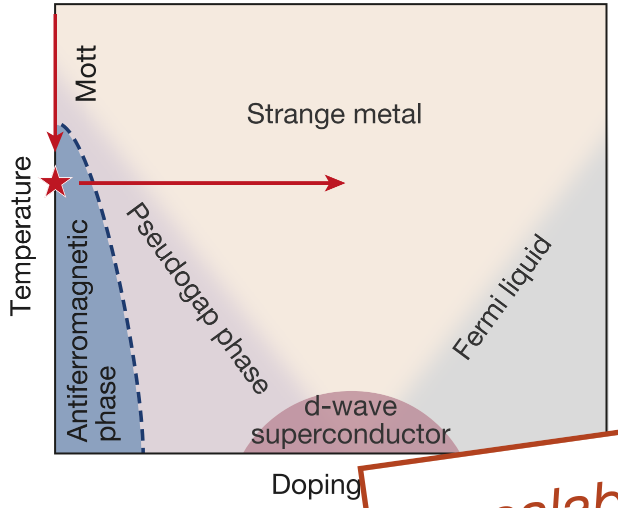


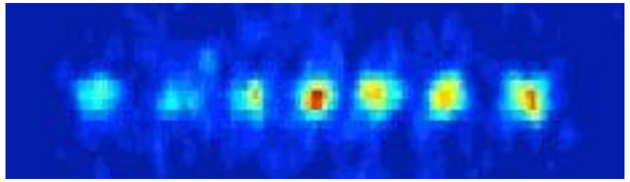
Fermi-Hubbard Model in 2D (high Tc)

… many-boo

- *scalable to large # particles*
- *restricted class of Hamiltonians*
- *… however with high fidelity*

Quantum Computing [Digital]

trapped ions



exp.: Innsbruck, Maryland, NIST, …

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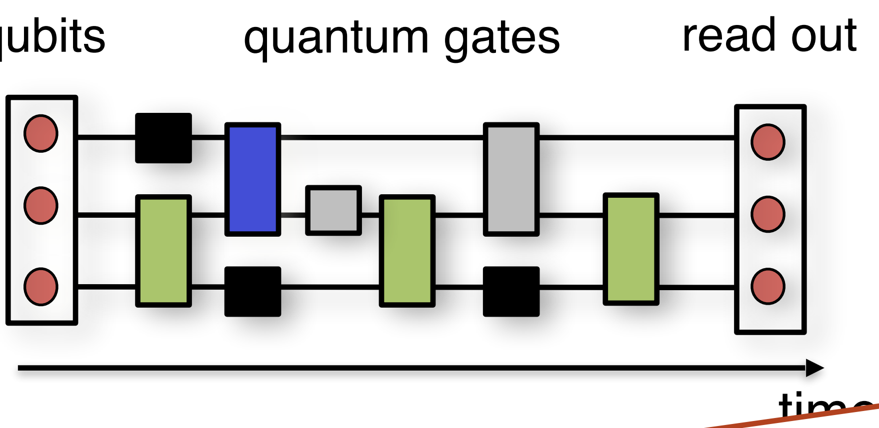

… demonst

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- *small # qubits*
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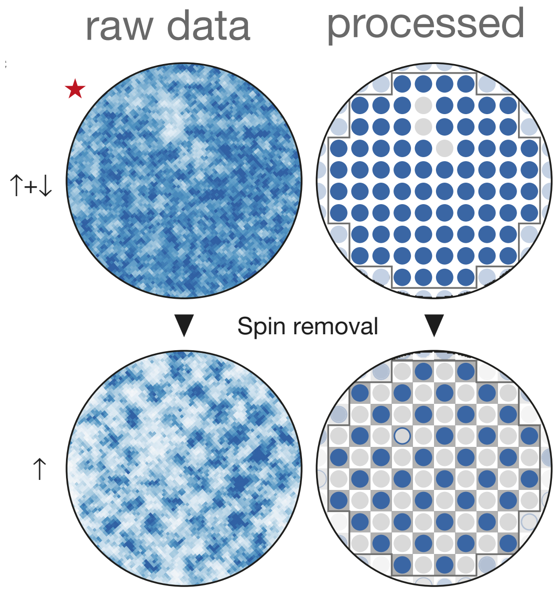
Quantum Simulation [Analog]

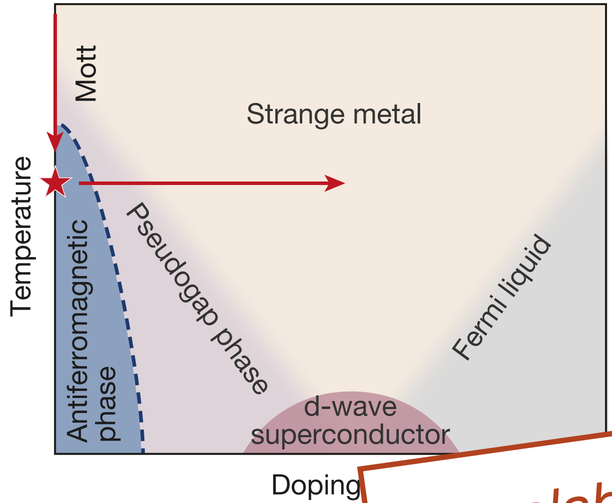
atoms in optical lattices

quantum gas microscope

`seeing single atom in a single shot'



© Greiner Lab



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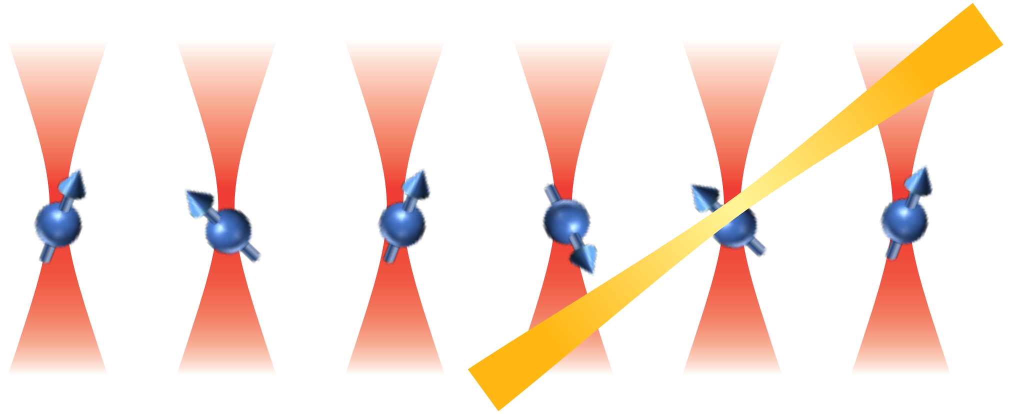
Programmable Analog Quantum Simulators

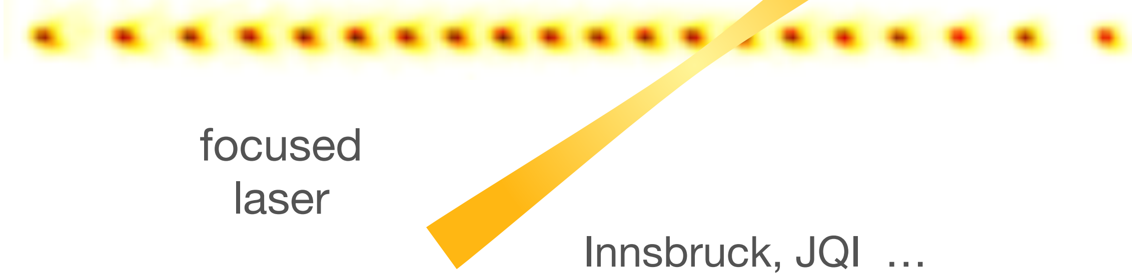
Rydberg Tweezer Arrays [1D,2D,3D]



Harvard - MIT, Palaiseau, JILA, Caltech, Wisconsin, Sandia, …

Trapped-Ions [1D, 2D]

… and single site control & readout



Innsbruck, JQI …

Engineered Spin Models & Hamiltonians

$$\hat{H} = \sum_i \frac{1}{2}\Omega_i \hat{\sigma}_x^i - \sum_i \Delta_i \hat{n}_i + \sum_{i<j} V_{ij}\hat{n}_i\hat{n}_j$$

$$V_{ij} = C_6 / r_{ij}^6$$

spin-spin interaction as Rydberg Van der Waals

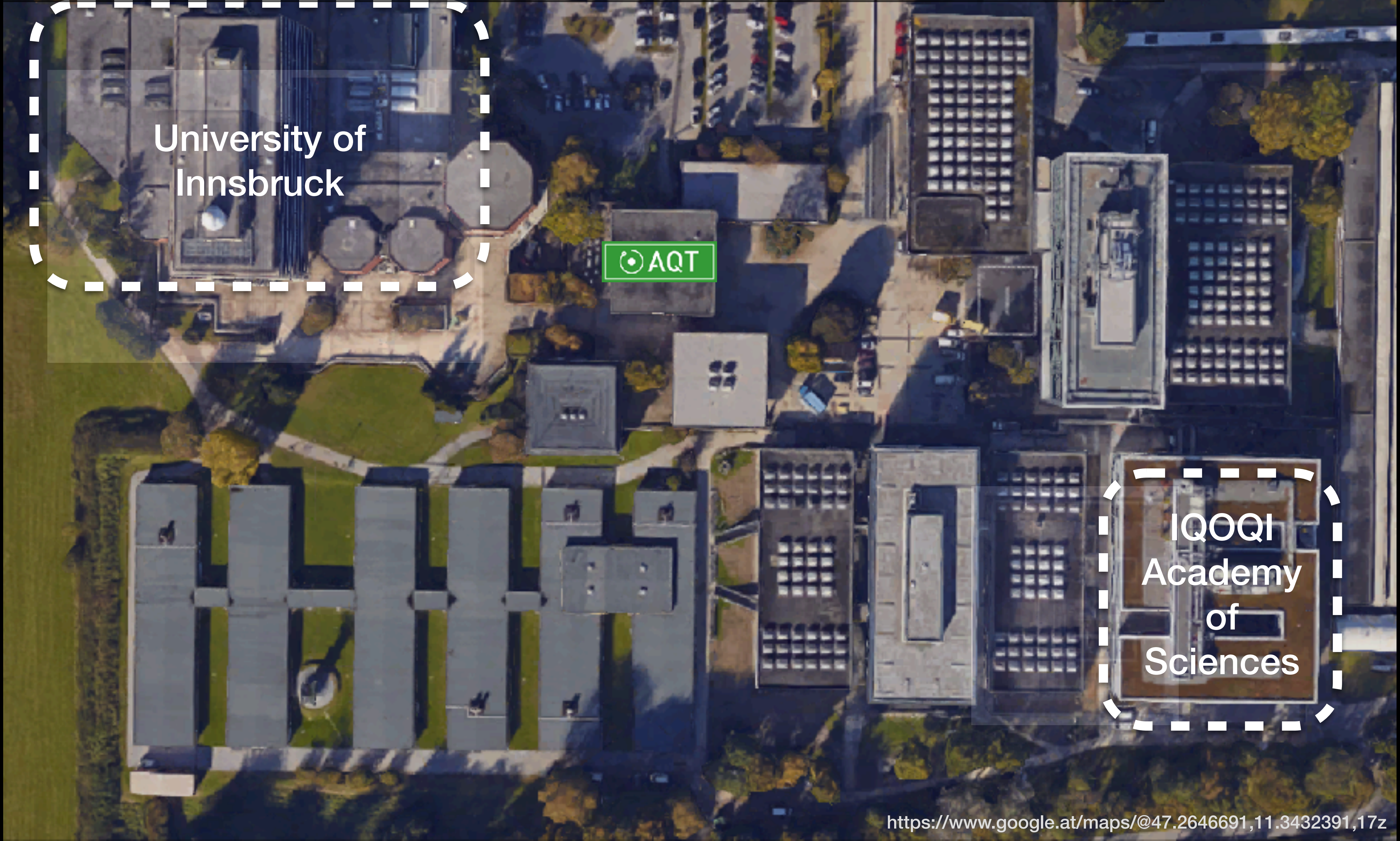
$$\hat{H}_{\text{Ising}} = \sum_{i,j} J_{ij}\hat{\sigma}_i^x\hat{\sigma}_j^x + B\sum_i \hat{\sigma}_i^z$$

$$J_{ij} \sim \frac{1}{|i-j|^\alpha} \quad \alpha = 0\ldots3$$ long range

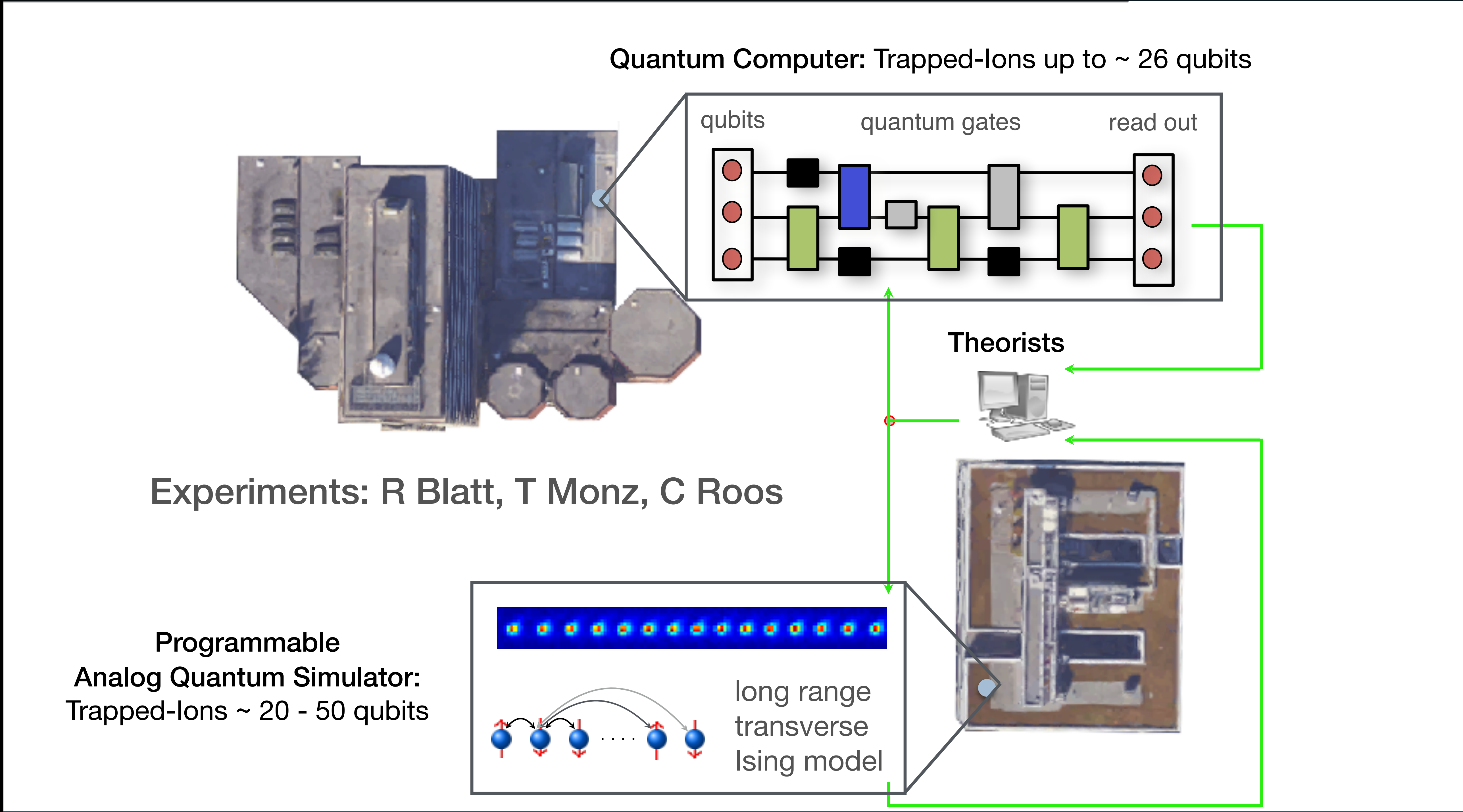
phonon-mediated spin-spin interaction

NISQ devices: few tens of atoms, scaling to ~ few hundred (no error correction)

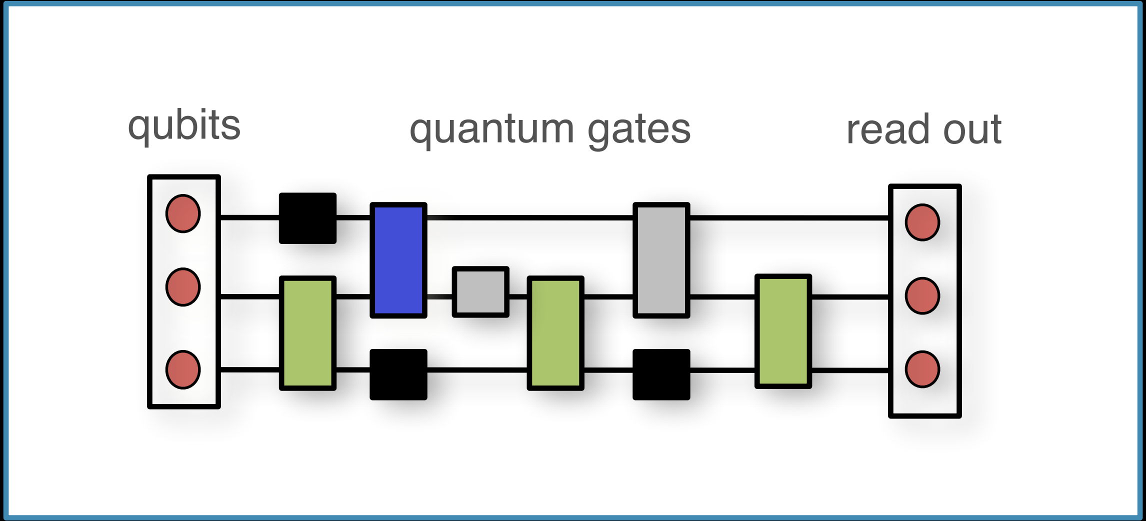
Innsbruck Quantum Cloud [since 2018]



Innsbruck Quantum Cloud [since 2018]

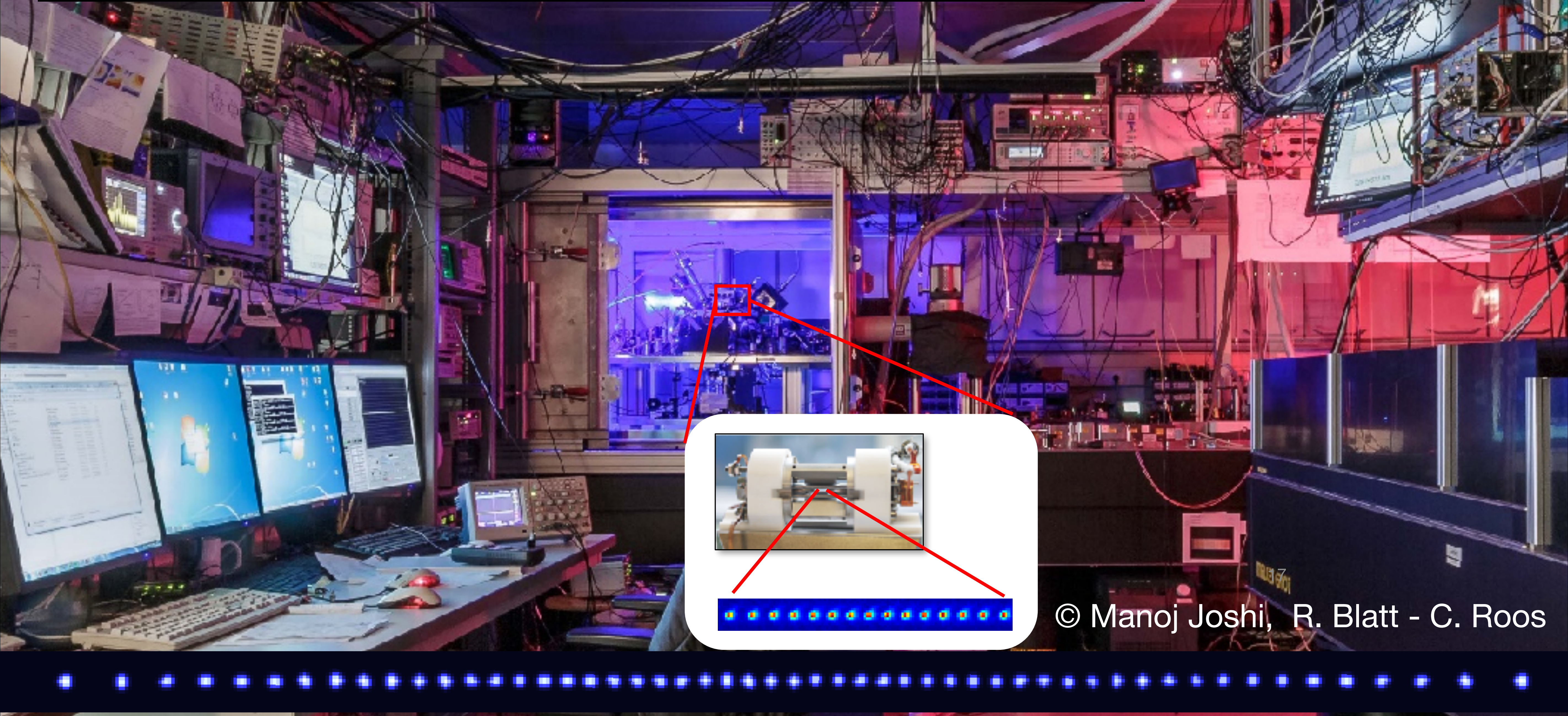


Trapped Ion Quantum Computer
@ R Blatt - T Monz UIBK-Labs



~26 qubits

10, 20 … 50 Qubit Trapped-Ion Programmable Quantum Simulator @ IQOQI-Labs



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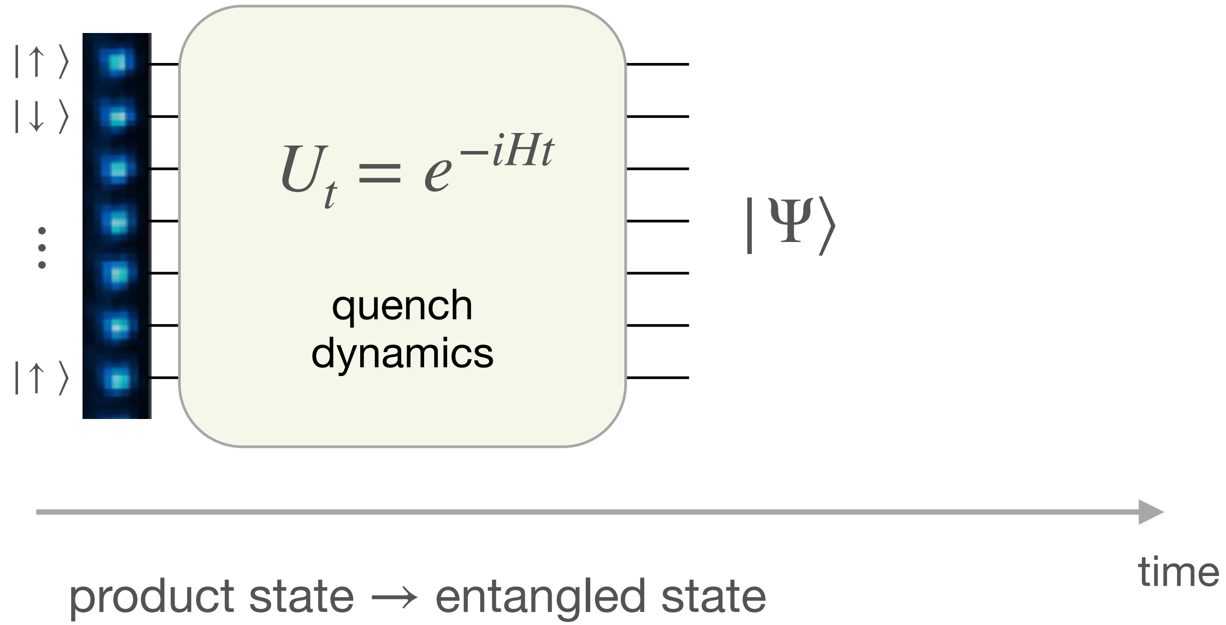
Analog Quantum Simulators

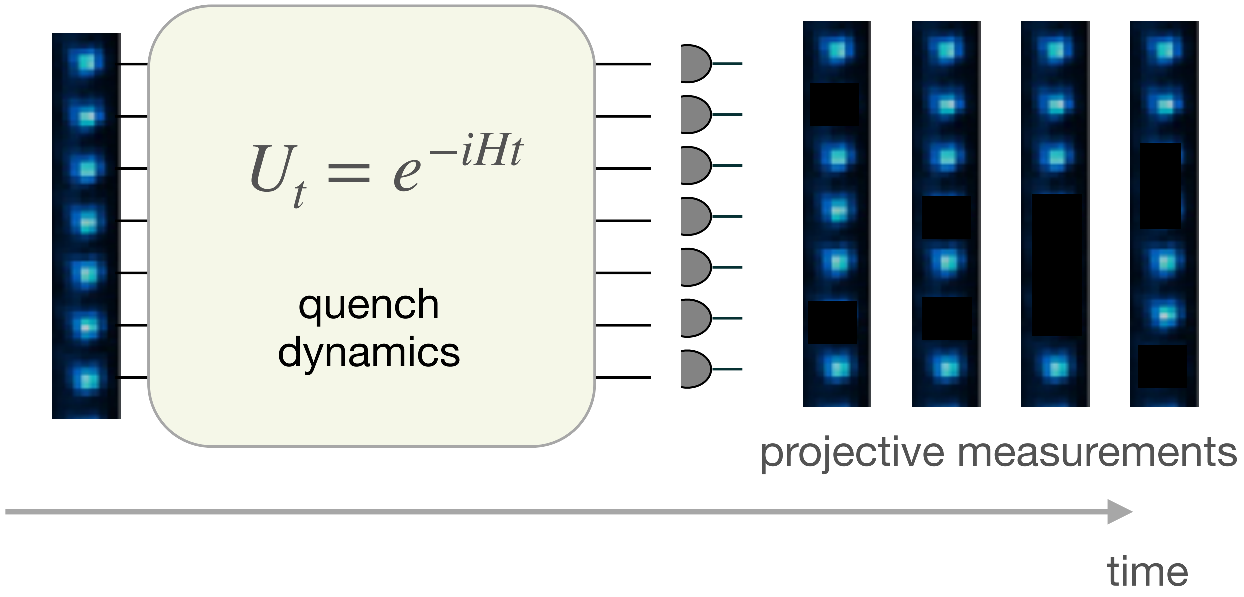
What physics can we do … ?



- isolated quantum system
- quantum many-body physics
- entanglement

Analog Quantum Simulators

What physics can we do … ?



Probability to see a specific configuration

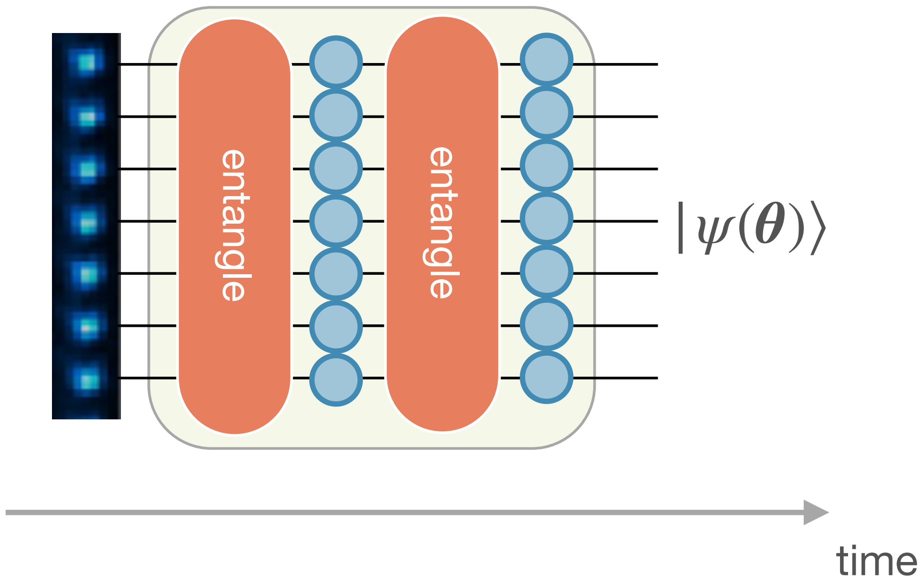
Repeat

$$P_\Psi(\mathbf{s})$$

quantum expectation value

`Programming' Quantum Simulators … Opportunities

programming quantum circuits



family of entangled states

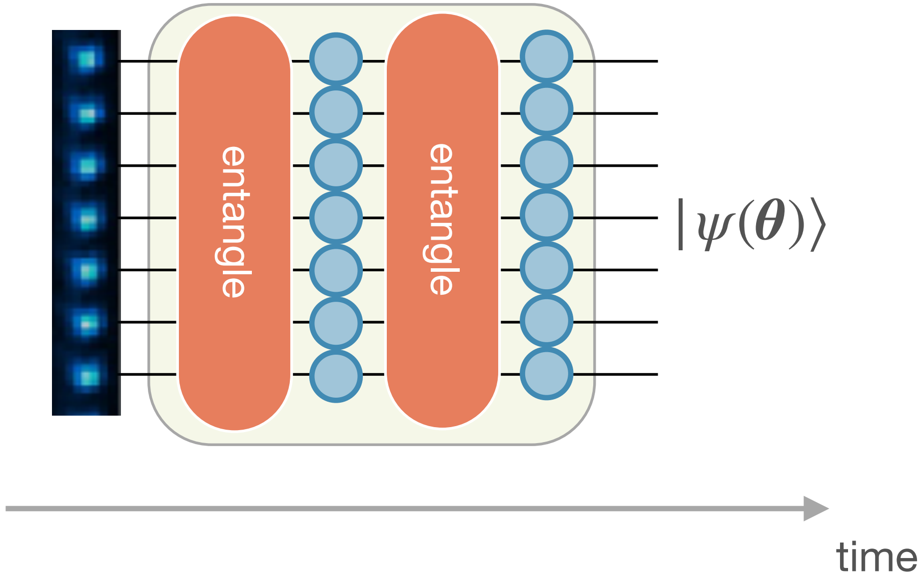
$$|\psi(\boldsymbol{\theta})\rangle = \hat{U}_N(\theta_N)\ldots\hat{U}_2(\theta_2)\hat{U}_1(\theta_1)|\psi_0\rangle$$

programming highly entangled quantum states from available quantum resources

… *program* interesting quantum states?

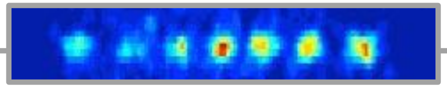
`Programming' Quantum Simulators

programming quantum circuits



family of entangled states

$$|\psi(\boldsymbol{\theta})\rangle = \hat{U}_N(\theta_N)\ldots\hat{U}_2(\theta_2)\hat{U}_1(\theta_1)|\psi_0\rangle$$



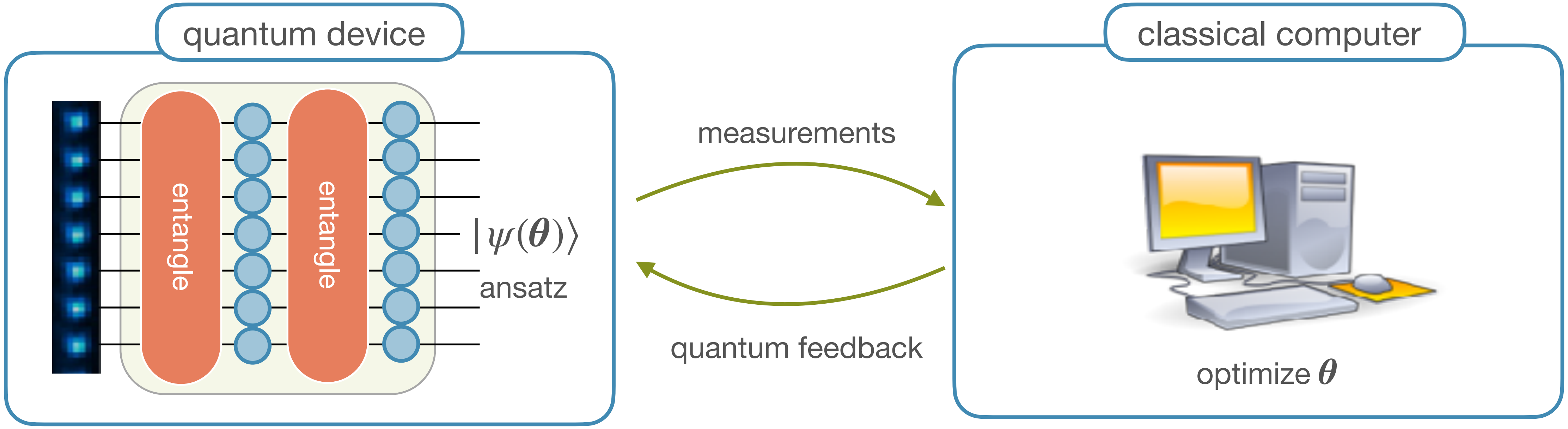
trapped ion quantum resources

$\hat{U}_1(\theta) = e^{-i\theta\sum_{ij} J_{ij}\hat{\sigma}_i^x\hat{\sigma}_j^x}$ entangle (Ising)

$\hat{U}_{2,i}(\theta) = e^{-i\theta\mathbf{n}\cdot\hat{\sigma}_i}$ local rotations

- in general not universal gate set
- scalable

`Programming' Quantum Simulators



Variational Classical-Quantum Algorithms

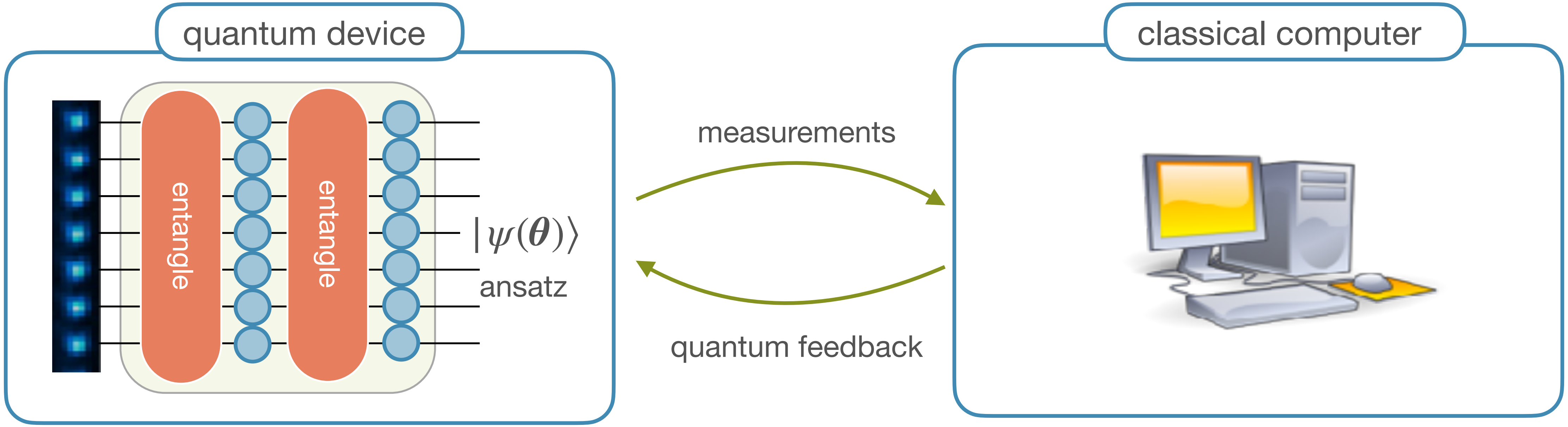
cost function: $\mathscr{C}(\boldsymbol{\theta}) = f(|\psi(\boldsymbol{\theta})\rangle) \to \text{opt}$

optimize on classical machine
variational parameters

evaluate on quantum machine
… efficiently!?

Review: M Cerezo, A Arrasmith, R Babbush, SC Benjamin, S Endo, K Fujii, JR McClean, K Mitarai, X Yuan, L Cincio, PJ Coles, Nature Reviews Physics 3, 625 (2021)

`Programming' Quantum Simulators



Variational Classical-Quantum Algorithms

target Hamiltonian (e.g. lattice model)

$$\hat{H}_T = \sum_{n\alpha} h_n^\alpha \hat{\sigma}_n^\alpha + \sum_{n\ell\alpha\beta} h_{n\ell}^{\alpha\beta} \hat{\sigma}_n^\alpha \hat{\sigma}_\ell^\beta + \ldots$$

Variational Quantum Eigensolver (see also QAOA)

$$\mathscr{C}(\theta) = \langle \psi(\boldsymbol{\theta}) | \hat{H}_T | \psi(\boldsymbol{\theta}) \rangle \to \text{min}$$

Hamiltonian never physically realized

… optimize quantum state

Review: M Cerezo, A Arrasmith, R Babbush, SC Benjamin, S Endo, K Fujii, JR McClean, K Mitarai, X Yuan, L Cincio, PJ Coles, Nature Reviews Physics 3, 625 (2021)

Classical - Quantum Feedback Loop

ARTICLE

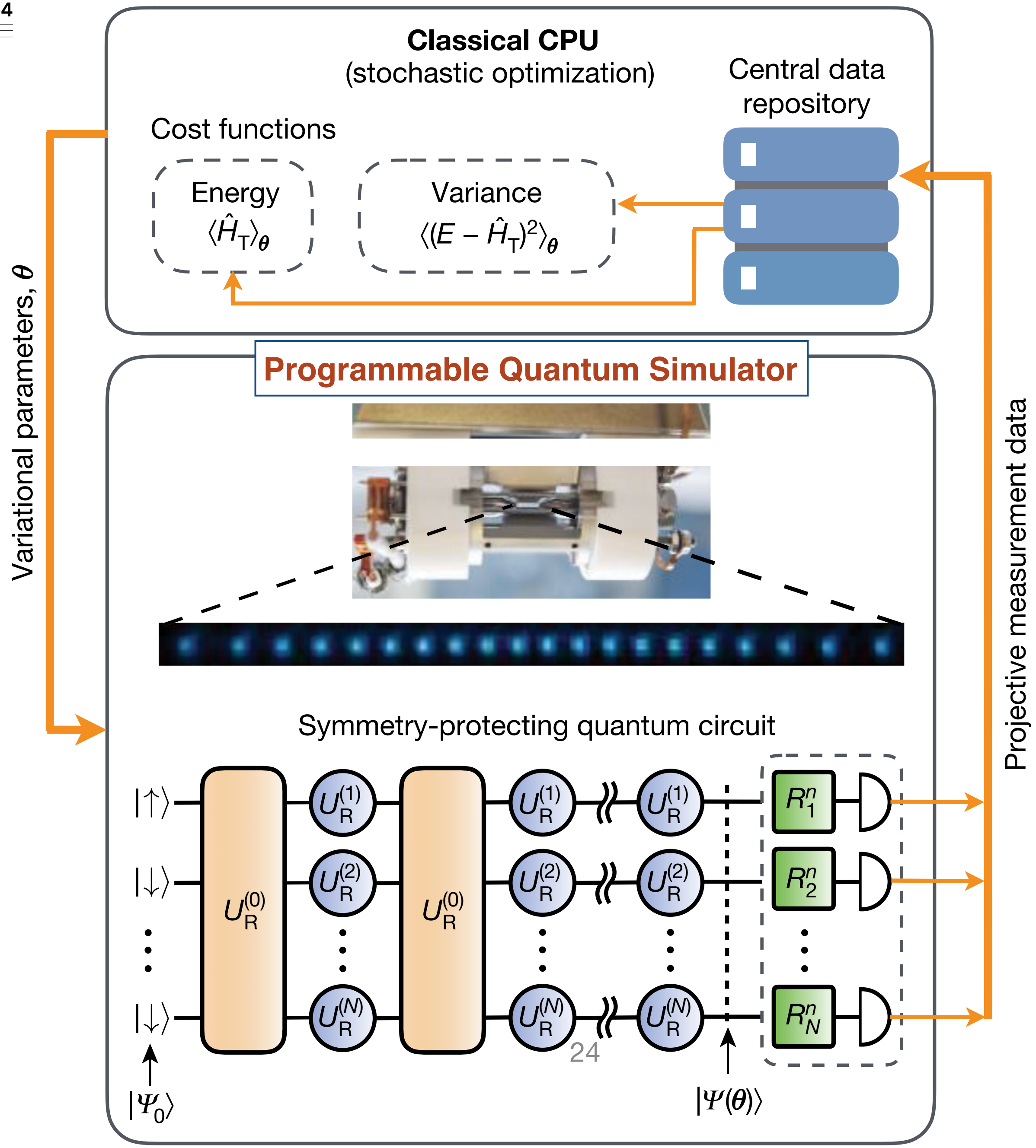
16 MAY 2019 | VOL 569 | NATURE | 355

https://doi.org/10.1038/s41586-019-1177-4

Self-verifying variational quantum simulation of lattice models

C. Kokail[1,2,3], C. Maier[1,2,3], R. van Bijnen[1,2,3], T. Brydges[1,2], M. K. Joshi[1,2], P. Jurcevic[1,2], C. A. Muschik[1,2], P. Silvi[1,2], R. Blatt[1,2], C. F. Roos[1,2] & P. Zoller[1,2]*

Rick van Bijnen (th-postdoc), Christine Maier (exp-PhD), Christian Kokail (th-PhD)



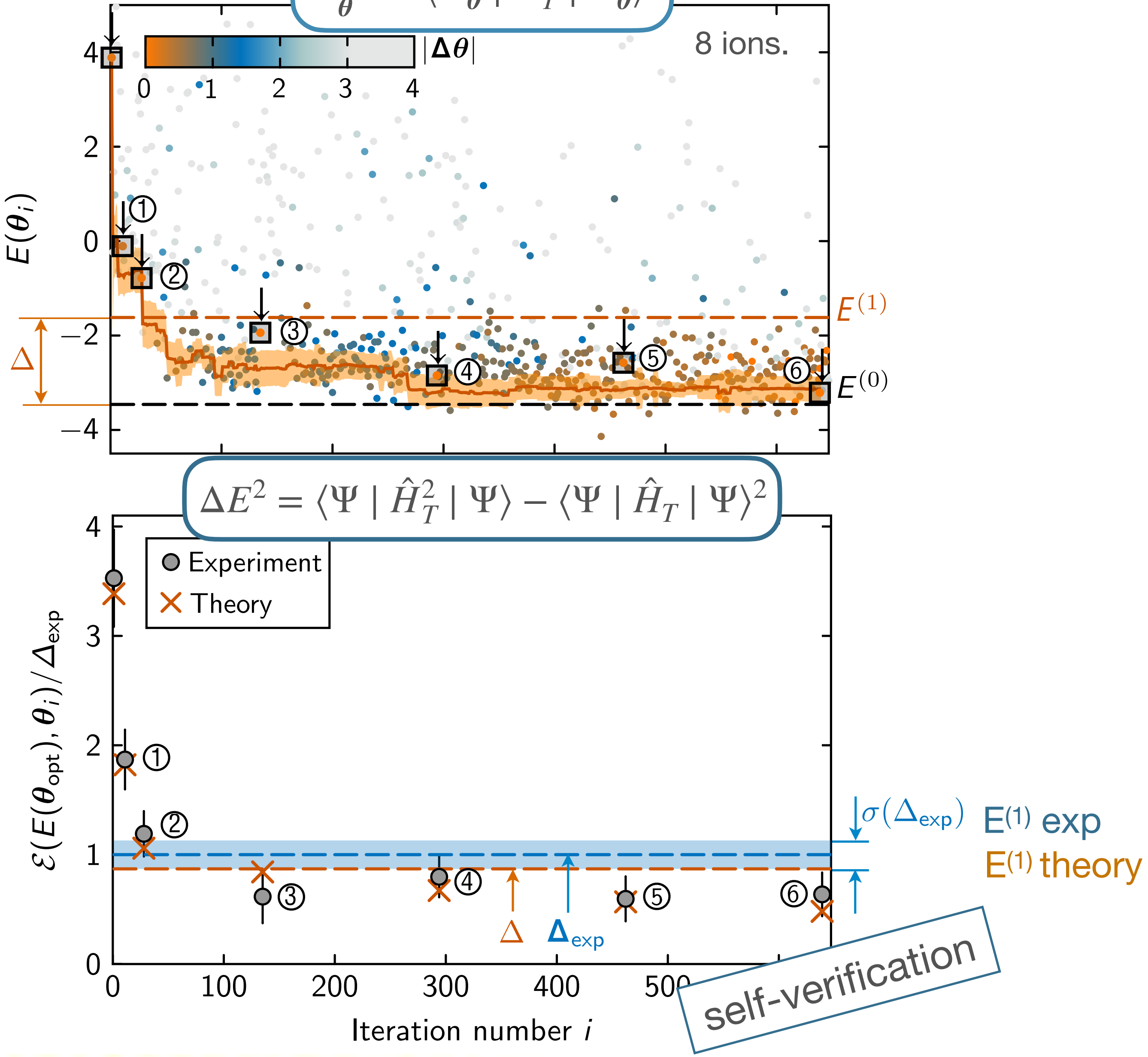
20 (now: 50) qubits, 10^5 call of PQS, circuit depth 6

Energy Optimization Trajectory for Ground State (VQE)


C Kokail   C Maier   R. van Bijnen

$$E_{\theta}^{(0)} = \langle \Psi_\theta \mid \hat{H}_T \mid \Psi_\theta \rangle$$

$$\Delta E^2 = \langle \Psi \mid \hat{H}_T^2 \mid \Psi \rangle - \langle \Psi \mid \hat{H}_T \mid \Psi \rangle^2$$

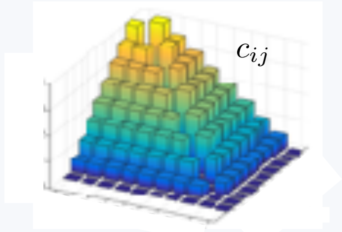



N=20 ions

Lattice Schwinger Model (1D QED)

$$H_S = J \sum c_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

long - range interaction



$$+w \sum \left(\hat{\sigma}_i^+ \hat{\sigma}_{i+1}^- + \hat{\sigma}_{i+1}^+ \hat{\sigma}_i^- \right)$$

particle - antiparticle creation/annihilation

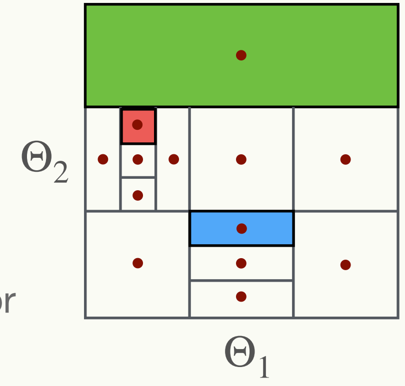
$$+m \sum c_i \hat{\sigma}_i^z + J \sum \tilde{c}_i \hat{\sigma}_i^z$$

effective particle masses

DIviding RECTangles (DIRECT)

global optimization
in noisy landscape

- 15 parameters
- circuit depth = 6
- budget: 10^5 calls to simulator



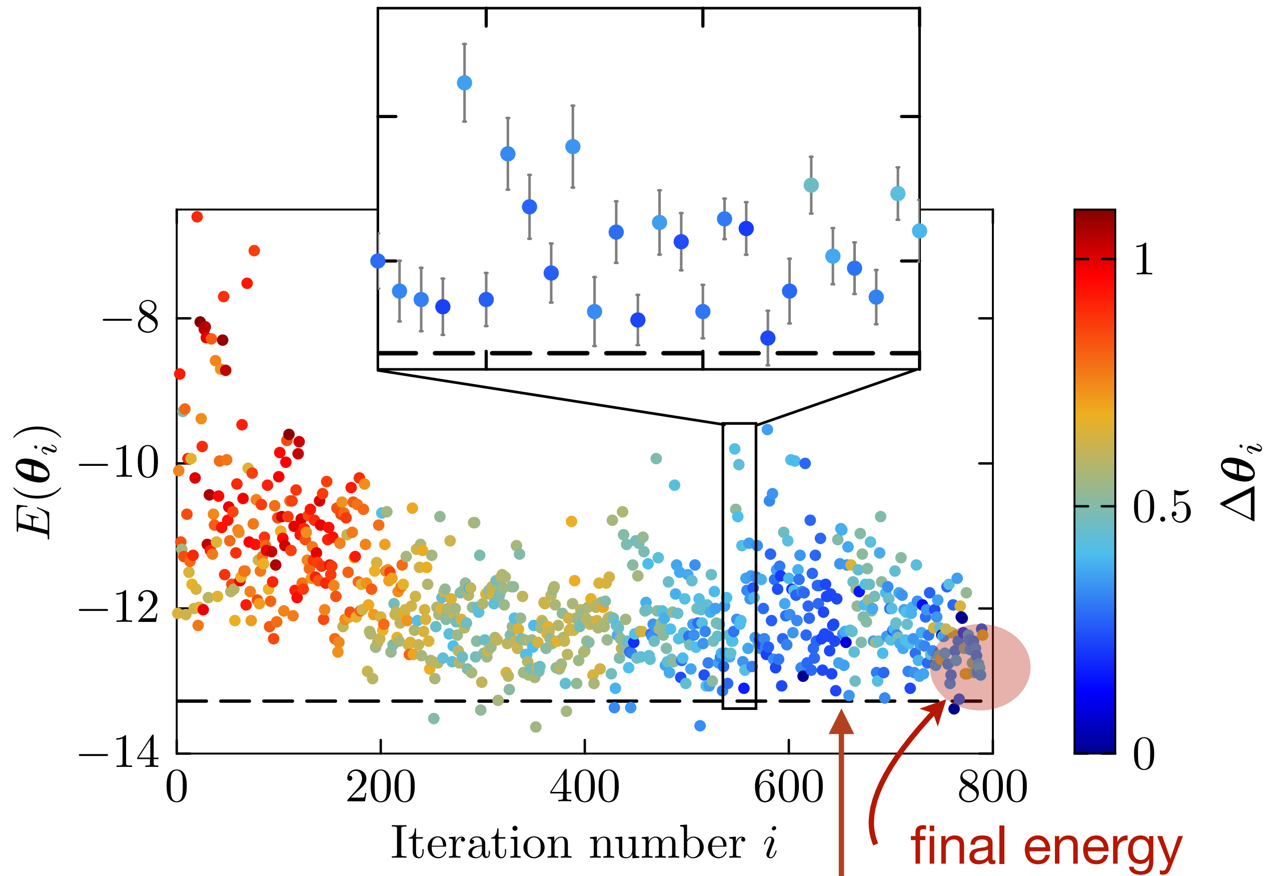
C Kokail, C Maier, R van Bijnen, T Brydges, MK Joshi, P Jurcevic, CA Muschik, P Silvi, R Blatt, CF Roos & P.Z., Nature (2019)

q-technology 2018

Experimental Energy Optimization Trajectory for Ground State (VQE)

C Kokail R van Bijnen M Joshi

N = 31 ions

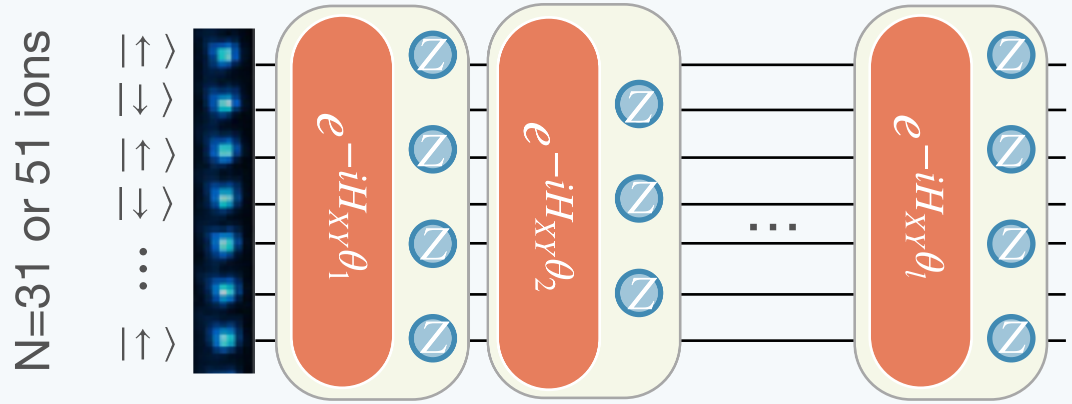


Heisenberg Model (spin-½)

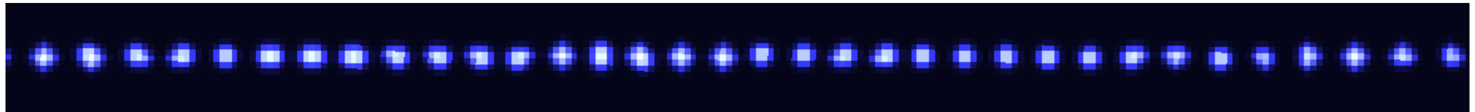
$$\hat{H} = J\sum_{i=1}^{N-1}\left(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y\right) + \Delta\sum_{i=1}^{N-1}\hat{S}_i^z\hat{S}_{i+1}^z + h\sum_{i=1}^{N}\hat{S}_i^z$$

$J = 1 \quad \Delta = 1 \quad h = 0.5$

VQE Circuit with Trapped Ion Resources



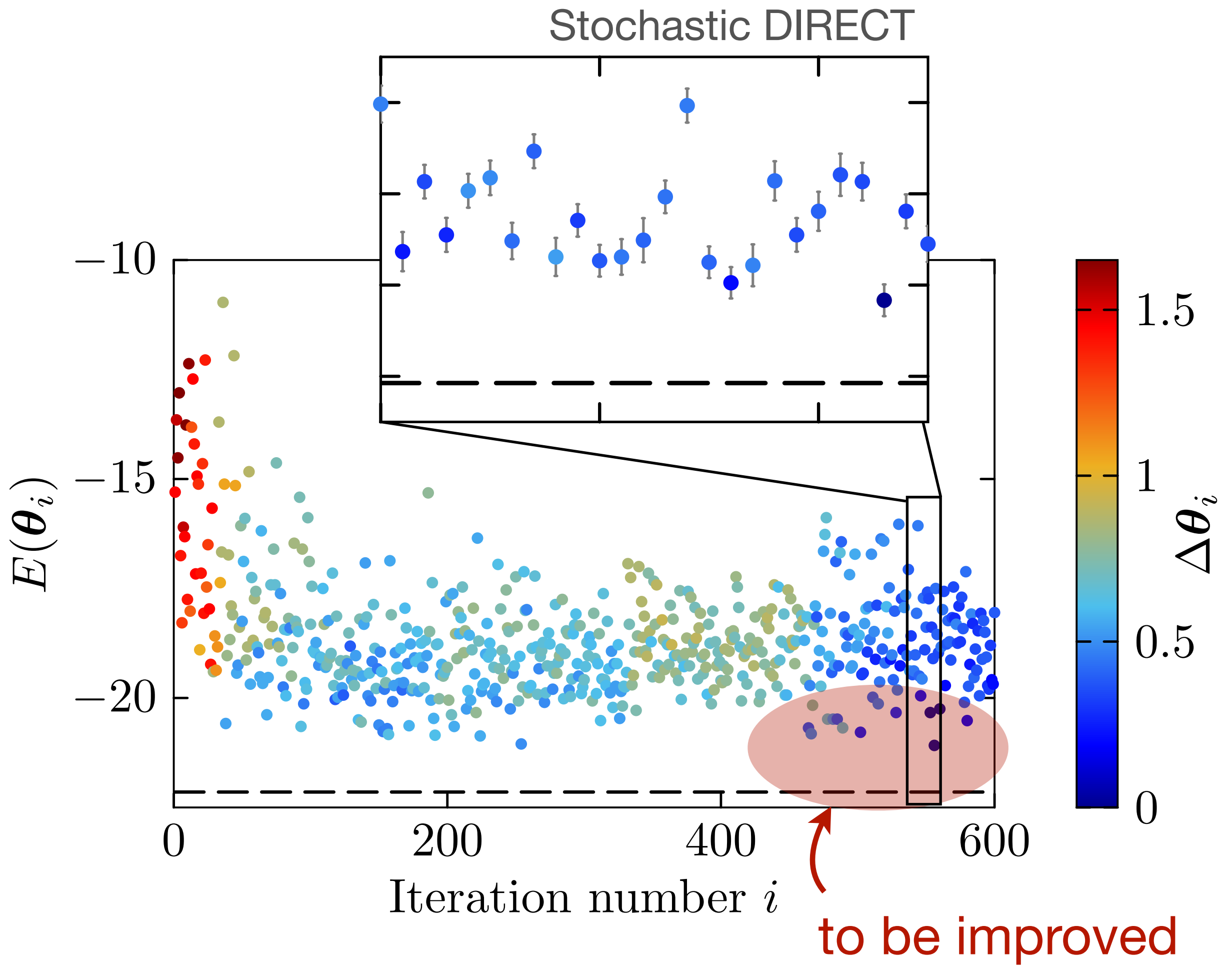
5 layers and 10 parameters



Experimental Energy Optimization Trajectory for Ground State (VQE)

Theory: C Kokail, R van Bijnen et al., unpublished
Experiment: M Joshi et al., unpublished

N = 51 ions



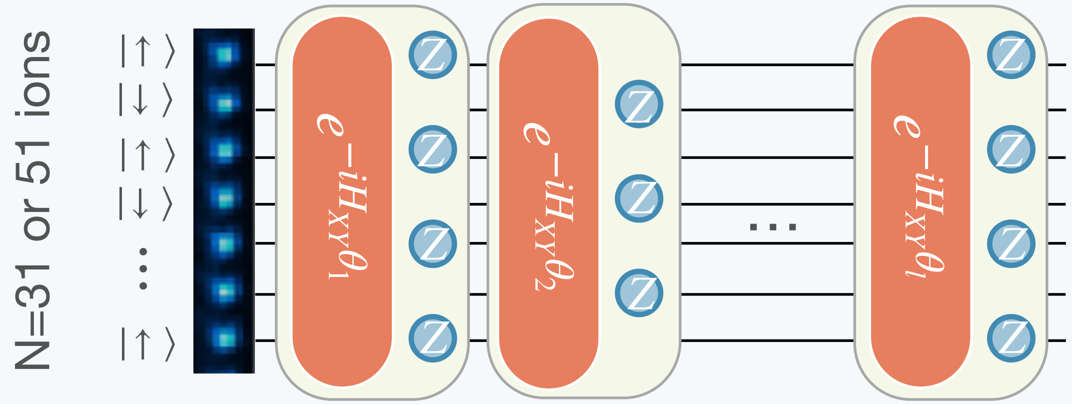
~ low temperature state T ~ few J

Heisenberg Model (spin-½)

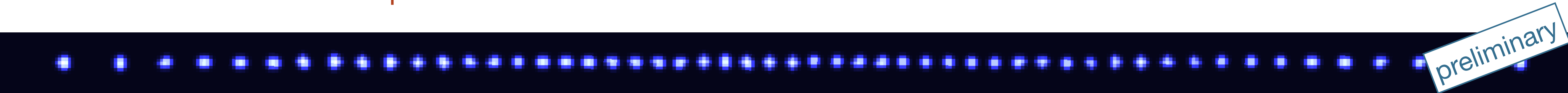
$$\hat{H} = J\sum_{i=1}^{N-1}\left(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y\right) + \Delta\sum_{i=1}^{N-1}\hat{S}_i^z\hat{S}_{i+1}^z + h\sum_{i=1}^{N}\hat{S}_i^z$$

$J = 1 \quad \Delta = 1 \quad h = 0.5$

VQE Circuit with Trapped Ion Resources



5 layers and 10 parameters



Lecture 1:

Introduction & Background Material

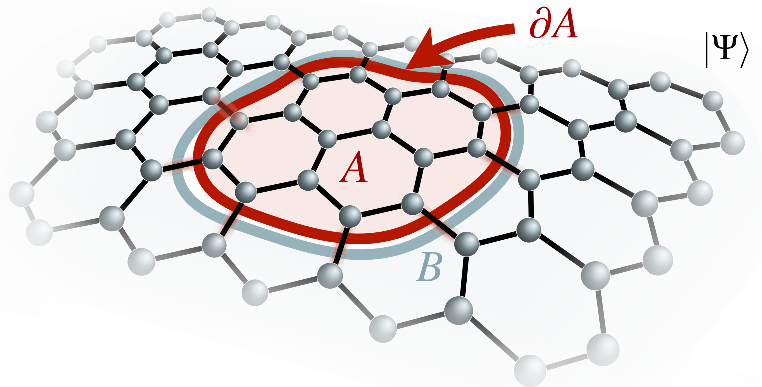
- Programmable Quantum Simulators - Atomic Platforms
- Characterizing Entanglement in Many-Body Systems

How to measure Entanglement

- Renyi Entanglement Entropy
- …

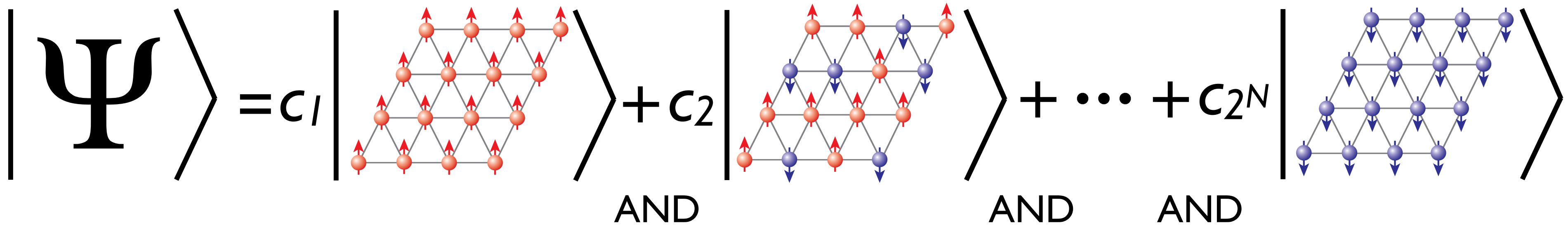
- quantum state tomography
- copies - quantum protocol
- randomized measurements & classical shadows

Entanglement in Quantum Many-Body Systems



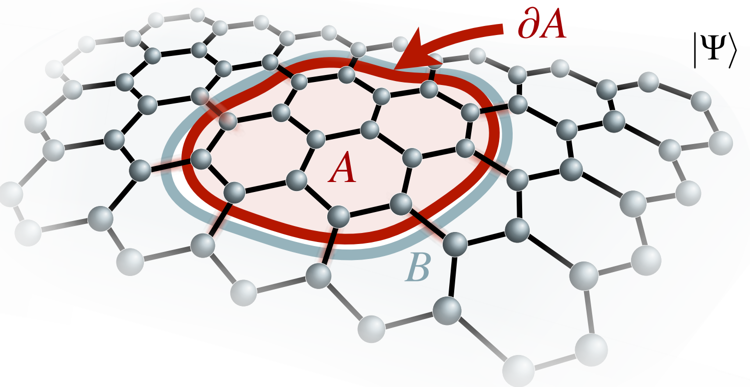
Pure N-body quantum state

$$|\Psi\rangle = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_N=0}^{1} c_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle \in \mathcal{H}_2^{\otimes N} \qquad i \in 0,1 \text{ or } \{\uparrow, \downarrow\}$$



Reviews: L. Amico, R. Fazio, A. Osterloh and V. Vedral, RMP (2008); J. Eisert, M. Cramer and M. B. Plenio, RMP (2010); P. Calabrese, J. Cardy and B. Doyon, JPA (2009); I. Peschel and V. Eisler, JPA (2009); O. Gühne, and G. Tóth, Phys. Rep. (2009); J. I Cirac, D Pérez-García, N Schuch, and F Verstraete, RMP (2021)

Entanglement in Quantum Many-Body Systems



Pure N-body quantum state

$$|\Psi\rangle = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\dots\sum_{i_N=0}^{1} c_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle \in \mathcal{H}_2^{\otimes N}$$

$i \in 0{,}1$ or $\{\uparrow,\downarrow\}$

bi-partite entanglement

$$|\Psi\rangle = \sum_{i=1}^{\mathcal{D}_A}\sum_{j=1}^{\mathcal{D}_B} c_{ij} |\boldsymbol{i}\rangle_A \otimes |\boldsymbol{j}\rangle_B$$

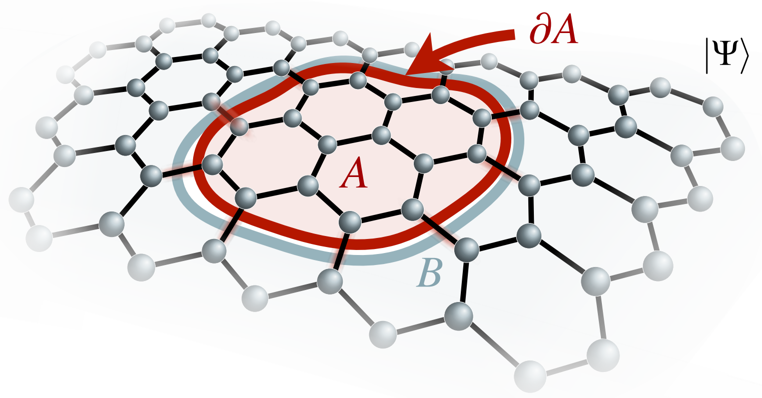
$$c_{ij} = \sum_{\alpha=1}^{\chi_A} U_{i\alpha}\lambda_\alpha \left[V^\dagger\right]_{\alpha j}$$

Singular Value Decomposition (SVD)

$$|\Phi_A^\alpha\rangle = \sum_{i=1}^{\mathcal{D}_A} U_{i\alpha} |i\rangle_A \qquad |\Phi_B^\alpha\rangle = \sum_{j=1}^{\mathcal{D}_B} \left[V^\dagger\right]_{\alpha j} |j\rangle_B$$

Reviews: L. Amico, R. Fazio, A. Osterloh and V. Vedral, RMP (2008); J. Eisert, M. Cramer and M. B. Plenio, RMP (2010); P. Calabrese, J. Cardy and B. Doyon, JPA (2009); I. Peschel and V. Eisler, JPA (2009); O. Gühne, and G. Tóth, Phys. Rep. (2009); J. I Cirac, D Pérez-García, N Schuch, and F Verstraete, RMP (2021)

Entanglement in Quantum Many-Body Systems



Pure N-body quantum state

$$|\Psi\rangle = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \dots \sum_{i_N=0}^{1} c_{i_1 i_2 \dots i_N} |i_1\rangle \otimes |i_2\rangle \otimes \dots \otimes |i_N\rangle \in \mathcal{H}_2^{\otimes N}$$

$i \in 0{,}1$ or $\{\uparrow, \downarrow\}$

Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi_A} \lambda_\alpha |\Phi_A^\alpha\rangle \otimes |\Phi_B^\alpha\rangle$$

χ_A Schmidt rank

$\chi_A = 1$ product state $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$

$\chi_A > 1$ entangled state $|\Psi\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$

reduced density matrix

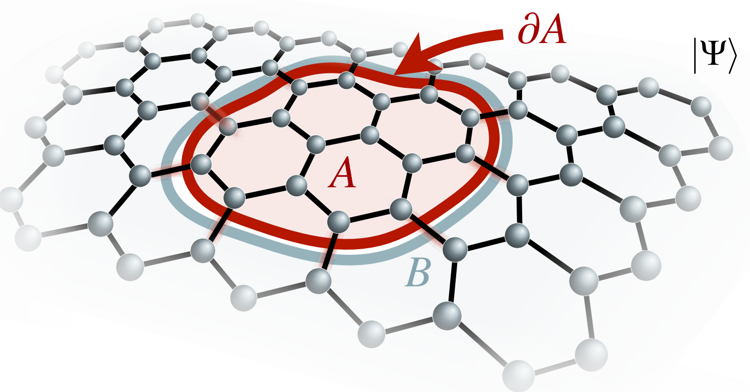
$$\rho_A = \mathrm{Tr}_B\left(|\Psi\rangle\langle\Psi|\right) = \sum_{\alpha=1}^{\chi_A} |\lambda_\alpha|^2 |\Phi_A^\alpha\rangle\langle\Phi_A^\alpha|$$

ρ_A has rank χ_A

Schmidt vectors
as eigenstates of ρ_A

How to measure?

e.g. more efficiently then tomography



Entanglement entropy & measurement of bipartite entanglement

Von Neumann

$$S_{VN}(\rho_A) = -\mathrm{Tr}\{\rho_A \log \rho_A\}$$

$\chi_A = 1$ product state $S_{VN}(\rho_A) = 0$

$\chi_A > 1$ entangled state $S_{VN}(\rho_A) > 0$

Rényi entropy

$$S_n(\rho_A) = \frac{1}{1-n} \log \mathrm{Tr}\{\rho_A^n\} \quad (n = 2, \ldots)$$

n=2

$$\mathrm{Tr}\{\rho_A^2\}$$

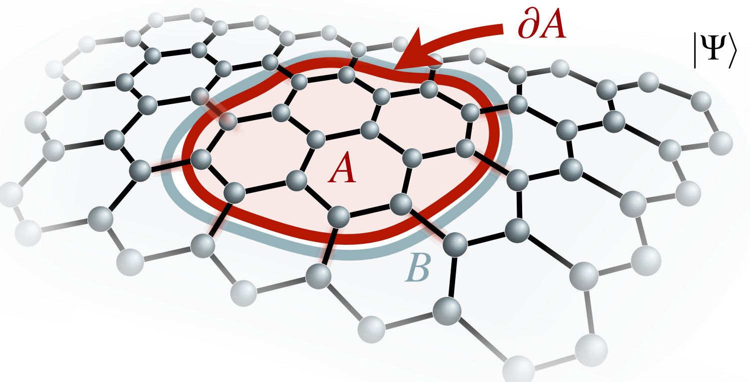
purity of reduced density matrix

$= 1$ product state

< 1 entangled state

properties:

- $S_{VN}(\rho)$ $[= \lim_{n \to 1} S_n(\rho)]$
- $S_{VN}(\rho) \geq S_2(\rho)$
- $S_{VN}(\rho) \geq 2S_2(\rho) - S_3(\rho)$



Entanglement entropy & measurement of bipartite entanglement

Von Neumann $S_{VN}(\rho_A) = -\mathrm{Tr}\{\rho_A \log \rho_A\}$

$\chi_A = 1$ product state $S_{VN}(\rho_A) = 0$

$\chi_A > 1$ entangled state $S_{VN}(\rho_A) > 0$

Rényi entropy $S_n(\rho_A) = \dfrac{1}{1-n} \log \mathrm{Tr}\{\rho_A^n\}$ $(n = 2, \ldots)$

n=2 $\mathrm{Tr}\{\rho_A^2\}$

$= 1$ product state

purity of reduced density matrix

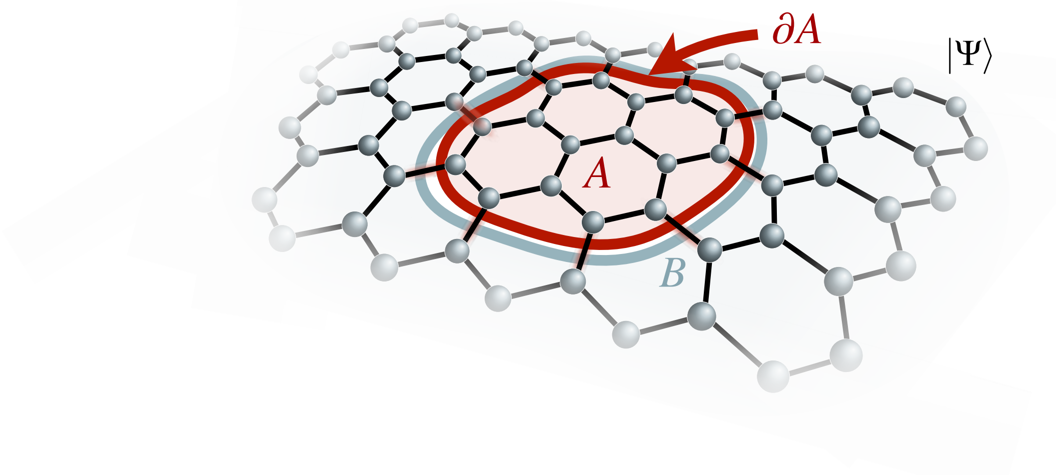
< 1 entangled state

nonlinear functional of density matrix

but expectation values are always linear: $\langle \hat{A} \rangle = \mathrm{Tr}[\hat{A}\rho]$:-(

How to measure?

e.g. more efficiently then tomography



Entanglement Hamiltonian and entanglement spectrum

$$\rho_A = \mathrm{Tr}_B \left[|\Psi\rangle\langle\Psi| \right] = \sum_{\alpha=1}^{\chi} e^{-\xi_\alpha} |\Phi_\alpha^A\rangle\langle\Phi_\alpha^A| = e^{-\tilde{H}_A}$$

mixed state: Gibbs ensemble with EH

entanglement spectrum (ES) — entanglement Hamiltonian (EH)

Why interesting?

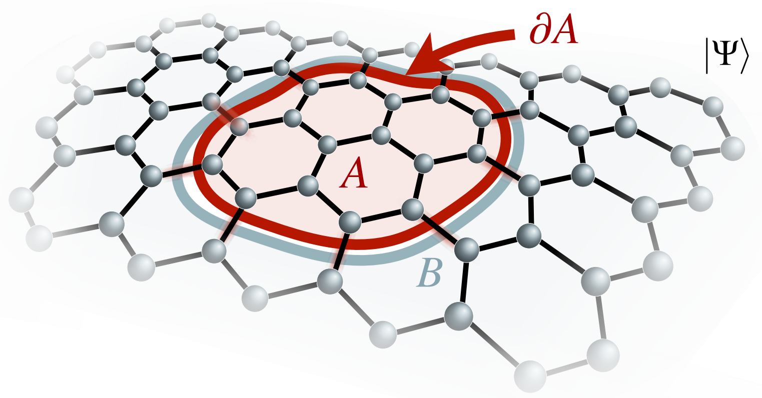
In quantum many-body problems the entanglement Hamiltonian $\tilde{H}_A$ often has a *simple* operator structure

↕

Can we *learn* operator structure of EH?

e.g. more efficiently then tomography

Entanglement spectroscopy?



Entanglement Hamiltonian and entanglement spectrum

$$\rho_A = \mathrm{Tr}_B\left[\,|\Psi\rangle\langle\Psi|\,\right] \quad = \sum_{\alpha=1}^{\chi} e^{-\xi_\alpha}|\Phi_\alpha^A\rangle\langle\Phi_\alpha^A| = e^{-\tilde{H}_A}$$

mixed state: Gibbs ensemble with EH

entanglement spectrum (ES) entanglement Hamiltonian (EH)

Why interesting?

- Entanglement measures
- fingerprint of topological order (Li-Haldane)
- detection of quantum phase transitions

… low-lying entanglement spectrum can be used as a "fingerprint" to identify topological order. [PRL 2008]

D. Haldane

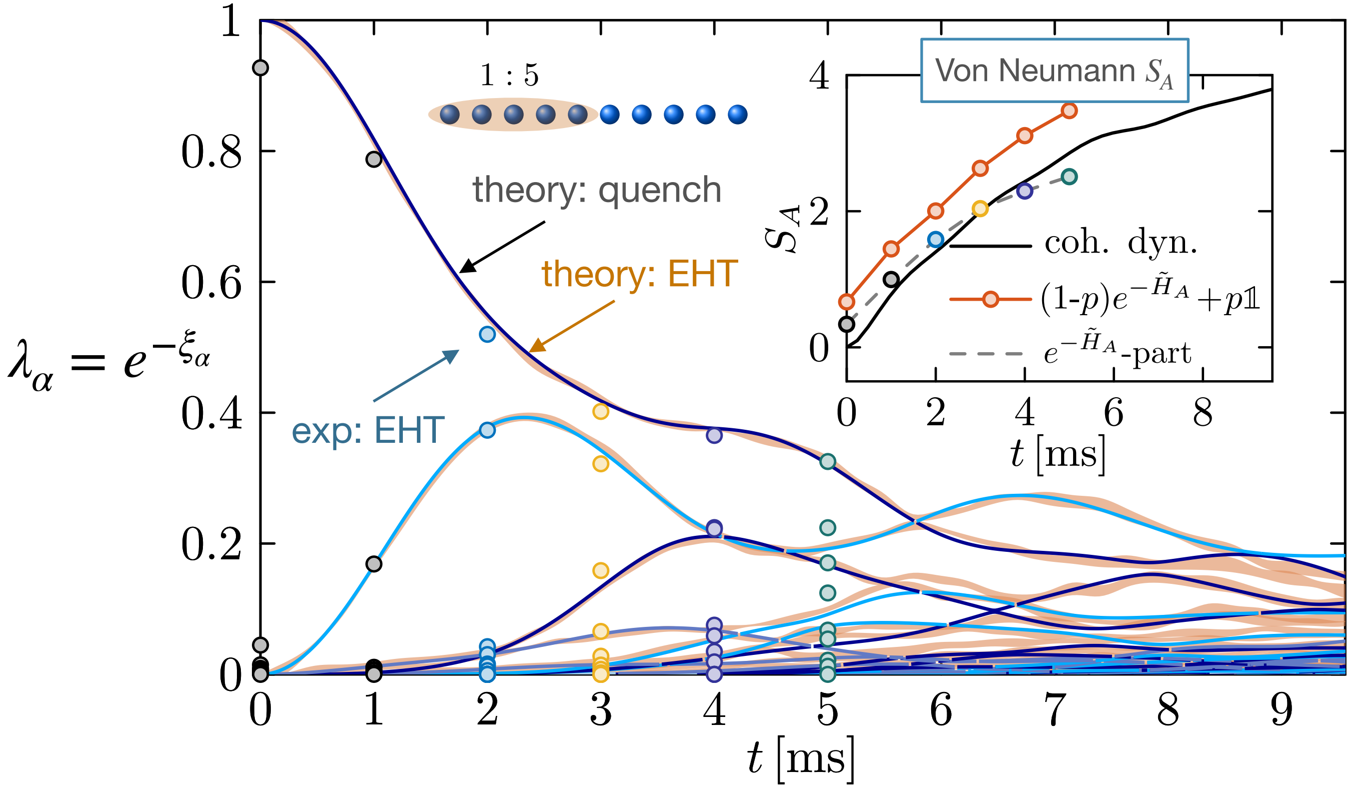
Can we *learn* operator structure of EH?

e.g. more efficiently then tomography

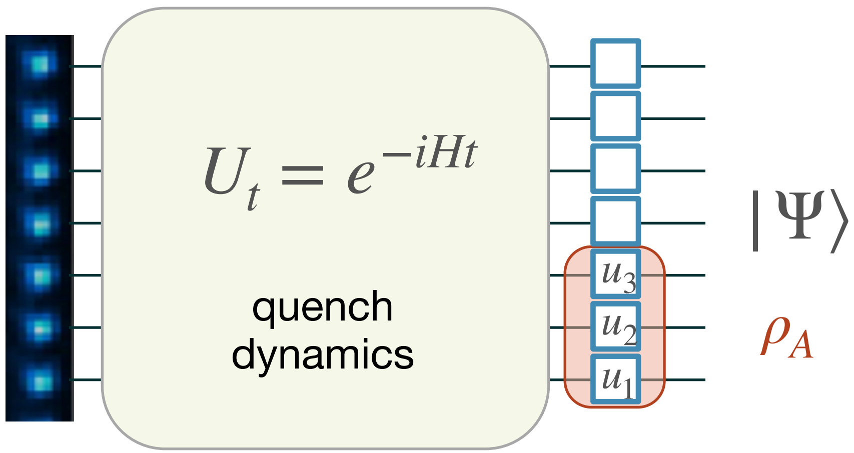
Entanglement spectroscopy?

Example: Entanglement Spectrum & Quench Dynamics Th+Exp

Sub-system [1:5] of 10 ions [similar data for 20 ions and subsystem [8:14]]



$$H = \sum_{i<j} \left(J_{ij}\sigma_i^+ \sigma_j^- + \text{ h.c.} \right) + B \sum_i \sigma_i^z$$



$$|\Psi\rangle = \sum_{\alpha=1}^{\chi} e^{-\xi_\alpha/2} |\Phi_\alpha^A\rangle \otimes |\Phi_\alpha^B\rangle$$

We wish to `see' the Schmidt values $\lambda_\alpha = e^{-\xi_\alpha}$ in quench dynamics in time

C Kokail, R van Bijnen, A Elben, B Vermersch, & P.Z, Nature Physics (2019); with experimental data from T. Brydges et al., Science (2019)

Lecture 1:

Introduction & Background Material

- Programmable Quantum Simulators - Atomic Platforms
- Characterizing Entanglement in Many-Body Systems

How to measure Entanglement

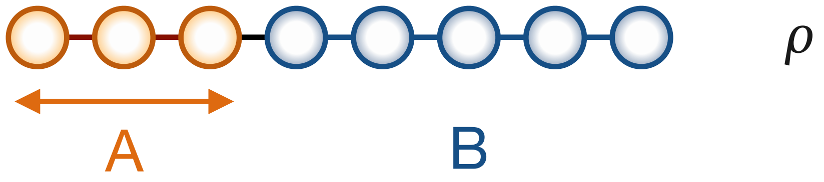
- Renyi Entanglement Entropy
- …

- quantum state tomography
- copies - quantum protocol
- randomized measurements & classical shadows

The randomized measurement toolbox,
A Elben, ST Flammia, HY Huang, R Kueng, J Preskill, B Vermersch & PZ,
Nature Review Physics (2022)

Measuring Renyi Entanglement Entropy

task: measure 2nd order Renyi entanglement entropy



$\mathrm{Tr}_A\, \rho_A^2$ Renyi entropy n=2

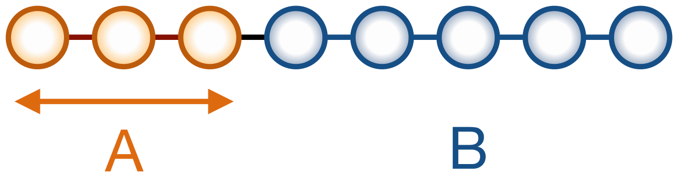
~ purity of subsystem

nonlinear functional of density matrix

but expectation values are always linear: $\langle \hat{A} \rangle = \mathrm{Tr}[\hat{A}\rho]$:-(

Measuring Renyi Entanglement Entropy

Protocol 0: Quantum State Tomography



$\mathrm{Tr}_A\, \rho_A^2$ Renyi entropy n=2

~ purity of subsystem

measurement data → ρ_A

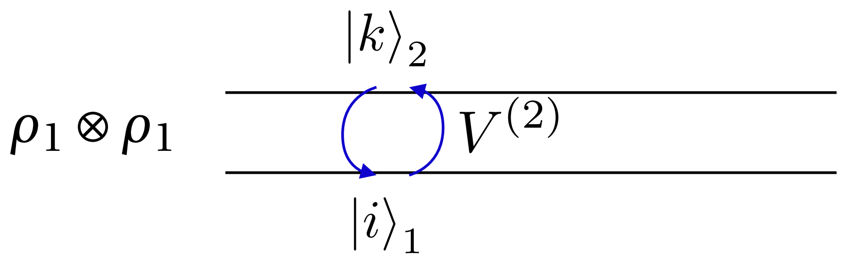
✓ expensive* $\sim \mathrm{rank}(\rho_A)\ 2^{N_A}$ (scales exponentially)

* except very small (sub)systems, or we know something about quantum state

Measuring Renyi Entanglement Entropy

Protocol 1: Copies of the quantum system [quantum protocol]

Example n=2:



swap operator

$$V^{(2)} |i\rangle_1 \otimes |k\rangle_2 = |k\rangle_1 \otimes |i\rangle_2$$

$$\mathrm{tr}\{V^{(2)} \rho_1 \otimes \rho_2\} = \mathrm{tr}\left\{ V^{(2)} \sum_{ijkl} \rho^{(1)}_{ij} \rho^{(2)}_{kl} |i\rangle \langle j| \otimes |k\rangle \langle \ell| \right\}$$

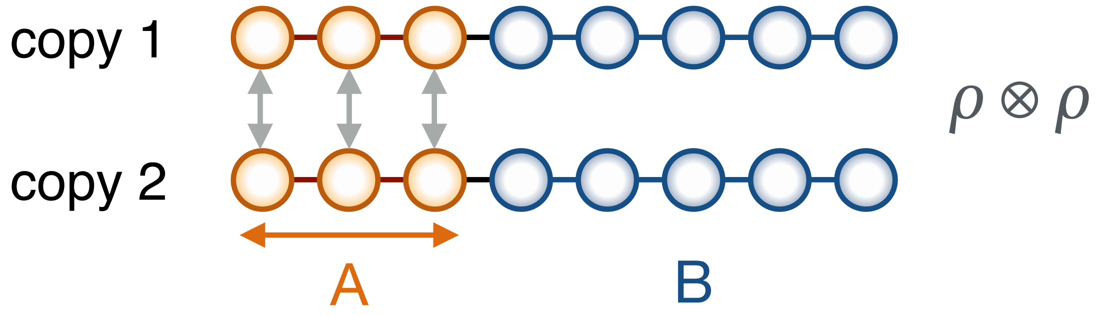
expectation value

$$= \mathrm{tr}\{\rho_1 \rho_2\}$$

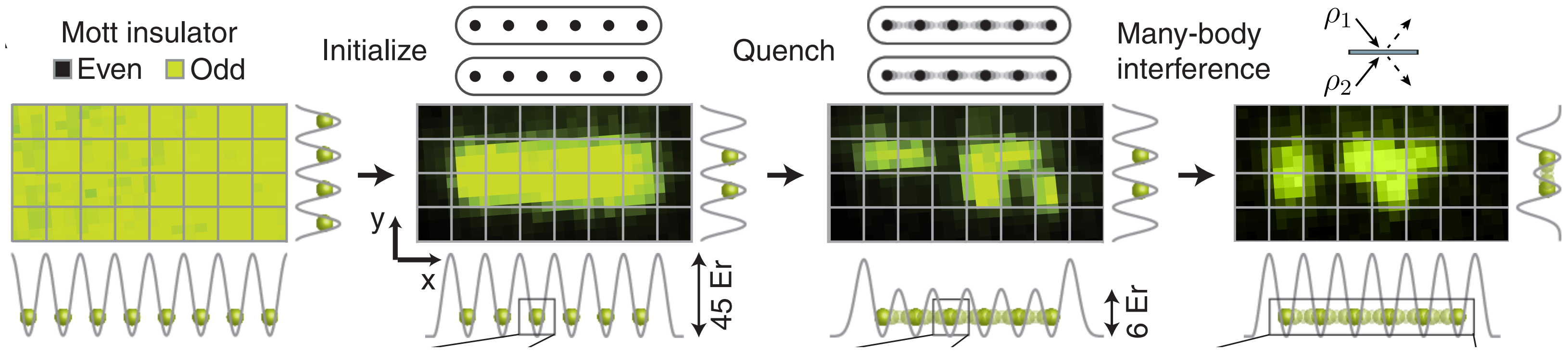
theory: AJ Daley, H Pichler, J Schachenmayer, PZ, PRL (2012); C Moura Alves & D Jaksch, PRL (2004); A. K. Ekert et al.PRL (2002).

Measuring Renyi Entanglement Entropy

Protocol 1: Copies of the quantum system [quantum protocol]



Controlled few-atom systems & quantum gas microscope



see Appendix with details of protocol

experiment: R Islam *et al*., Nature (2015); AM Kaufmann *et al.*, Science (2016) [Greiner Group]

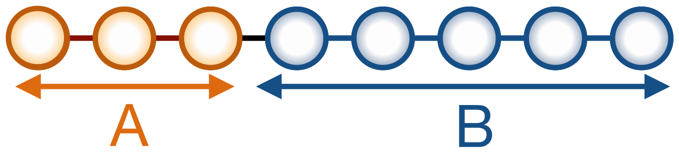
experiment: R Islam *et al*., Nature (2015); AM Kaufmann *et al.*, Science (2016) [Greiner Group]

Measuring Renyi Entanglement Entropy


A Elben B Vermersch

Protocol 2: Single copy of quantum system

single system



$\rho \otimes \rho$?

virtual copy*
(replica trick)

how?

$\mathrm{Tr}_A \, \rho_A^2$ …

from Statistical Correlations
in Random Measurements

signal is in the noise

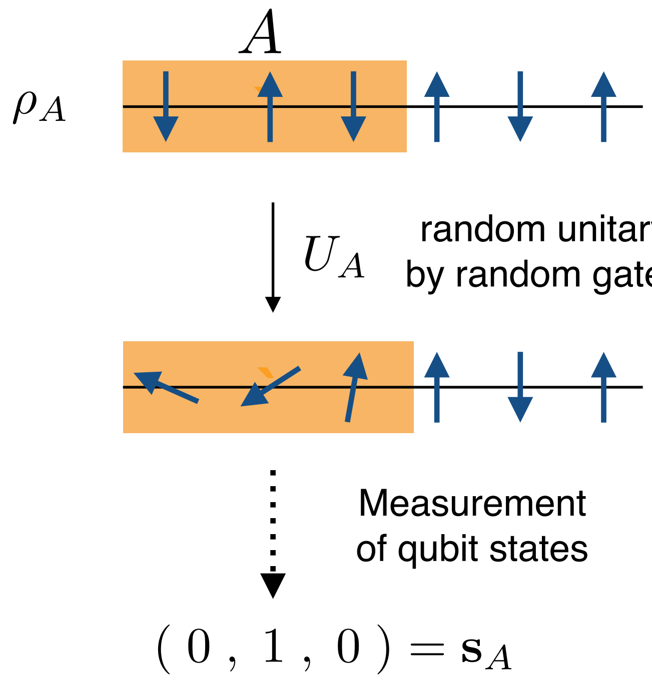
* in contrast to real copies, virtual copies are legal in quantum mechanics

Statistical Correlations in Random Measurements

Protocol for a chain of qubits:

Random measurement

$$P_U(\mathbf{s}_A) = \mathrm{Tr}\left[\boldsymbol{U_A \rho_A U_A^\dagger} \left|\mathbf{s}_A\right\rangle \left\langle \mathbf{s}_A\right|\right]$$



Average over the Circular Unitary Ensemble (CUE)

$$\overline{P_U(\mathbf{s}_A)} = \frac{1}{N_{\mathcal{H}_A}} \qquad \overline{P_U(\mathbf{s}_A)^2} = \frac{1 + \mathrm{Tr}\left[\rho_A^2\right]}{N_{\mathcal{H}_A}(N_{\mathcal{H}_A}+1)}$$

Hilbertspace dimension of A

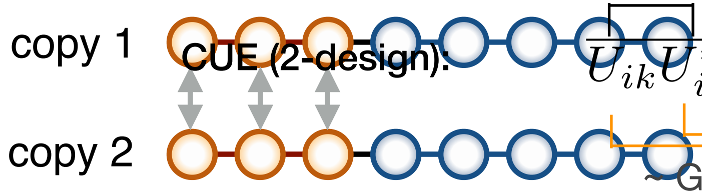
Virtual copies:

$$\overline{P_U(\mathbf{s}_A)^2} = \overline{\mathrm{Tr}_{1\oplus 2}\left[\ldots U_A \rho_A U_A^\dagger \otimes U_A \rho_A U_A^\dagger\right]} = \frac{\mathrm{Tr}_{1\oplus 2}\left[(1+S)\rho_A \otimes \rho_A\right]}{N_{\mathcal{H}_A}(N_{\mathcal{H}_A}+1)}$$

CUE (2-design):

$$\overline{U_{ik} U_{il}^* U_{im} U_{in}^*} = \frac{\delta_{kl}\delta_{mn} + \delta_{kn}\delta_{ml}}{N_{\mathcal{H}_A}(N_{\mathcal{H}_A}+1)}$$

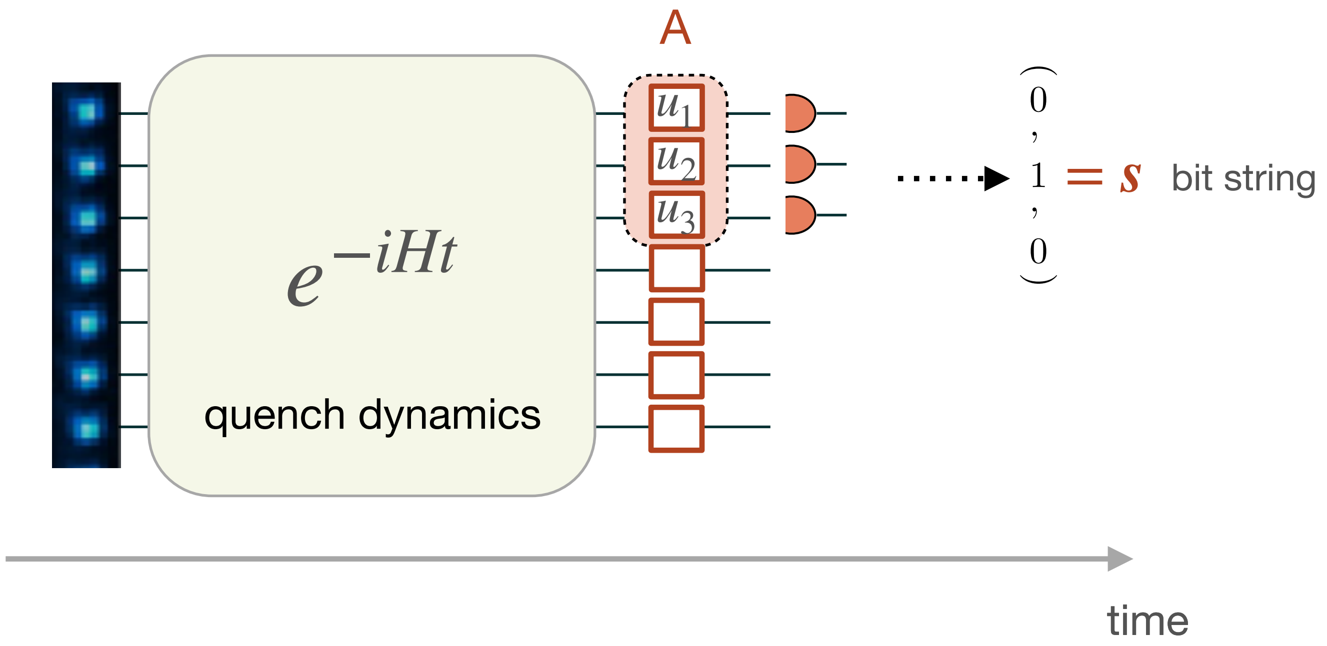
~Gaussian



S van Enk, C Beenakker (PRL 2012)

Randomized Measurements: Local Random Unitaries

Measurement post-processing



$$P_U(\mathbf{s}) = \text{Tr}\left[U \rho_A U^\dagger |\mathbf{s}\rangle\langle\mathbf{s}|\right]$$

$$U = \bigotimes_{i \in A} u_i \qquad u_i \in \text{CUE}(d)$$

evenly distributed on Bloch sphere

purity - Renyi entropy

$\text{Tr}\,\rho_A^2 = \mathbb{E}_{U\sim\text{CUE}}[\hat{P}_2]$ with $\hat{P}_2 = 2^{|A|} \sum_{s,s'} (-2)^{-D[s,s']} P_U(s) P_U(s')$

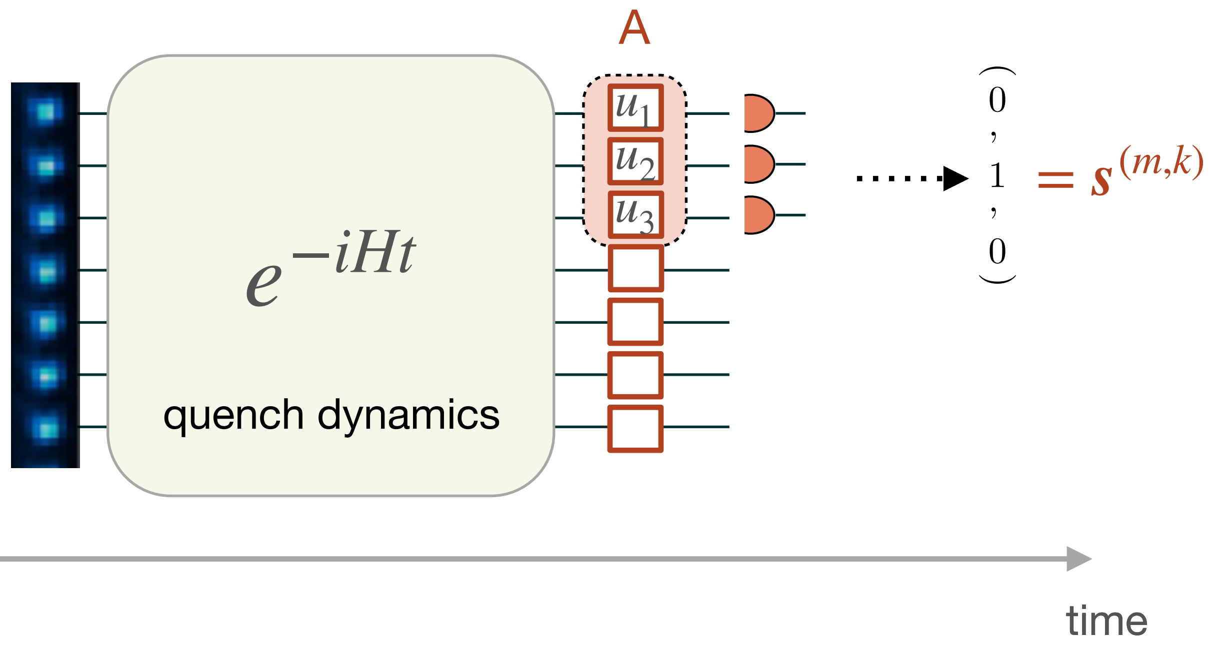
Hamming distance   cross-correlation

Random *single spin* rotations are sufficient!

T Brydges, A Elben et al., Science (2019)
A Elben, B Vermersch, et al., PRA (2019)

Randomized Measurements: Local Random Unitaries

Measurement post-processing



$$P_U(\mathbf{s}) = \text{Tr}\left[U\rho_A U^\dagger |\mathbf{s}\rangle\langle\mathbf{s}|\right]$$

$$U = \bigotimes_{i\in A} u_i \qquad u_i \in \text{CUE}(d)$$

evenly distributed on Bloch sphere

purity - Renyi entropy

$$\text{Tr}\,\rho_A^2 = \mathbb{E}_{U\sim\text{CUE}}[\hat{P}_2] \quad \text{with} \quad \hat{P}_2 = \frac{1}{MK(K-1)}\sum_{m=1}^{M}\sum_{k\neq k'=1}^{K}(-2)^{-D[s^{(m,k)},s^{(m,k')}]}$$

$k, k' = 1,\ldots,N_U \equiv K$ random unitaries

$m = 1,\ldots,N_M \equiv M$ measurements

$N_U \times N_M$ # exp runs

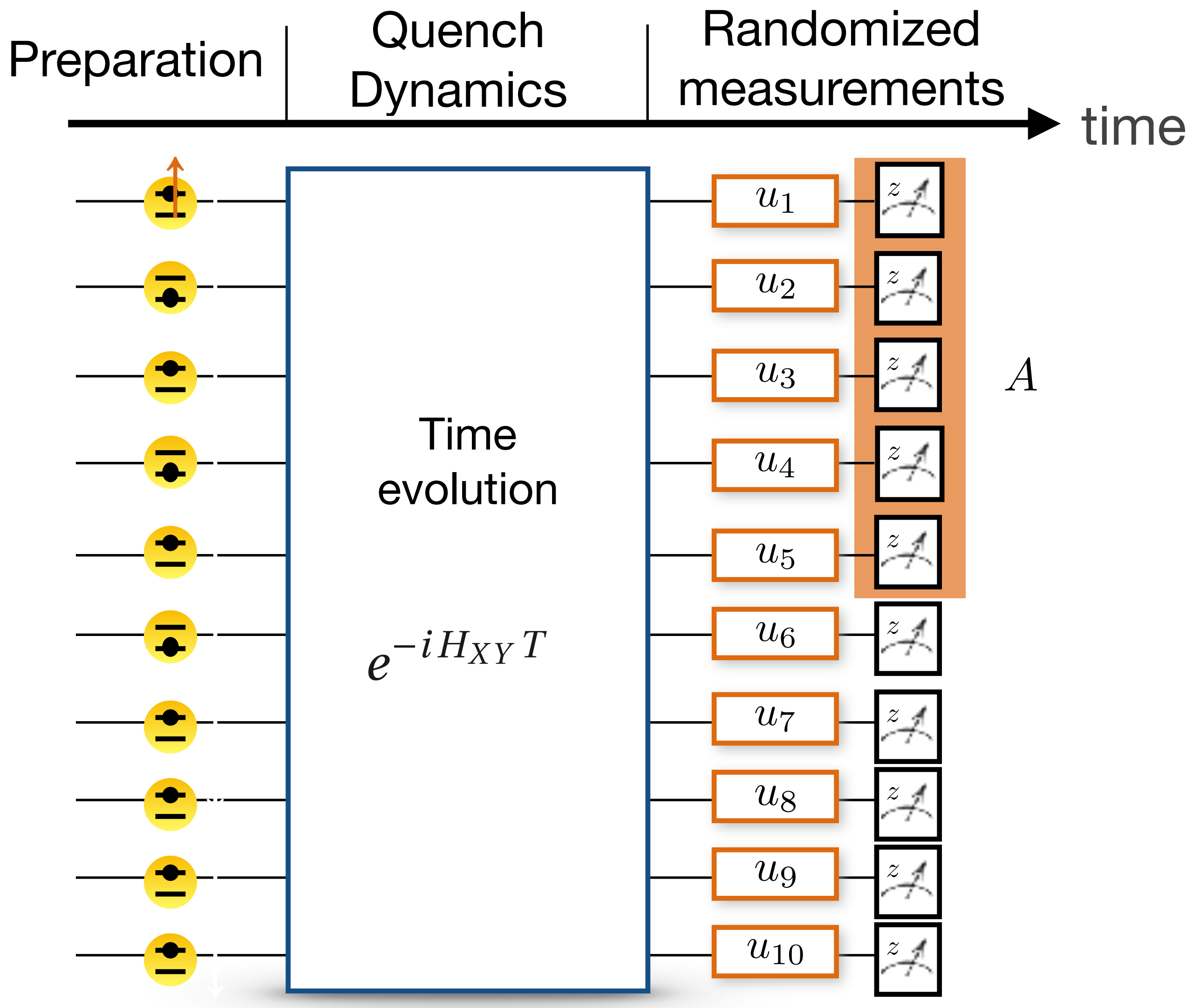
Features:

- local operations & measurements
- scaling with #unitaries and #measurements?

T Brydges, A Elben et al., Science (2019)

Proof → A Elben, B Vermersch, et al., PRA (2019)

Example 1: Experiment —Entanglement in Quench Dynamics



Hamiltonian

$$H_{XY} = \hbar \sum_{i<j} J_{ij} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)$$

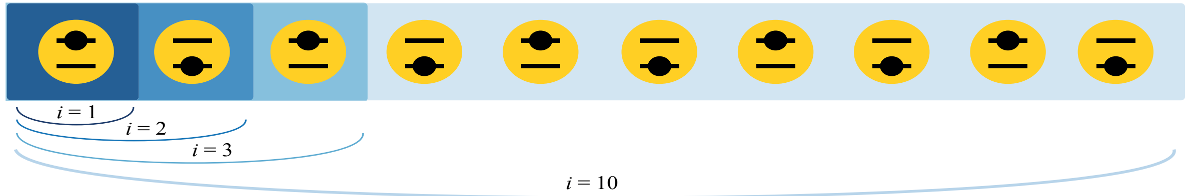
long range interaction

$$+ \hbar \sum_j (B + b_j) \sigma_j^z$$

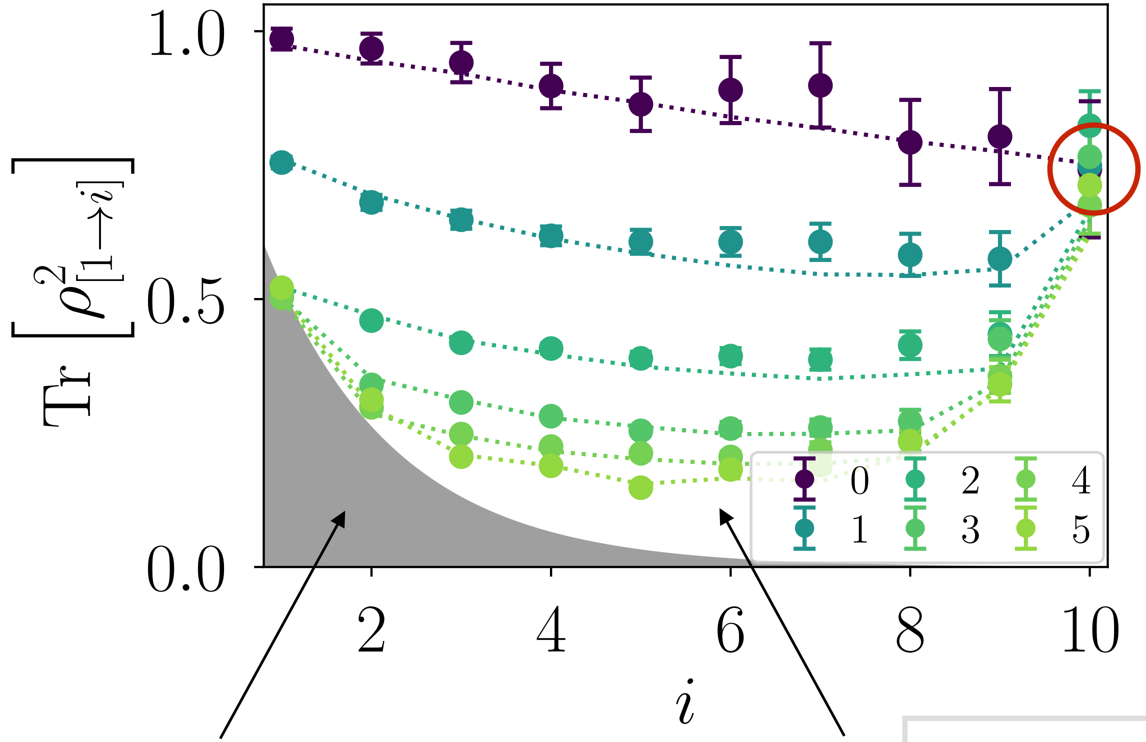
local disorder potentials

long range interaction local disorder potentials

10 Ions [no disorder]

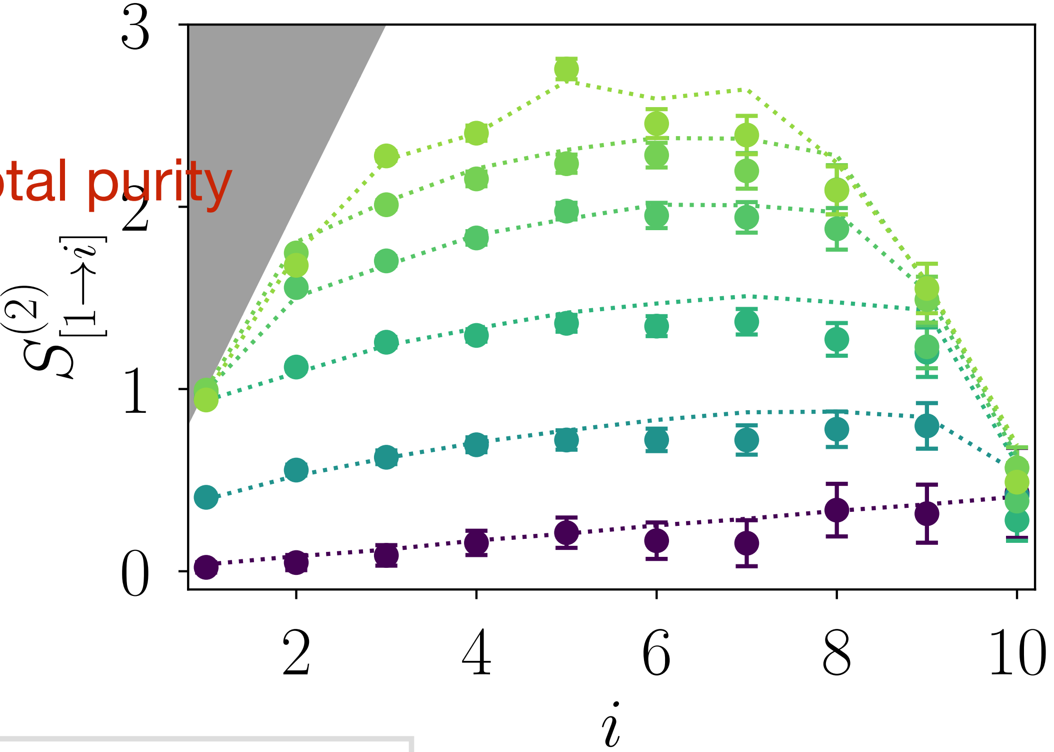


purity $\mathrm{Tr}[\rho_A^2]$



=0.76 total purity

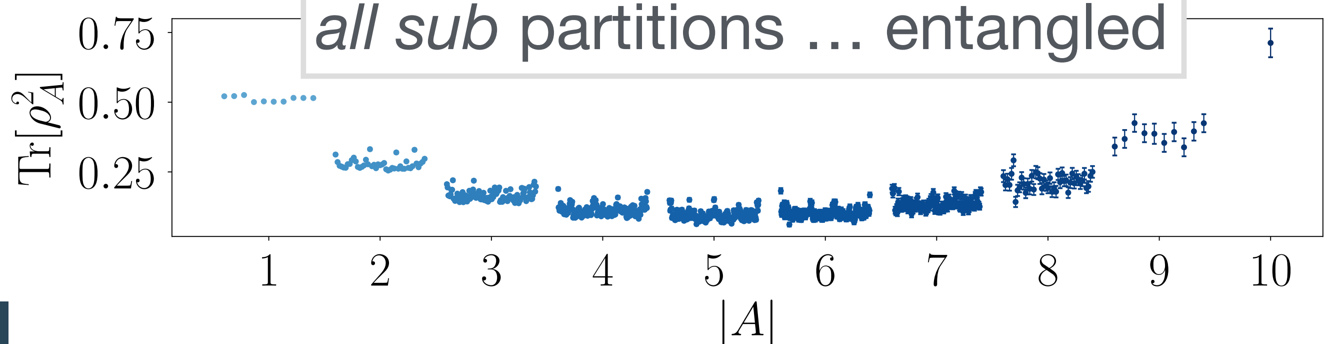
Renyi entropy



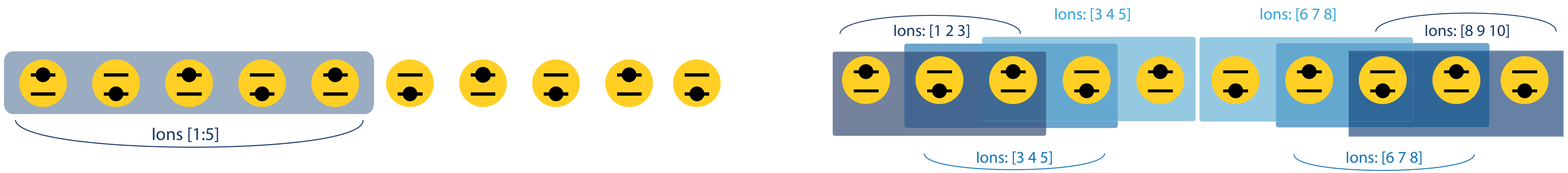
ms

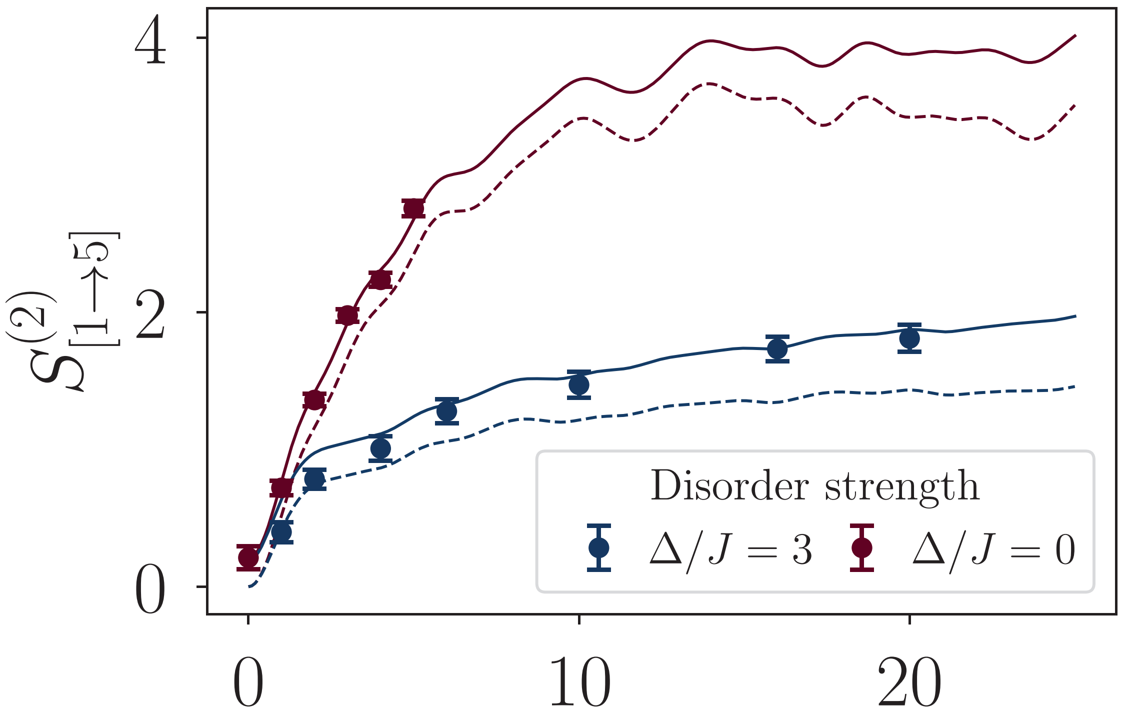
maximally mixed

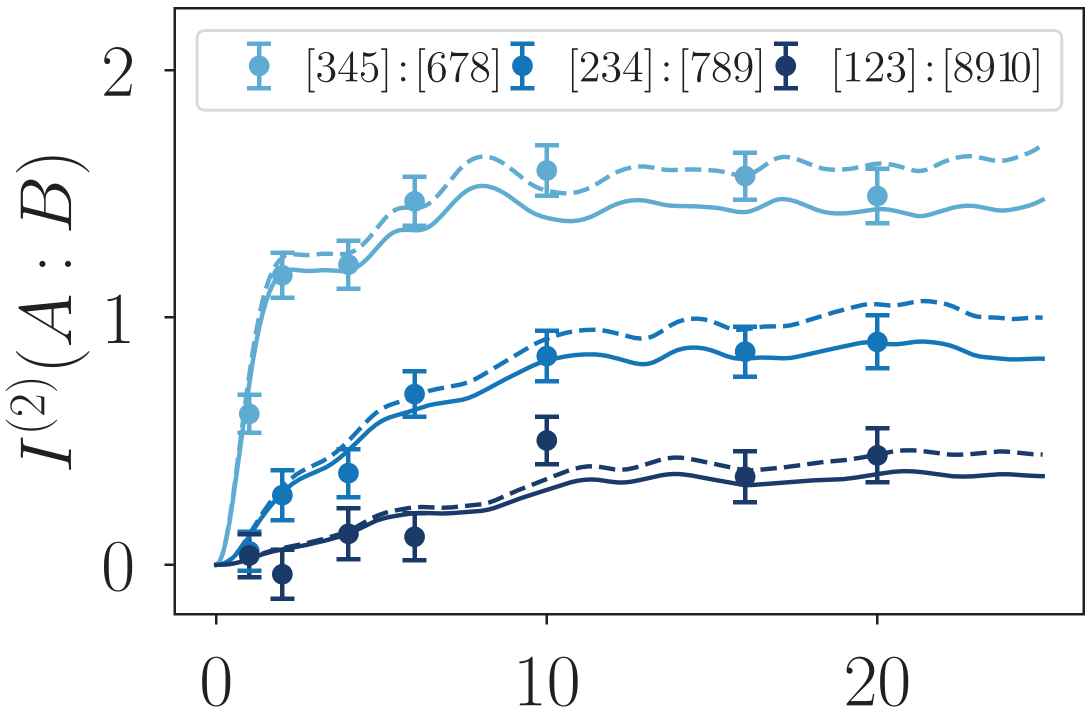
all sub partitions … entangled



10 Ions [disorder]





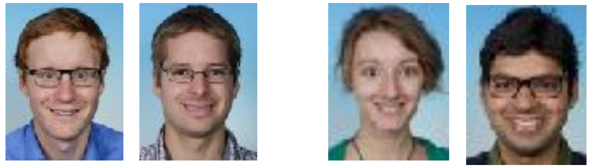


$$H_{XY} = \hbar \sum_{i<j} J_{ij}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+) + \hbar \sum_j (B + b_j)\sigma_j^z$$

long range interaction

local disorder potentials

Example 2:

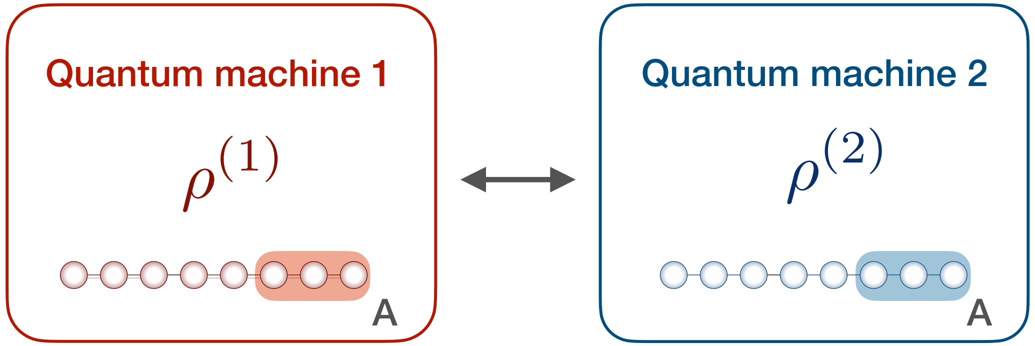


A Elben → Caltech B Vermersch → Grenoble T Brydges MK Joshi

Editors' Suggestion Featured in Physics PHYSICAL REVIEW LETTERS **124,** 010504 (2020) published 6 January 2020

Cross-Platform Verification of Intermediate Scale Quantum Devices

Andreas Elben, Benoît Vermersch, Rick van Bijnen, Christian Kokail, Tiff Brydges, Christine Maier, Manoj K. Joshi, Rainer Blatt, Christian F. Roos, and Peter Zoller



density matrix overlaps

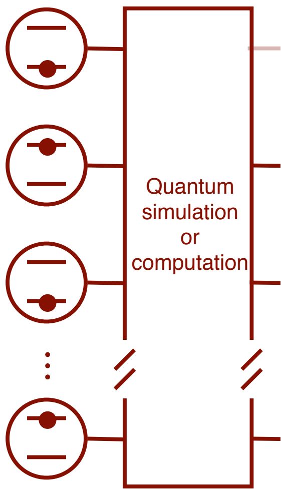
Fidelity

$$\mathcal{F}(\rho_A^{(1)}, \rho_A^{(2)}) = \frac{\mathrm{Tr}\left[\rho_A^{(1)}\rho_A^{(2)}\right]}{\max\left(\mathrm{Tr}\left[(\rho_A^{(1)})^2\right], \mathrm{Tr}\left[(\rho_A^{(2)})^2\right]\right)}$$

purity of subsystem = Renyi entanglement entropy

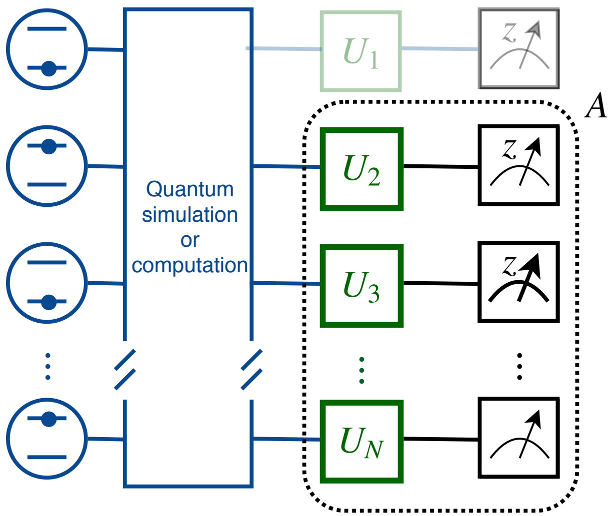
Cross-Correlation of Randomized Measurements

Platform 1



Platform 2

Same random unitaries



Classical link
+
more 'efficient' than tomography

$P_U^{(1)}(\mathbf{s}_A) = \mathrm{Tr}\left[U\rho_1 U^\dagger |\mathbf{s}_A\rangle\langle\mathbf{s}_A|\right]$

$P_U^{(2)}(\mathbf{s}_A) = \mathrm{Tr}\left[U\rho_2 U^\dagger |\mathbf{s}_A\rangle\langle\mathbf{s}_A|\right]$

Purity 1

$\overline{\left(P_U^{(1)}(s_A)\right)^2} \sim \mathrm{Tr}[\rho_{1,A}^2]$

$\mathrm{Tr}[\rho_{1,A}\rho_{2,A}] = 2^{N_A} \sum_{s_A,s'_A} (-2)^{-D[s_A,s'_A]} \overline{P_U^{(1)}(s'_A) P_U^{(2)}(s_A)}$

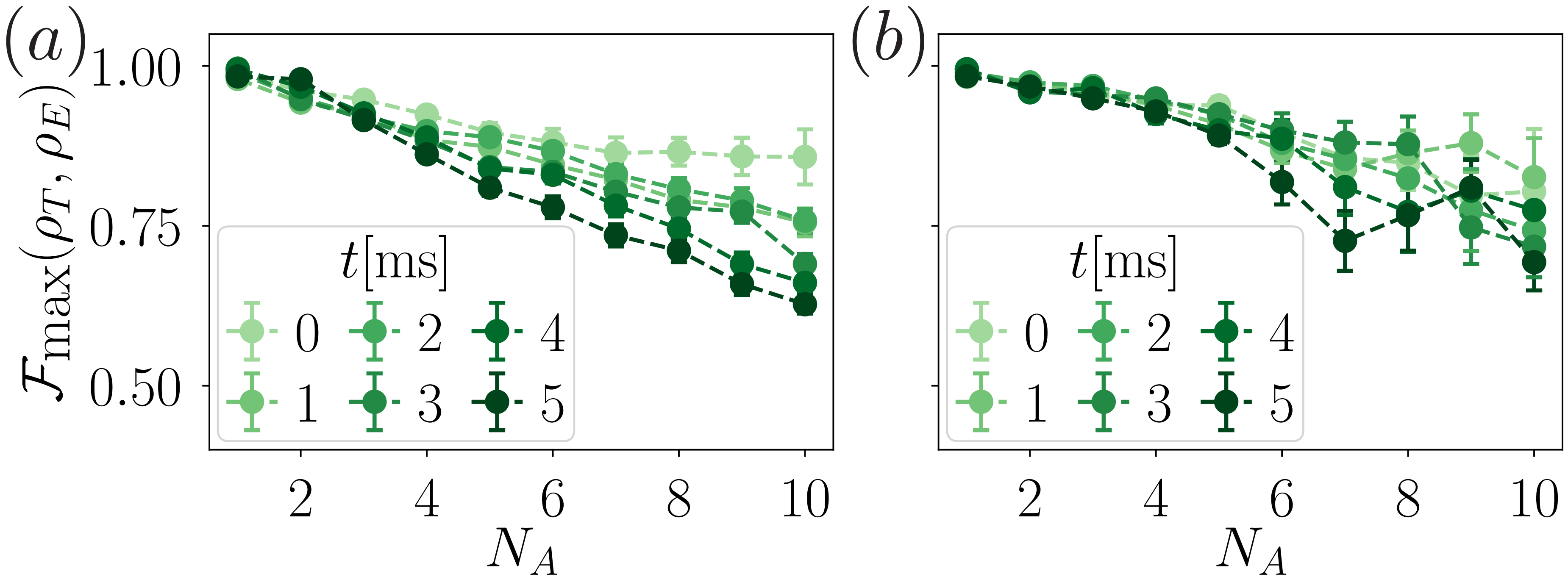
Overlap

Purity 2

$\overline{\left(P_U^{(2)}(s_A)\right)^2} \sim \mathrm{Tr}[\rho_{2,A}^2]$

Theory vs. Experiment Fidelities: `Emulating X-Platform'

Measured fidelities $\mathcal{F}_{\max}(\rho_E, \rho_T)$ vs. partition size N_A (total system 10 qubits) for Neel states evolved with H_{XY} ($J_0 = 420 s^{-1}, \alpha = 1.24$) for various times.



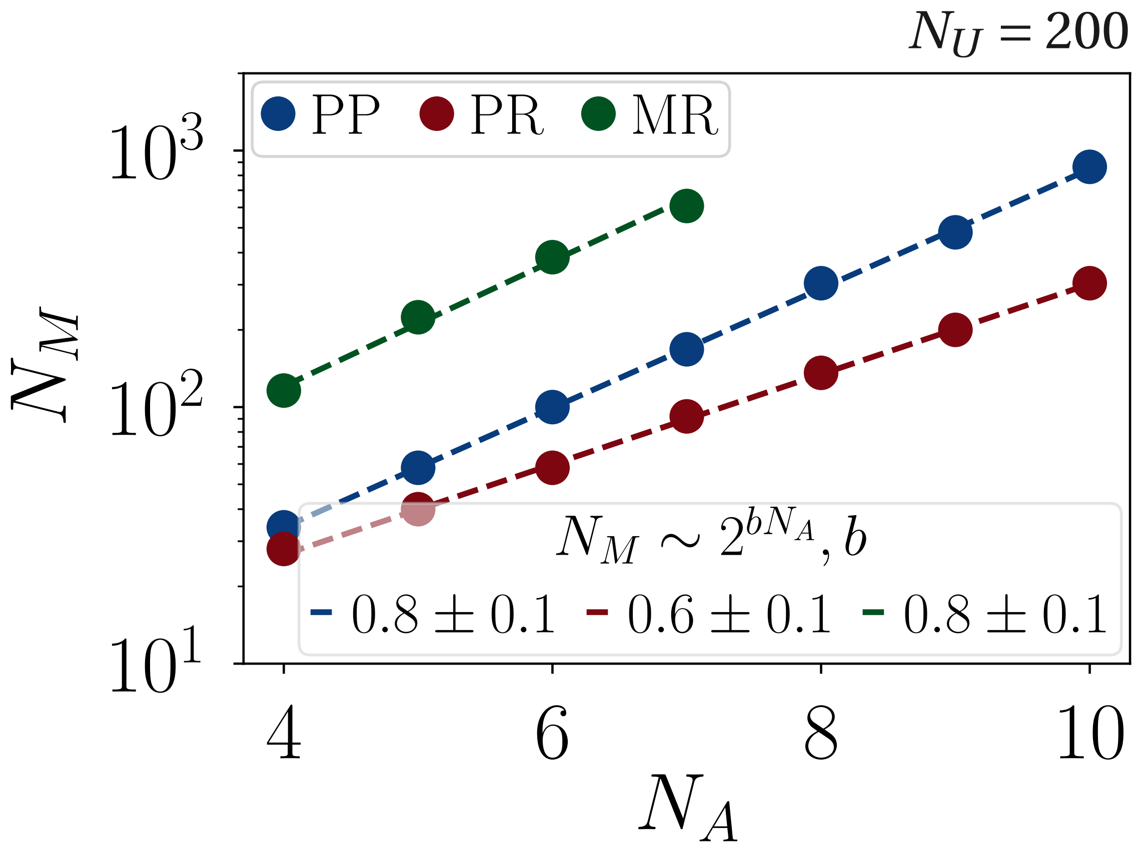
Theory states ρ_T are obtained with (a) unitary dynamics and including (b) decoherence effects.

$N_U = 500, \; N_M = 150$

Data taken in context of: T Brydges, A Elben, P Jurcevic, B Vermersch, C Maier, BP Lanyon, P. Z., R Blatt, CF Roos, Science 2019

Scaling of the required number of measurements [numerical results]

Minimal number of required measurements N_M to estimate $(\mathscr{F}_{\max}(\rho_A, \rho_A))_e$ for error $\epsilon = 0.05$ vs. number qubits N_A for $N_U = 100$.



PP: pure product state
PR: pure Haar random state
MR: mixed random states

Results:

- Scaling statistical error

$$|[\mathscr{F}_{\max}(\rho_A, \rho_A)]_e - 1| \sim 1/(N_M \sqrt{N_U})$$

  for $N_M \lesssim D_A = 2^{N_A}$ and $N_U \gg 1$,

- Scaling experimental runs

$$N_U N_M \sim 2^{bN_A}$$

  with $b \lesssim 1$ vs. full tomography $b \geq 2$

Appendix:

Behind the stage of Variational Quantum Simulation

- VQS of the Schwinger Model
- The Classical Optimization Algorithm

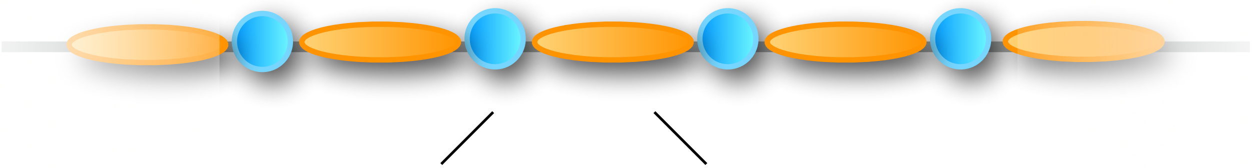
Self-verifying variational quantum simulation of lattice models
C. Kokail, C. Maier, R. van Bijnen, T. Brydges, M. K. Joshi, P. Jurcevic, C. A. Muschik, P. Silvi, R. Blatt, C. F. Roos & P. Zoller
Nature volume 569, pages 355–360 (2019)

C. Maier P. Jurcevic → IBM M K Joshi

Use QuantumPU = True

C. Kokail R. van Bijnen

Variational Quantum Simulation of Schwinger Model (1D QED)



fermions
(electron-positron)

gauge bosons
(vector potential)

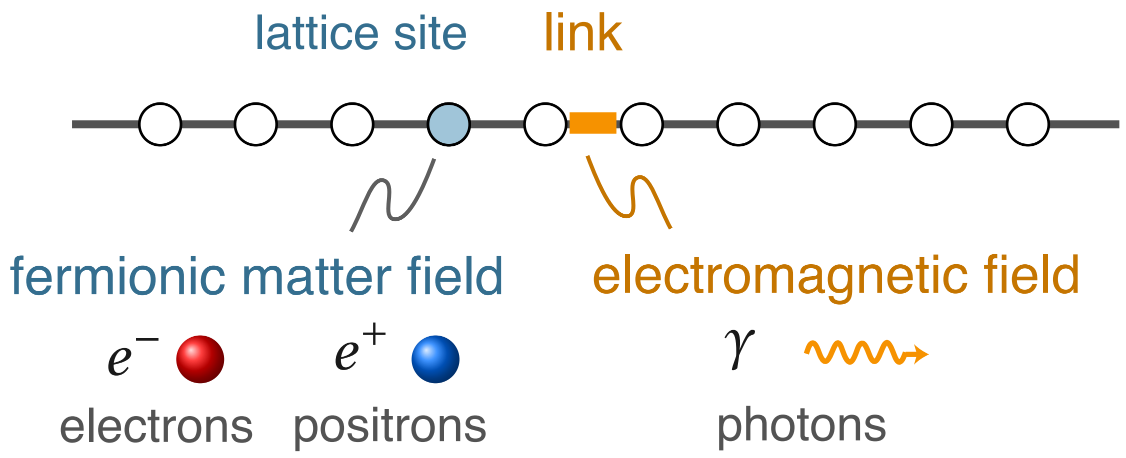
- ground state
- self-verification

- excited states
- quantum phase transitions

test bed

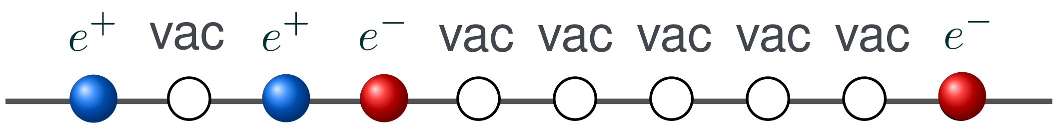
The Lattice Schwinger Model

Quantum Electrodynamics in 1D



The Lattice Schwinger Model

Quantum Electrodynamics in 1D



Staggered Fermions

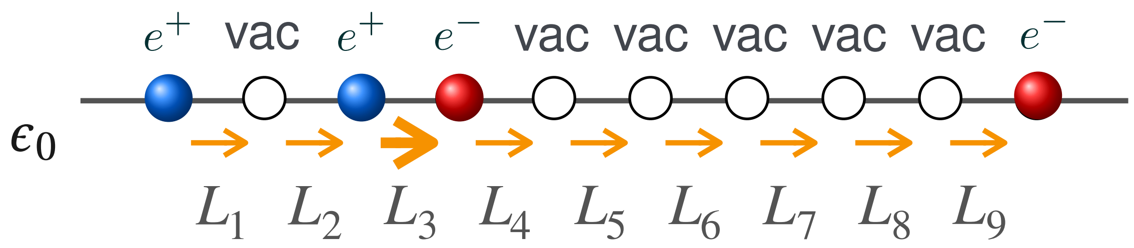
even		2		4		6		8		10
odd	1		3		5		7		9	

e^- electrons on even lattice sites

e^+ positrons on odd lattice sites

The Lattice Schwinger Model

Quantum Electrodynamics in 1D

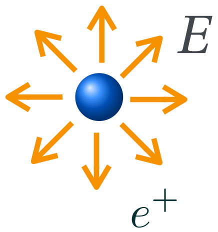


Schwinger Hamiltonian (Kogut-Susskind)

$$\hat{H}_{\mathrm{S}} = w \sum_{n=1}^{N-1} \left[\sigma_n^+ e^{i\hat{\Theta}_n} \sigma_{n+1}^- + \mathrm{H.c.} \right] + \frac{m}{2} \sum_{n=1}^{N} (-1)^n \sigma_n^z + J \sum_{n=1}^{N-1} \hat{L}_n^2$$

Gauss Law

$$\nabla \cdot E = \rho$$



1D: fermion configuration $\Longleftrightarrow$ field

$$\hat{L}_n - \hat{L}_{n-1} = \frac{1}{2} \left[\hat{\sigma}_n^z + (-1)^n \right]$$

Jordan-Wigner: fermions → spins

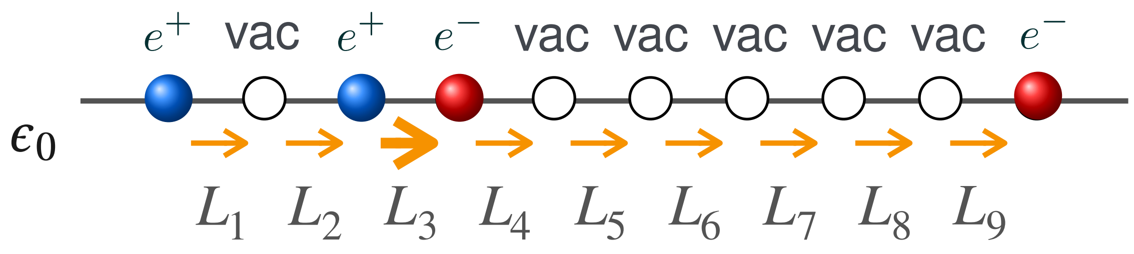
odd sites: e^+ $\hat{=} \downarrow$
vac $\hat{=} \uparrow$

even sites: e^- $\hat{=} \uparrow$
vac $\hat{=} \downarrow$

Kogut-Susskind encoding

The Lattice Schwinger Model

Quantum Electrodynamics in 1D



Schwinger Hamiltonian (Kogut-Susskind)

$$\hat{H}_S = +\frac{m}{2}\sum_{n=1}^{N}(-1)^n\hat{\sigma}_n^z + w\sum_{n=1}^{N-1}\left[\hat{\sigma}_n^+\hat{\sigma}_{n+1}^- + \text{h.c.}\right] + J\sum_{n=1}^{N-1}\left[\epsilon_0 + \frac{1}{2}\sum_{m=1}^{n}\left[\hat{\sigma}_m^z + (-1)^m\right]\right]^2$$

mass

$$\epsilon_0 = 0$$

$$\hat{L}_n - \hat{L}_{n-1} = \frac{1}{2}\left[\hat{\sigma}_n^z + (-1)^n\right]$$

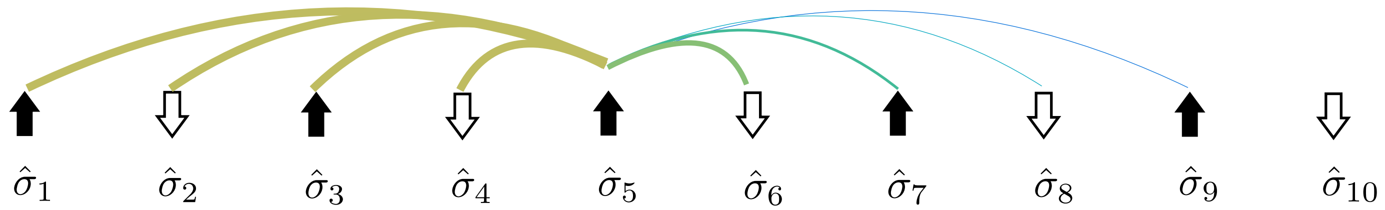
Symmetries of Schwinger Model

$$\left[\hat{H}_S, \hat{S}^z_{\text{tot}}\right] = 0 \quad \left[\hat{H}_S, \hat{C}\hat{P}\right] = 0 \quad \text{for} \quad s^z_{\text{tot}} = 0$$

total magnetization

charge-parity

exotic long range couplings



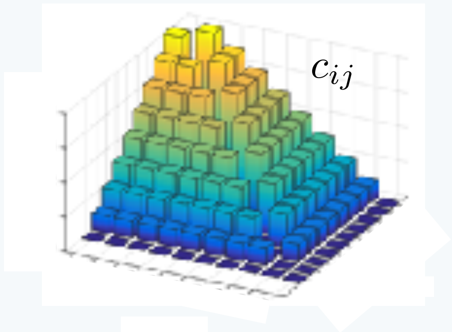
Variational Quantum Simulation of Lattice Models

Target

Schwinger Model

$$\hat{H}_S = J \sum_{i<j} c_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

long - range interaction



$$+ \, w \sum_i \left(\hat{\sigma}_i^+ \hat{\sigma}_{i+1}^- + \hat{\sigma}_{i+1}^+ \hat{\sigma}_i^- \right)$$

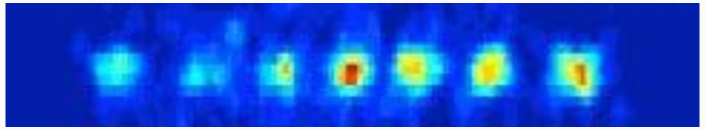
particle - antiparticle creation/annihilation

$$+ \, m \sum_i c_i \hat{\sigma}_i^z + J \sum_i \tilde{c}_i \hat{\sigma}_i^z$$

effective particle masses

Quantum Resource

ions



Analog QS

$$\hat{U}_1(\theta) = e^{-i\theta \sum_{ij} J_{ij} \hat{\sigma}_i^x \hat{\sigma}_j^x}$$ entangling

$$\hat{U}_{2,i}(\theta) = e^{-i\theta \vec{n} \cdot \hat{\vec{\sigma}}_i}$$ local rotations

native toolbox

classical computer

generate entangled states:

$$|\psi(\Theta)\rangle = \hat{U}_N(\theta_N) \ldots \hat{U}_2(\theta_2) \hat{U}_1(\theta_1) |\psi_0\rangle$$

Remarks:

- symmetries ⟵ *match* the symmetries ⟶ symmetries
- *global* parameter optimization with *noisy* data (fixed budget allocation)

Matching Symmetries of Target and Resource

Target

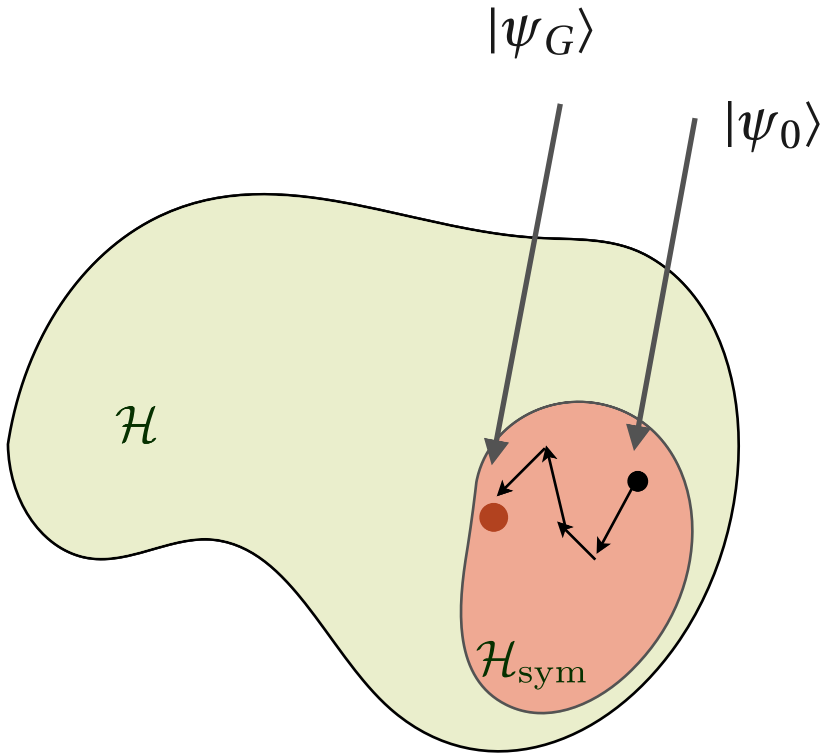
Symmetries of Schwinger Model

$$[\hat{H}_S, \hat{S}^z_{\text{tot}}] = 0 \quad [\hat{H}_S, \hat{C}\hat{P}] = 0 \quad \text{for} \quad s^z_{\text{tot}} = 0$$

total magnetization

charge-parity

[+ approximate translational symmetry]



Confine searches to the subspaces selected by symmetry

Matching Symmetries of Target and Resource

Target

Symmetries of Schwinger Model

$$[\hat{H}_S, \hat{S}^z_{\text{tot}}] = 0 \quad [\hat{H}_S, \hat{C}\hat{P}] = 0 \quad \text{for} \quad s^z_{\text{tot}} = 0$$

total magnetization

charge-parity

Quantum Resource

Symmetries of Ion Analog QS

Flip-Flop

$$\hat{H}_{XY} = \sum_{i,j} J_{ij} \left(\hat{\sigma}_i^+ \hat{\sigma}_j^- + \hat{\sigma}_i^- \hat{\sigma}_j^+ \right)$$

preserves the total magnetization

hardware

Single Spin Rotation

$$R_i(\boldsymbol{\theta}_i) = e^{-i\boldsymbol{\theta}_i \cdot \frac{1}{2}\hat{\boldsymbol{\sigma}}_i}$$

We perform CP-symmetric single qubit rotations:

$$(\Theta^x, \Theta^y, \Theta^z)_n = (\Theta^x, -\Theta^y, -\Theta^z)_{N-n+1}.$$

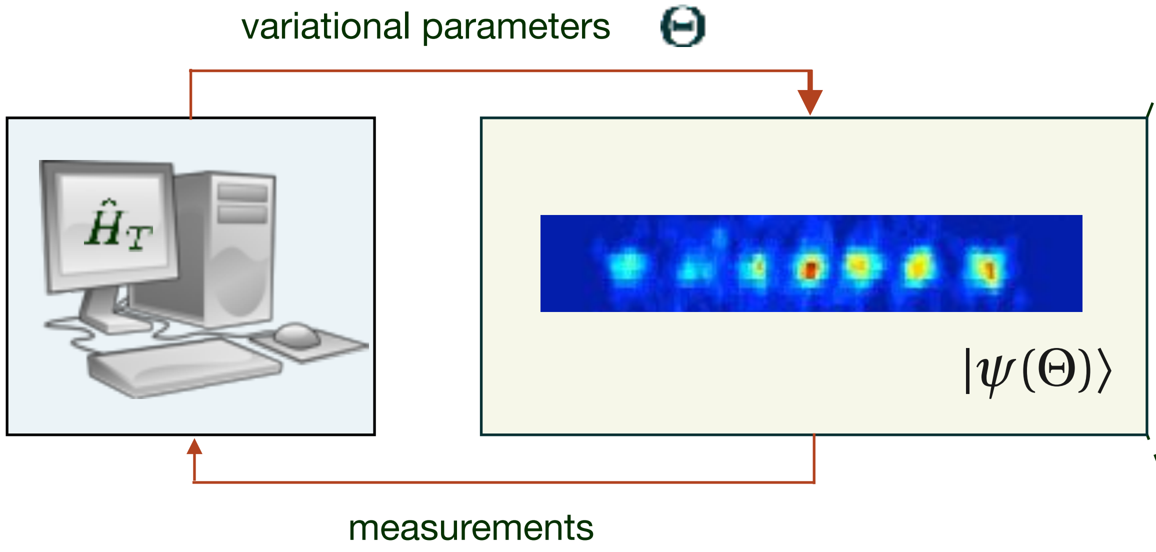
angles left - right half of chain linked

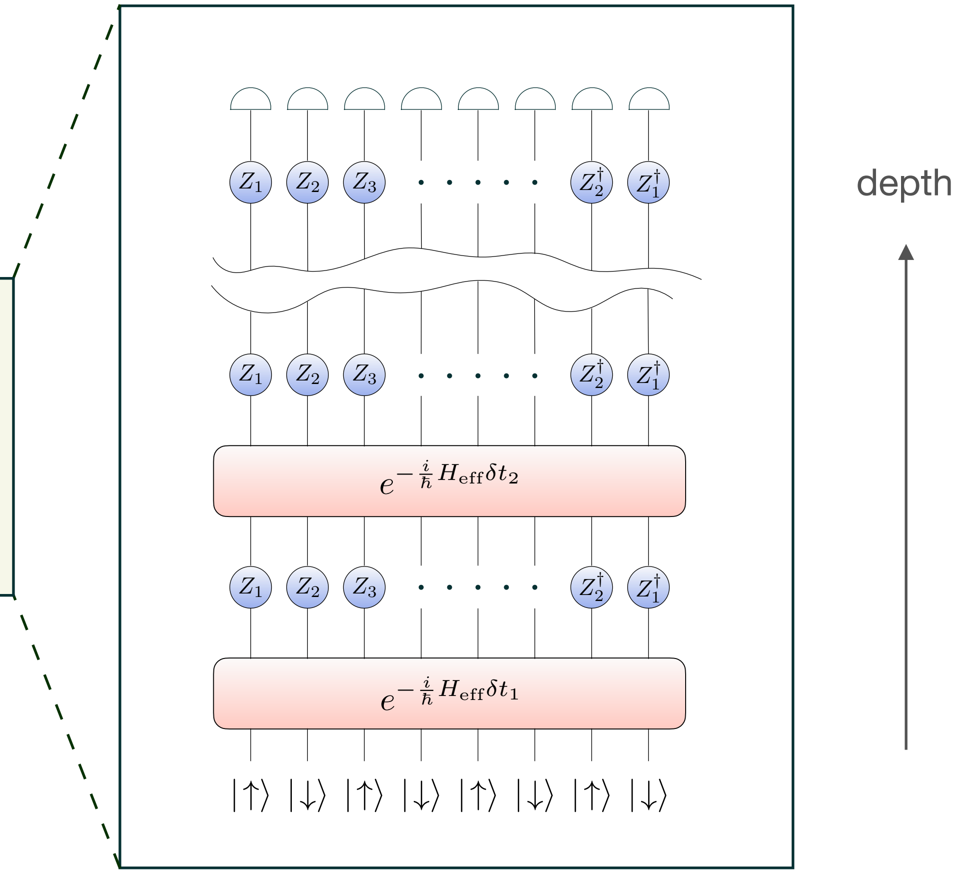
$$CPR(\boldsymbol{\Theta})[CP]^\dagger = \exp\left\{-i\boldsymbol{\Theta}_n \cdot CP\boldsymbol{\sigma}_n[CP]^\dagger\right\}$$
$$= \exp\left\{-i\boldsymbol{\Theta}_n \cdot (\sigma^x, -\sigma^y, -\sigma^z)^T_{N-n+1}\right\}.$$

software

Variational Quantum Simulation — *Symmetry-Protected*

The Feedback Loop





symmetry-protecting circuit

Schwinger Hamiltonian

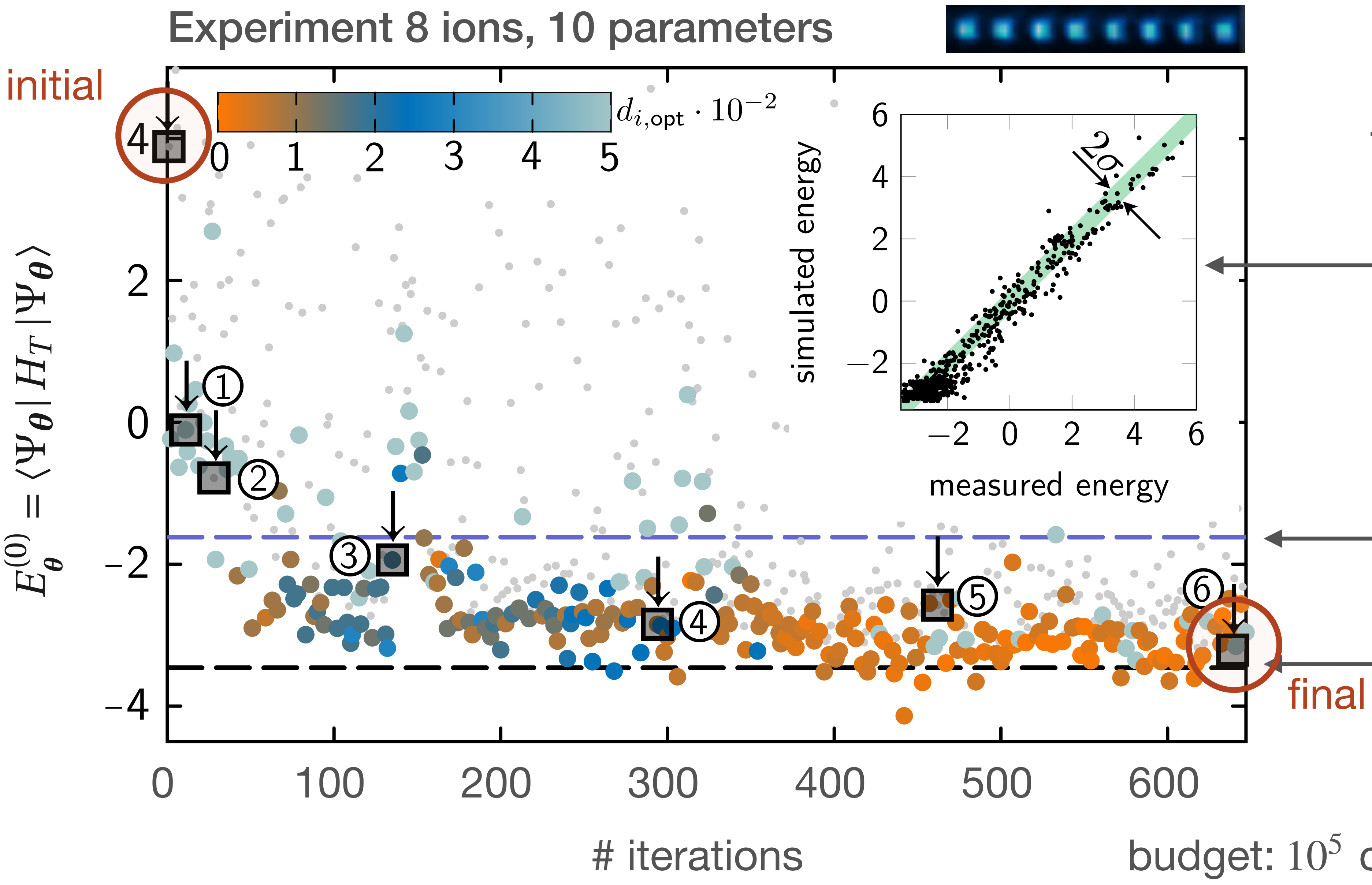
$$\hat{H}_S = \ldots$$

goal: prepare ground state

$$\langle\psi(\theta)|\,\hat{H}_S\,|\psi(\theta)\rangle \rightarrow \text{min}$$ (cost function)

Optimization Trajectory for Schwinger Ground State]

Experiment 8 ions, 10 parameters



comparison theory vs. experiment

experiment vs. theory

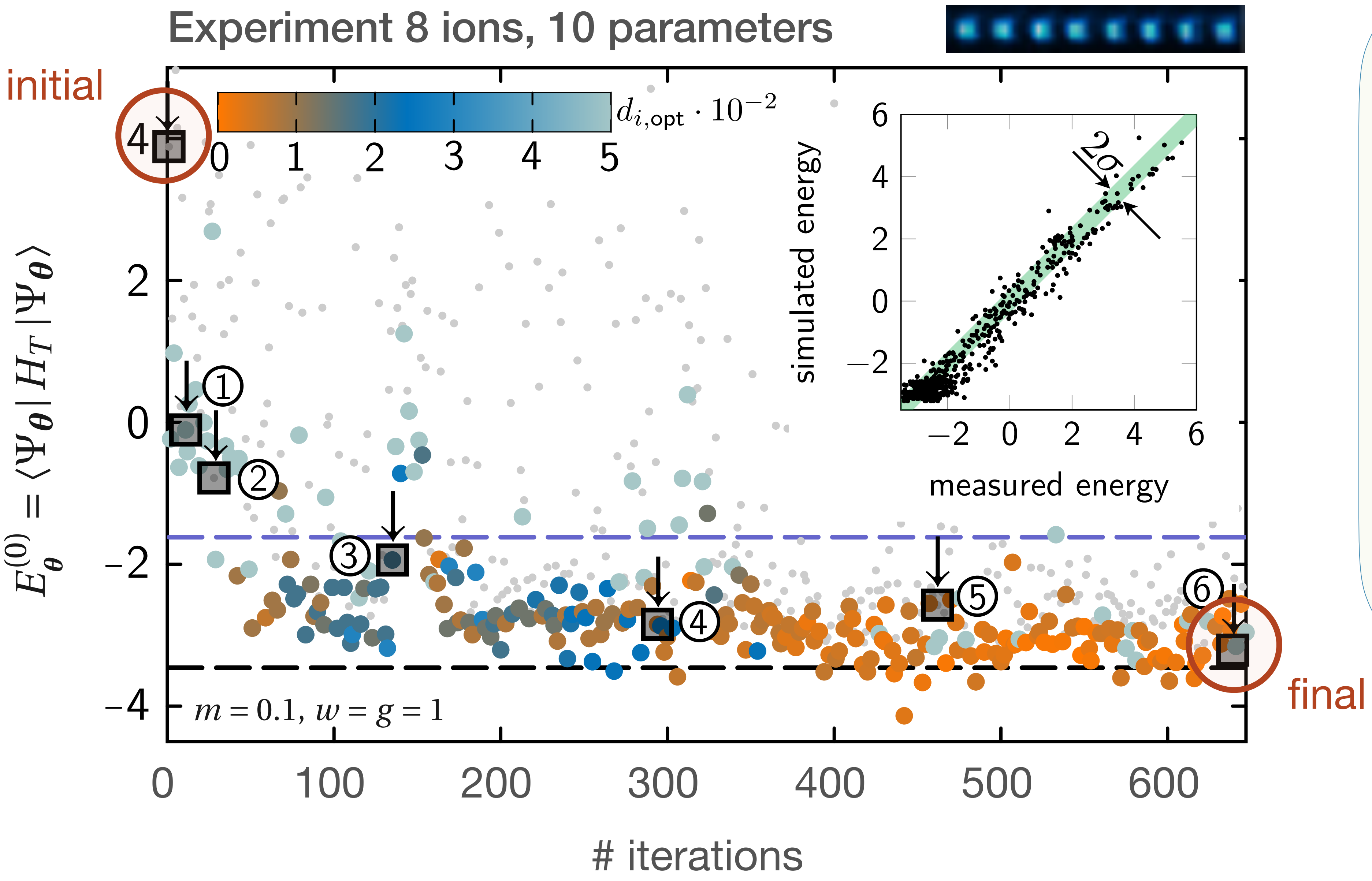
$E^{(1)}$ first excited state

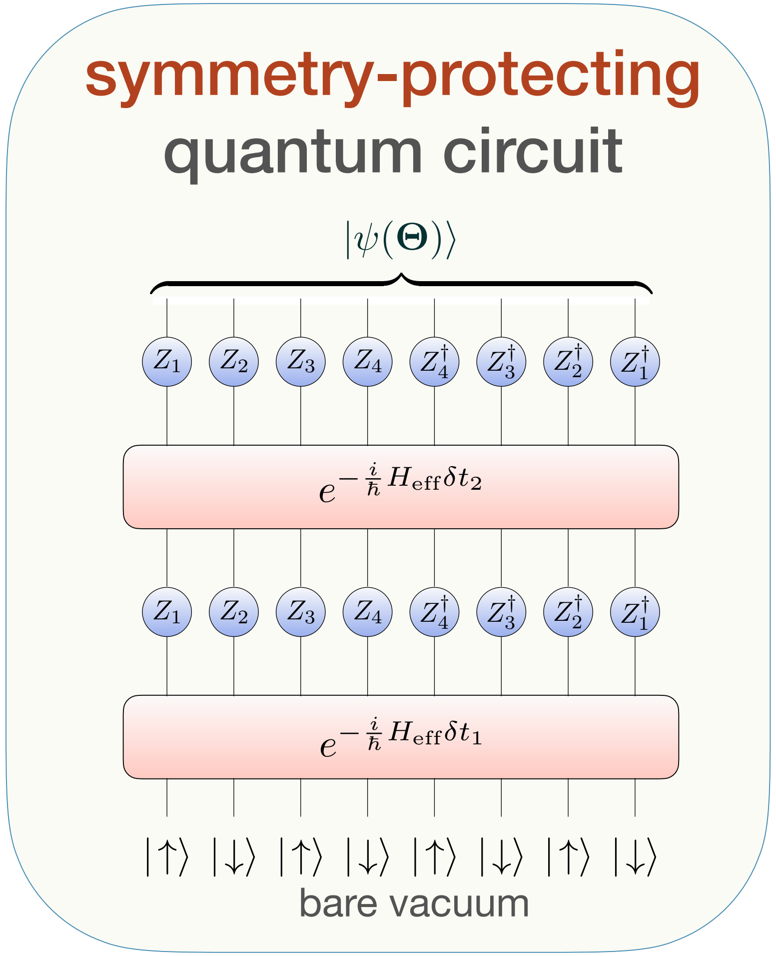
Δ gap

$E^{(0)}$ ground state (exact)

budget: 10^5 calls to the quantum simulator

Optimization Trajectory for Schwinger Ground State

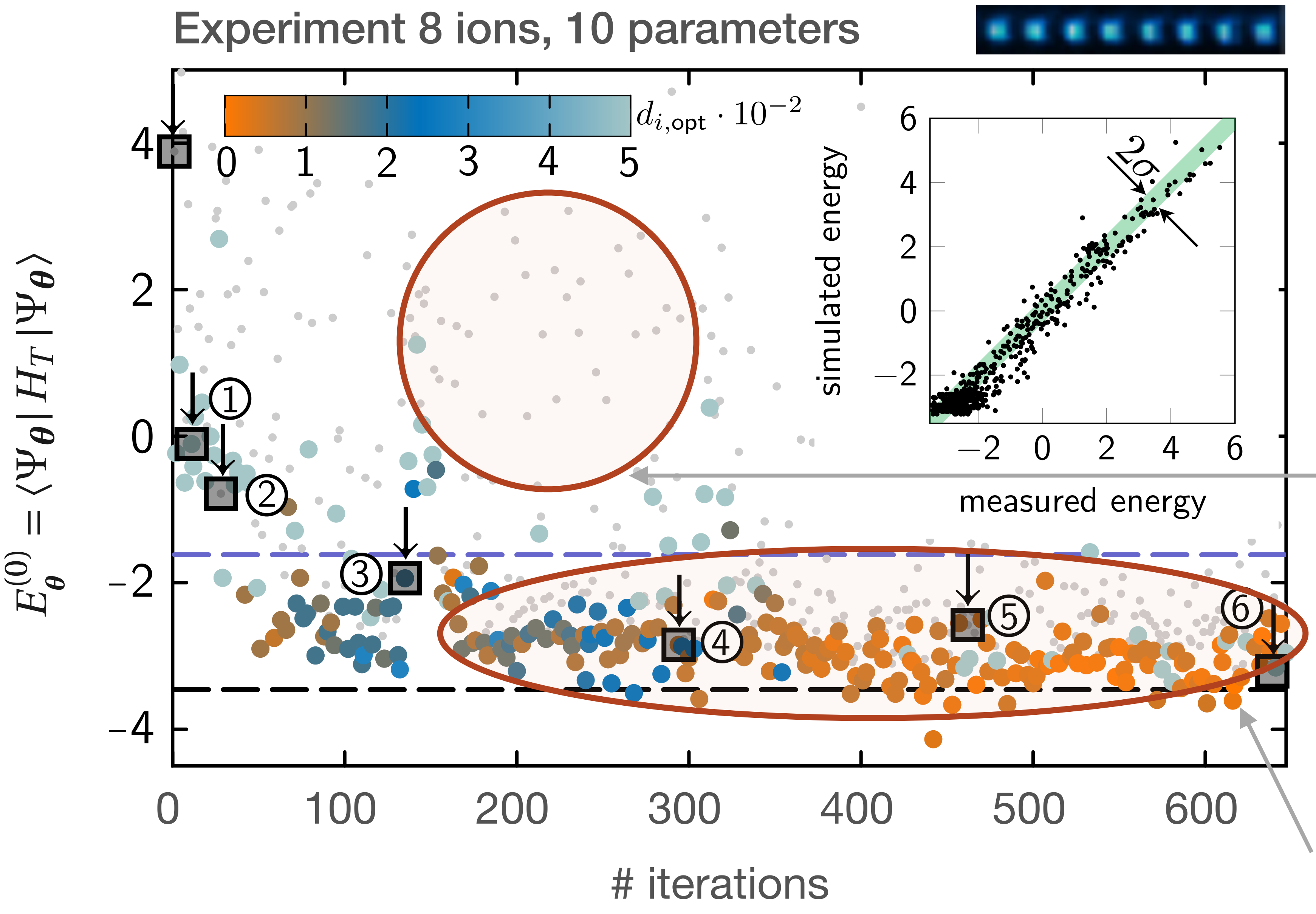




8 ions, 10 parameters

Optimization Trajectory for Schwinger Ground State

Experiment 8 ions, 10 parameters



global **s**tochastic **DI**viding **RECT**angles parameter search

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exploration

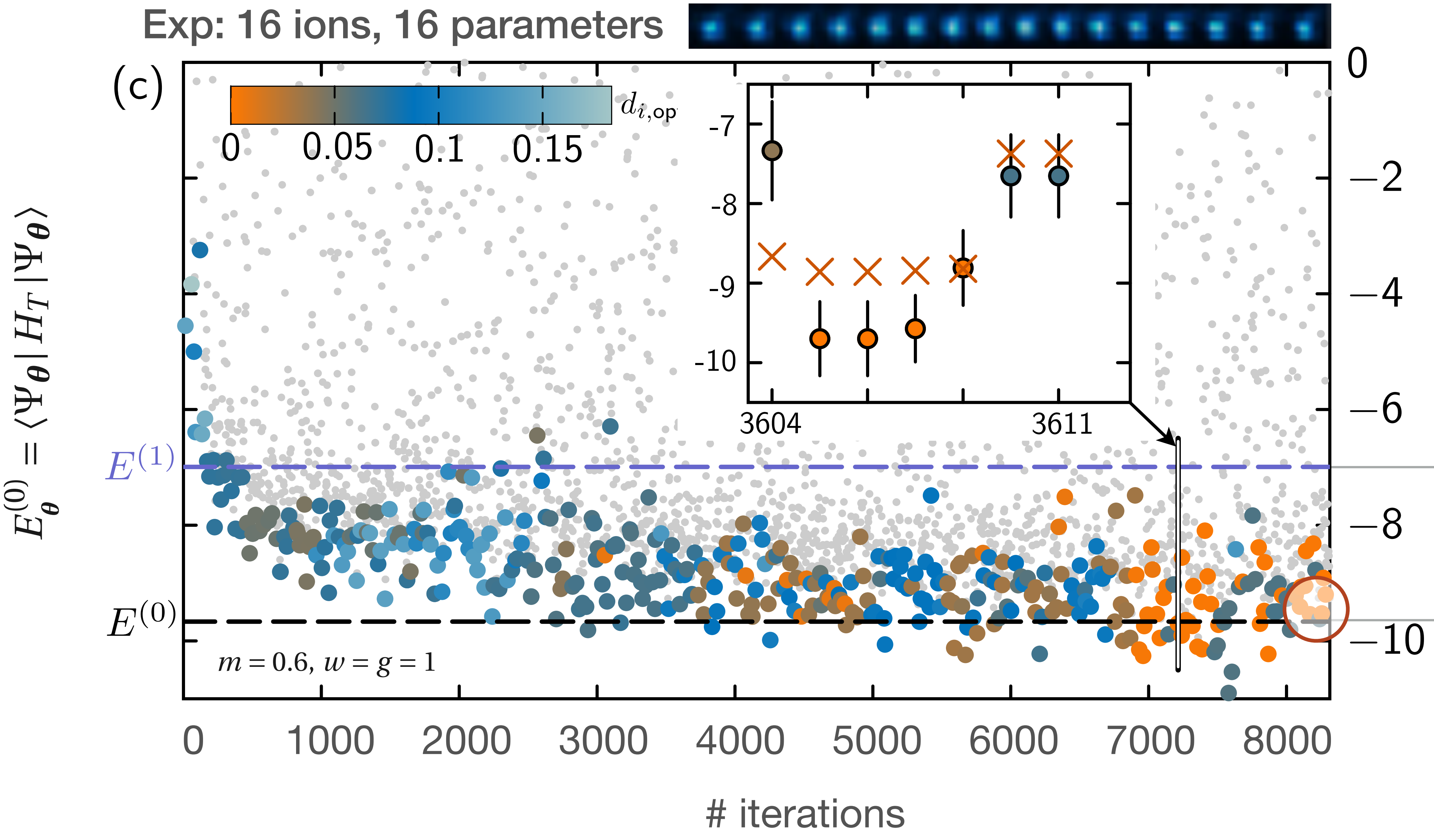
vs.

refinement

color-coded distance to final state

$$d_{i,\text{final}} = \frac{1}{2}[1 - \boldsymbol{\theta}_i^T \cdot \boldsymbol{\theta}_\text{final} / (|\boldsymbol{\theta}_i||\boldsymbol{\theta}_\text{final}|)]$$

Optimization Trajectory for Schwinger Ground State

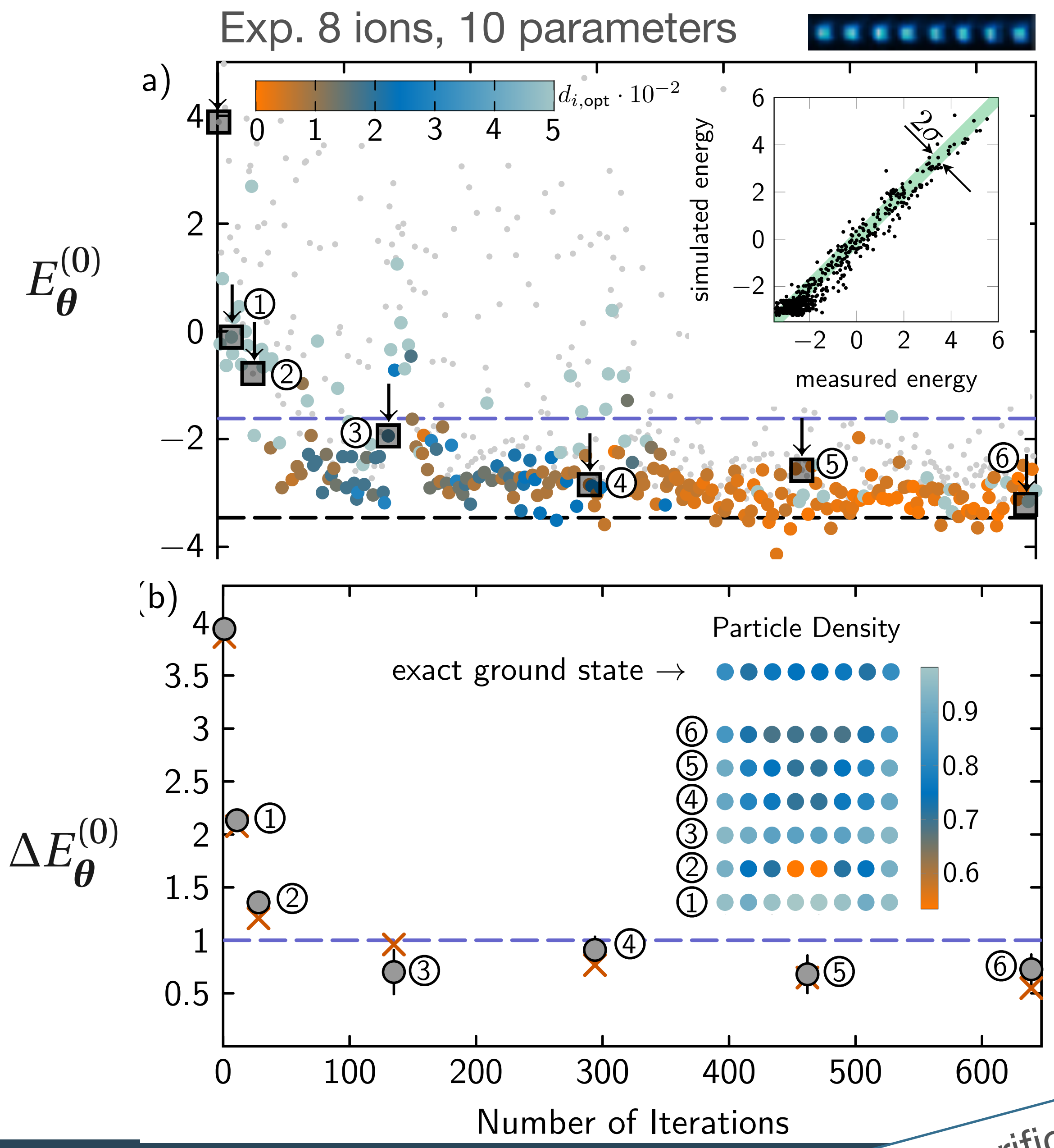


fidelity wave fct

$$\mathscr{F} = |\langle \Psi_G | \Psi^{\mathrm{Sim}}_{\boldsymbol{\theta}} \rangle|^2$$

= 87%

Measurement of Error Bars ['Algorithmic Error']



- we can not only compute the optimal energy

$$E_{\theta}^{(0)} = \langle \Psi_{\theta} \mid \hat{H}_T \mid \Psi_{\theta} \rangle \rightarrow \text{min}$$

and wave function,

- but also *measure the error bar* as energy variance

$$\left(\Delta E_{\theta}^{(0)}\right)^2 = \langle \Psi_{\theta} \mid \left(E_{\theta}^{(0)} - \hat{H}_T\right)^2 \mid \Psi_{\theta} \rangle \geq 0$$

algorithmic error
(vs. projection noise)

=0 for eigenstate

expensive

and monitor convergence with # iterations, and/or increasing depth of quantum circuit

Quantum Phase Transition in Schwinger Ground State

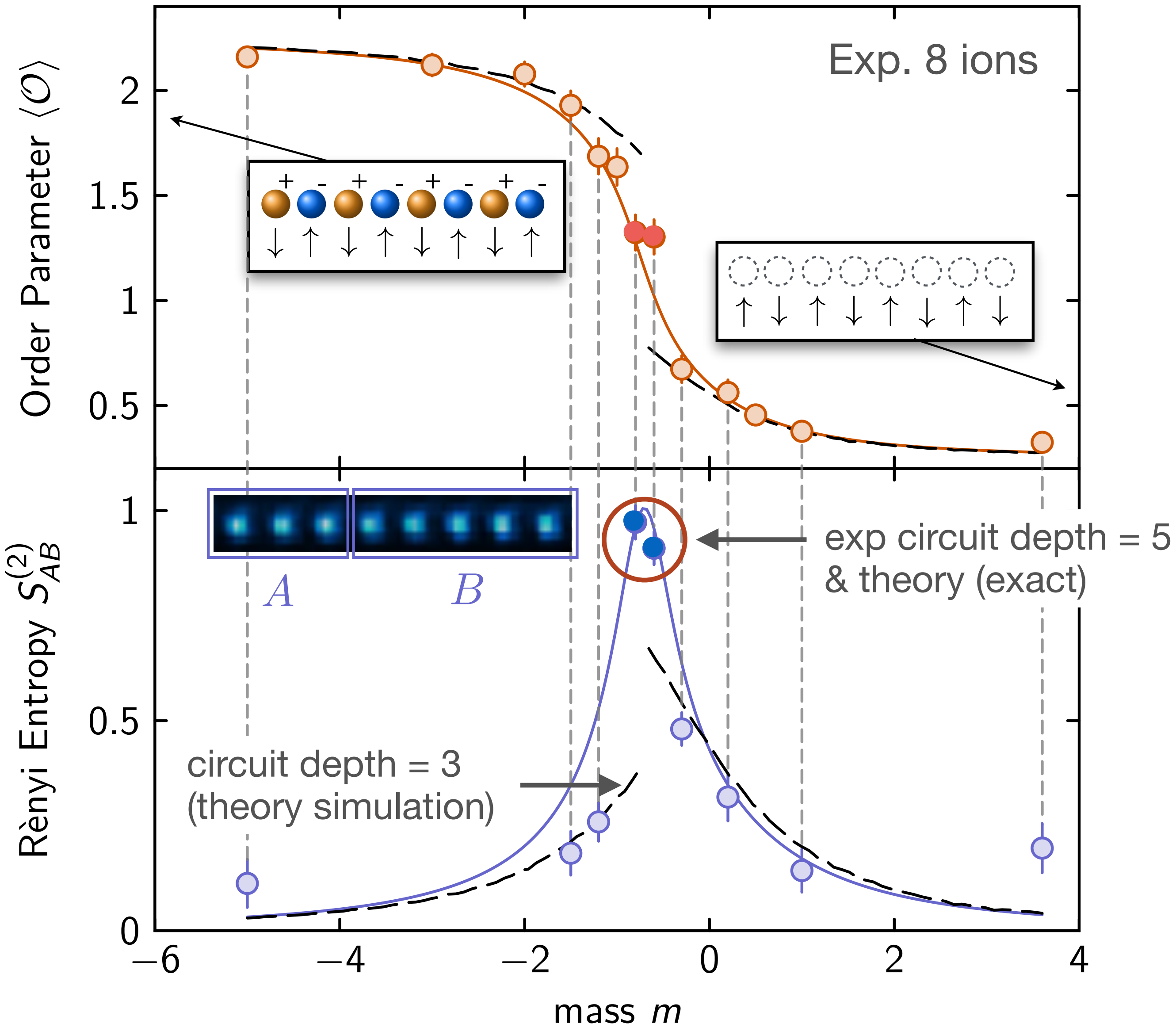
- order parameter

$$\langle \mathcal{O} \rangle \sim \sum_{i,j>i} \langle (1+(-1)^i \sigma_i^z)(1+(-1)^j \sigma_j^z) \rangle$$

$$-\infty < m < +\infty$$

- *Measure* entanglement entropy (Renyi) across QPT

test convergence of entanglement entropy with circuit depth



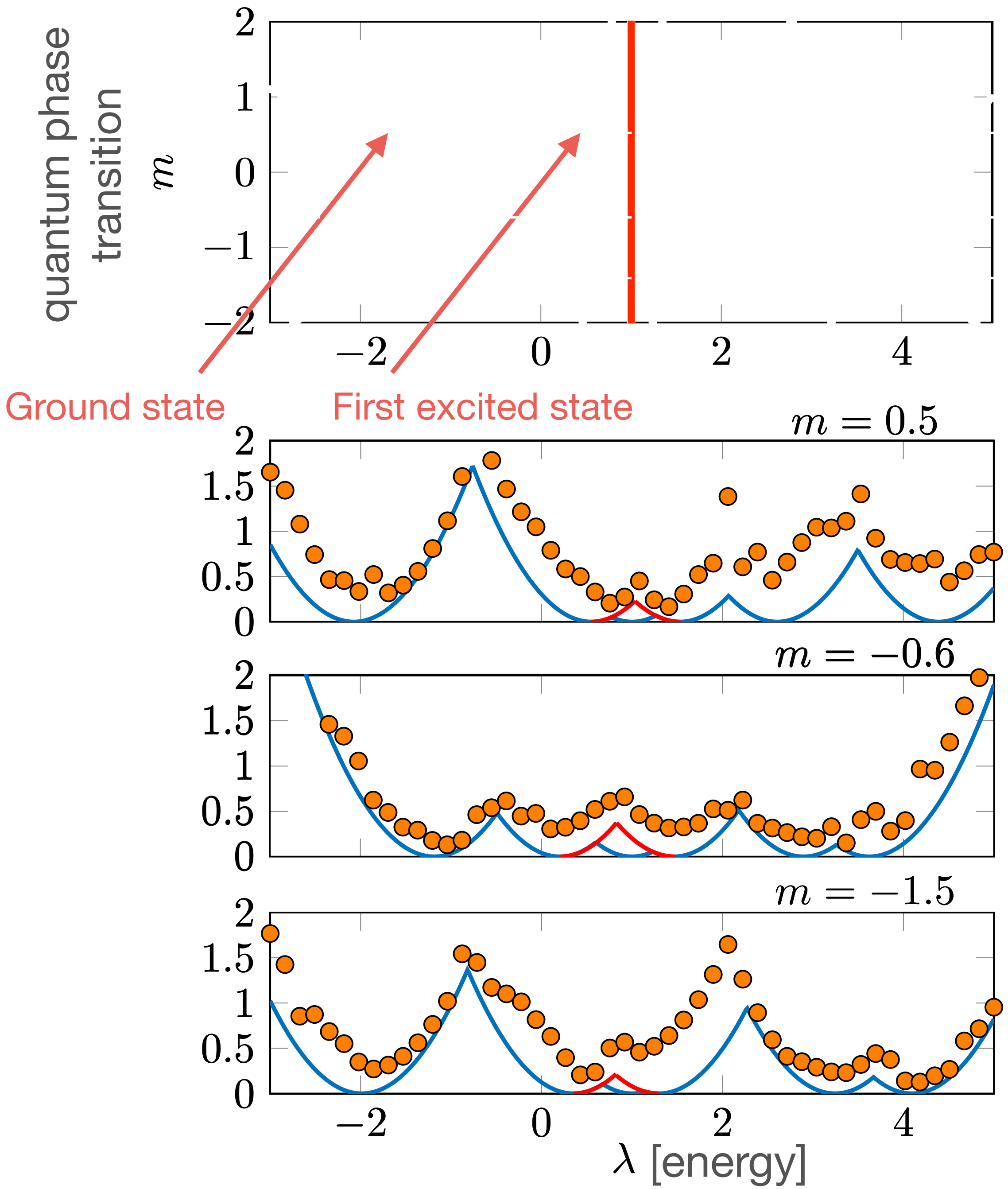
Excited States & Self-Validation

- Cost function

$$\mathscr{E}(E) = \min_{\boldsymbol{\theta}} \left[\langle \Psi_{\boldsymbol{\theta}} \mid \left(E - \hat{H}_T\right)^2 \mid \Psi_{\boldsymbol{\theta}} \rangle \right] \geq 0$$

as a function of the energy parameter *E* has minima at eigenstates. The value at minimum gives the error.

expensive



Excited States & Self-Validation

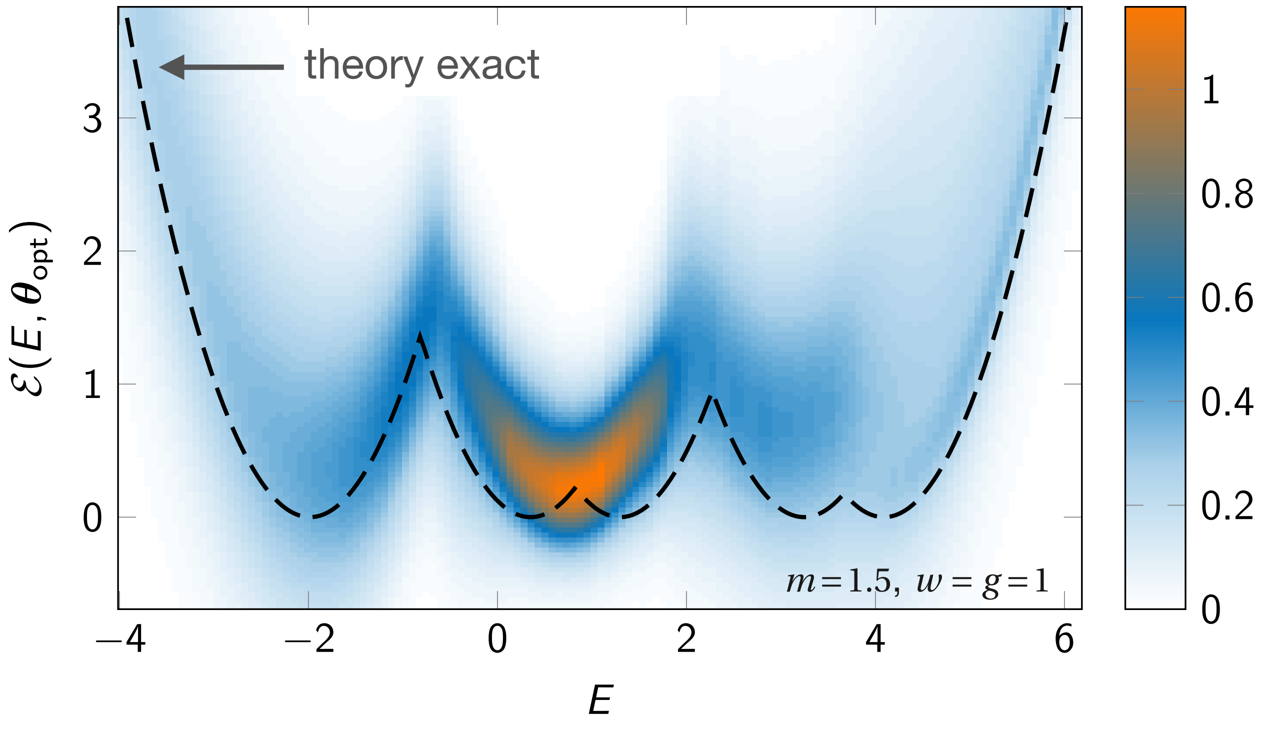
- **Cost function**

$$\mathcal{E}(E) = \min_{\boldsymbol{\theta}} \left[\langle \Psi_{\boldsymbol{\theta}} | \left(E - \hat{H}_T\right)^2 | \Psi_{\boldsymbol{\theta}} \rangle \right] \geq 0$$

as a function of the energy parameter *E* has minima at eigenstates. The value at minimum gives the error.

expensive

Experiment 4 ions



circuit of depth = 7

projective measurements = 5×10^4

The Classical Optimization Algorithm (Overview)

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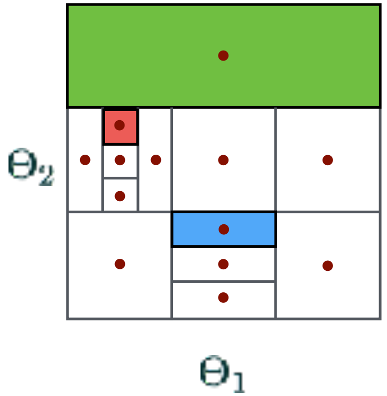
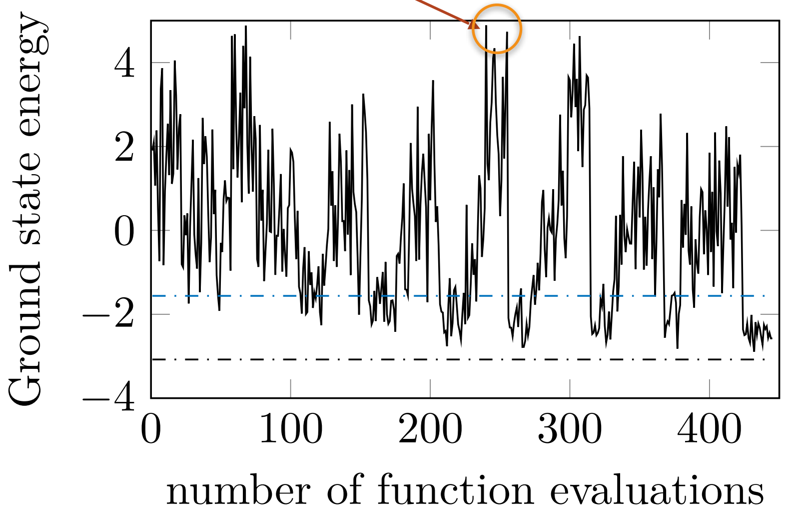
Stochastic DIRECT search

- Global optimization problem with many local minima
- Very noisy problem
- We cannot use gradients
- Optimization with error bars requires elements from decision theory: Optimal Computational Budget Allocation (OCBA)

➡ **DI**viding **RECT**angles (**DIRECT**) global optimization algorithm



Identifying promising regions in a 2D search space



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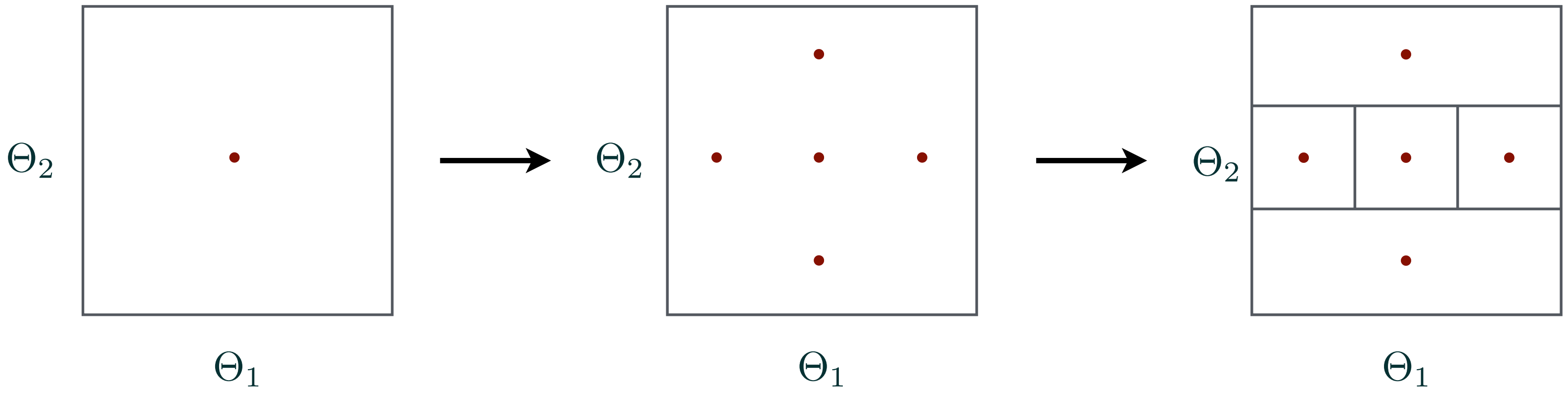
The Classical Optimization Algorithm: Stochastic DIRECT Search



Initialization: sample the center of the hyper-cube

sample points along each dimension in parameter space

divide into smaller hyper-cubes

- **DIviding RECTangles algorithm** (DIRECT) divides search space into ever smaller rectangles, where each rectangle is represented by a single sampling point.
- Each sampling point is an energy measurement as representative for the entire rectangle.
- Upper right: We have to decide, which region of the search space, which of these rectangles, warrants a closer look, i.e. which one should we divide further?

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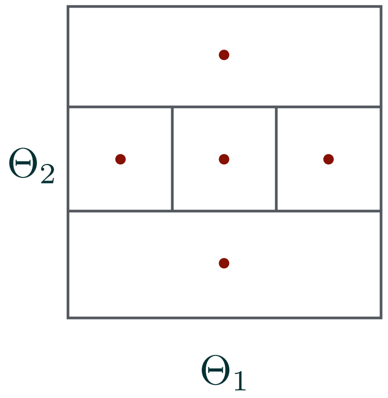
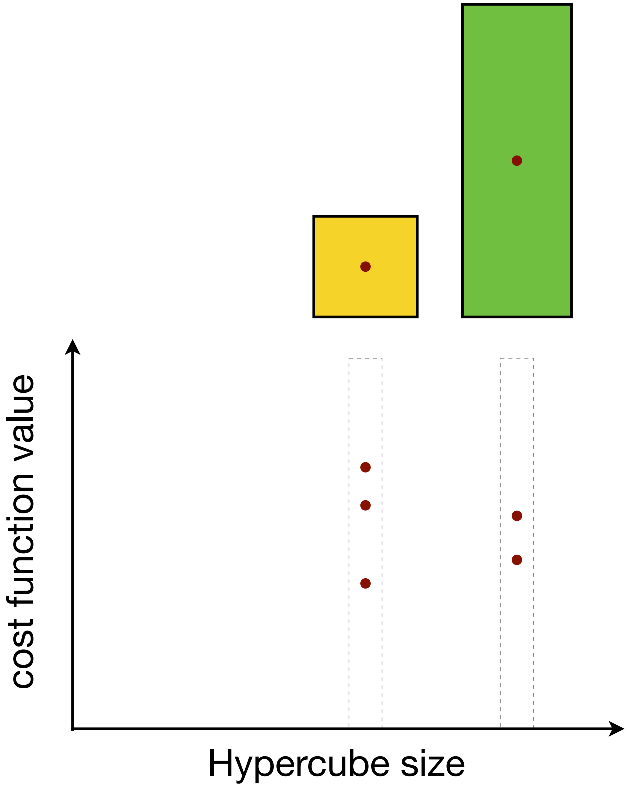


divide into smaller hyper-cubes

Identifying potentially optimal cubes:



- Properties of promising rectangle:
  - sampled energy value
  - large rectangles = lot of unexplored territory for minimum.
- Taking both criteria into account:
  - plot energy values vs. size of each rectangle
  - select all points in lower convex hull as maximally promising

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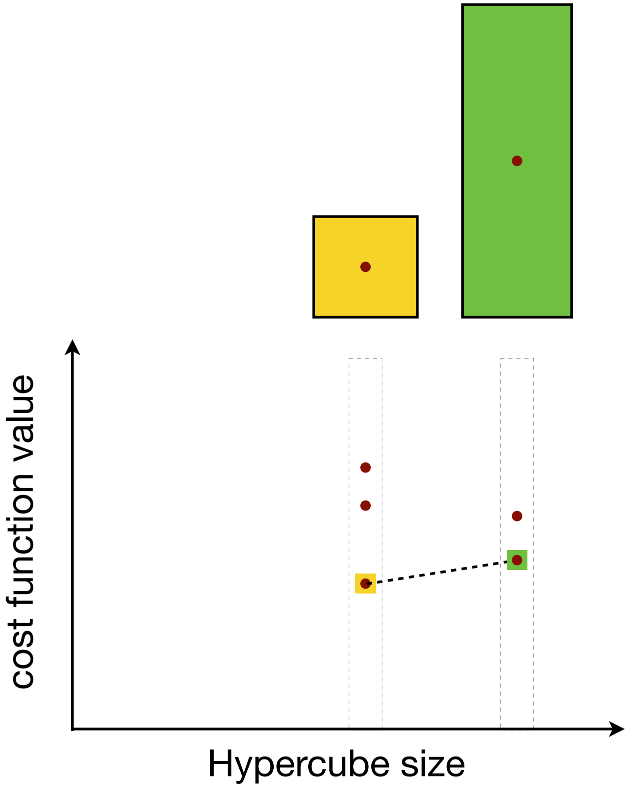

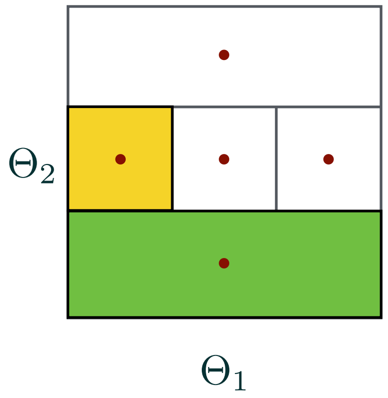

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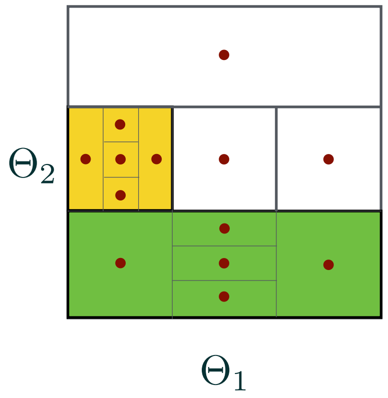

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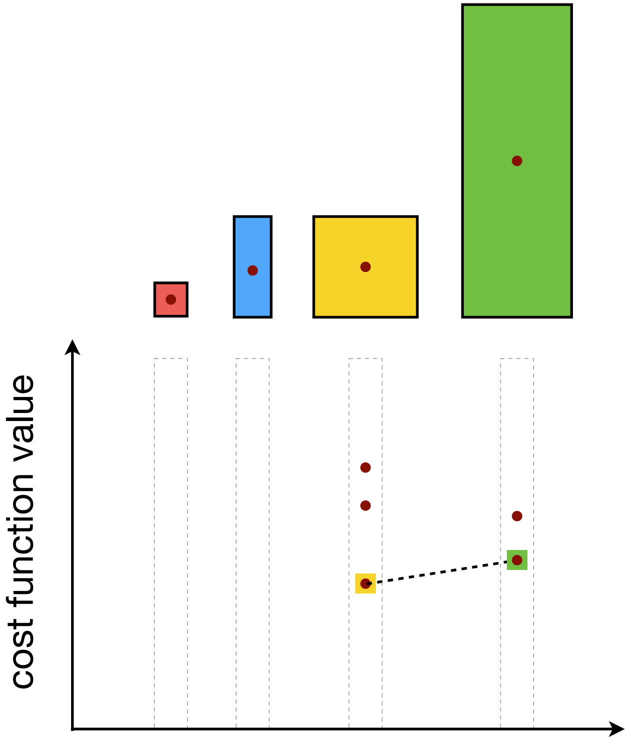

- Select and subdivide these rectangles & repeat
- Note
  - The convex hull will always contain at least one rectangle of largest size, i.e in each step we always take the largest region and subdivide it into smaller regions
  - This guarantees that, eventually, an infinitely fine sampling of the subspace everywhere, and we are guaranteed to find the global minimum.

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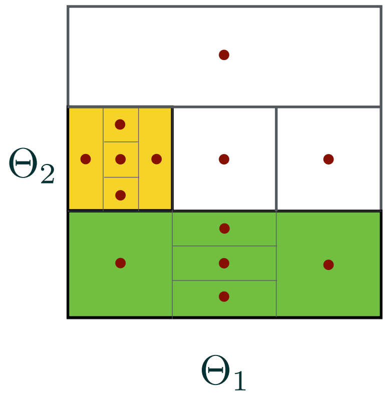

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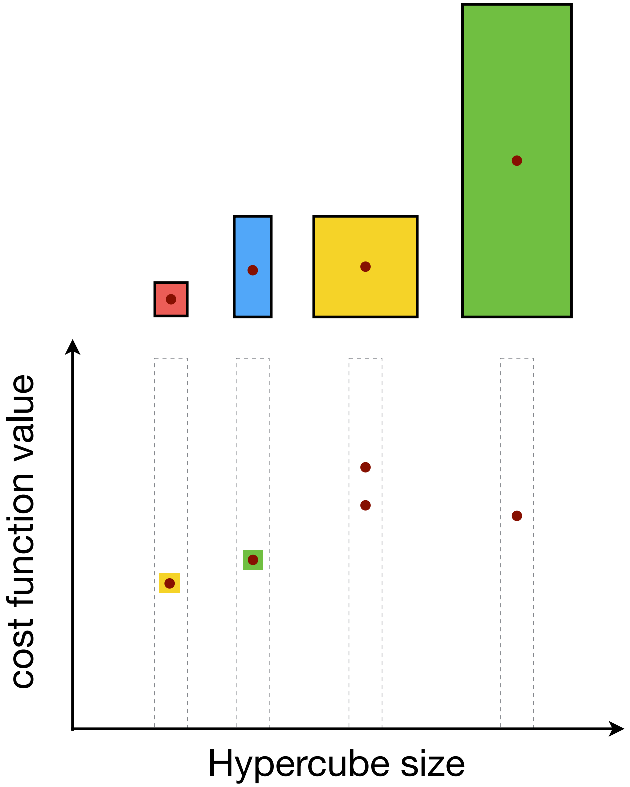

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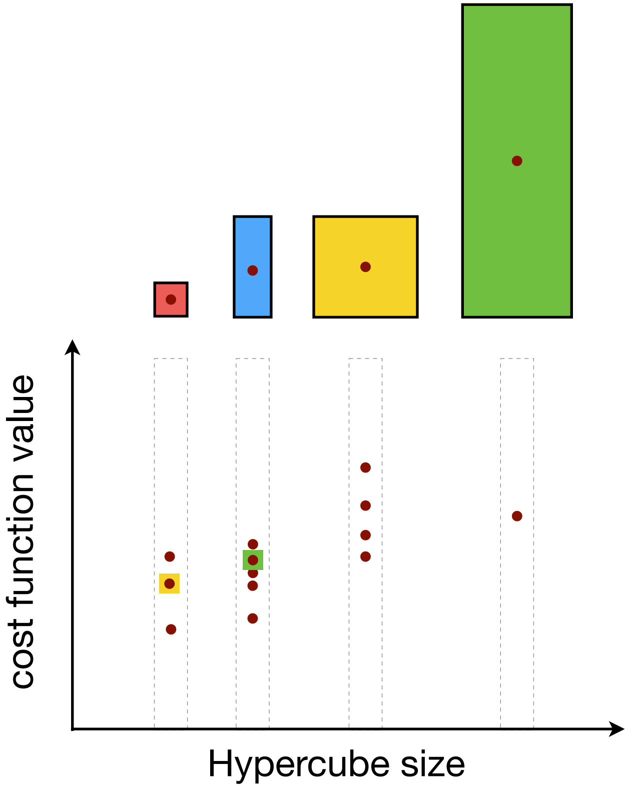

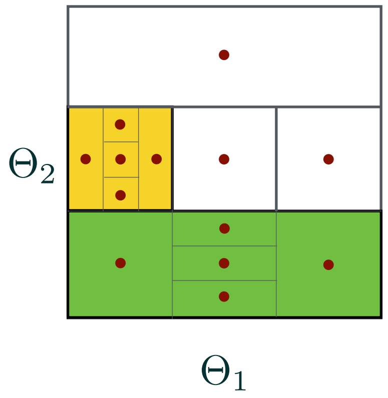

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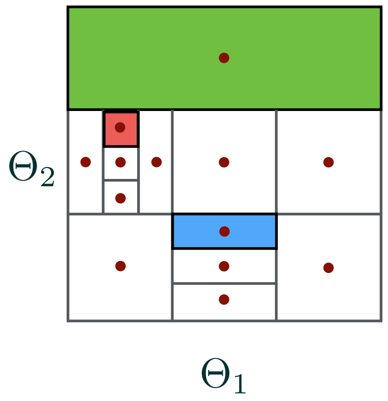

divide into smaller hyper-cubes

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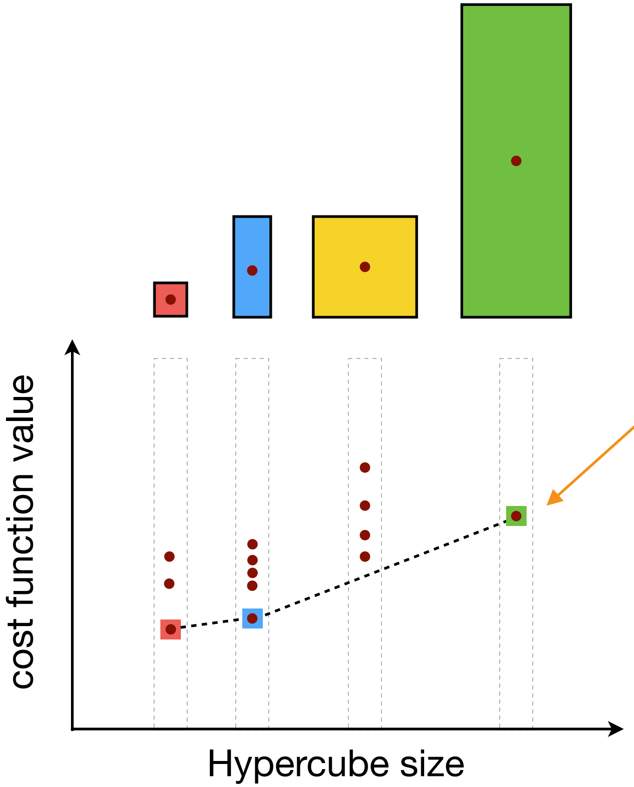

One of the largest boxes is **always** added to the convex hull

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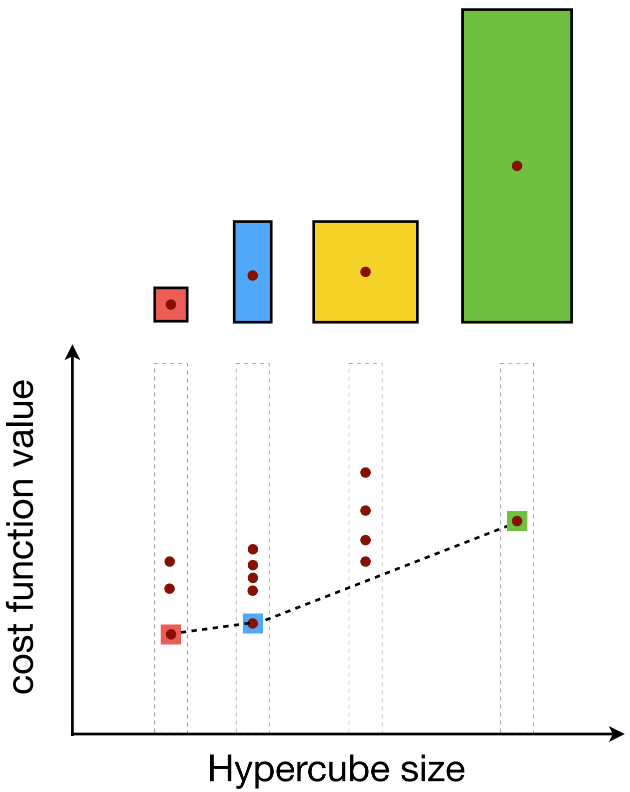
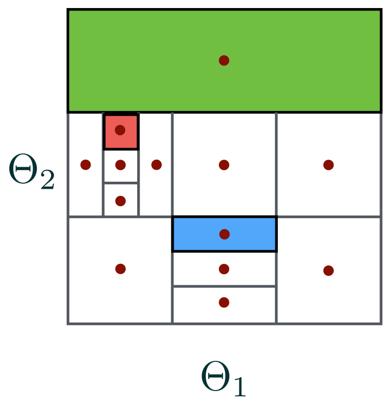
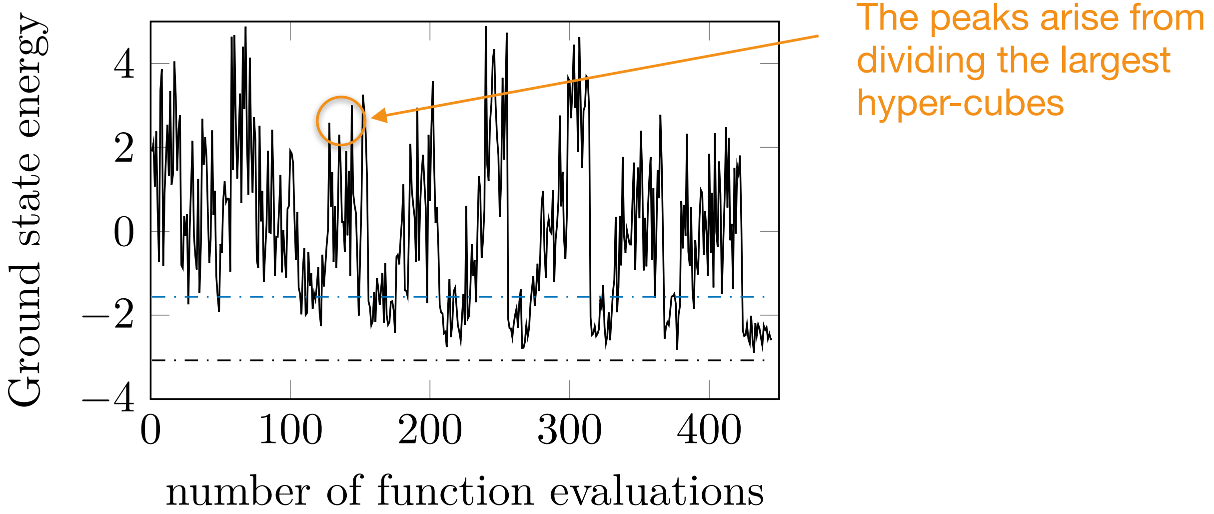




divide into smaller hyper-cubes



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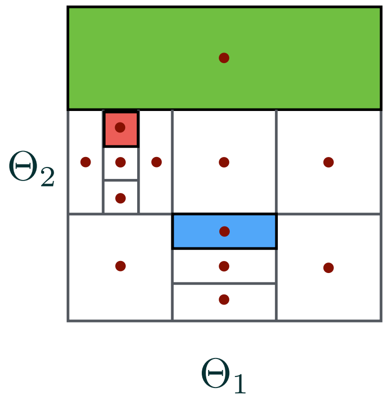
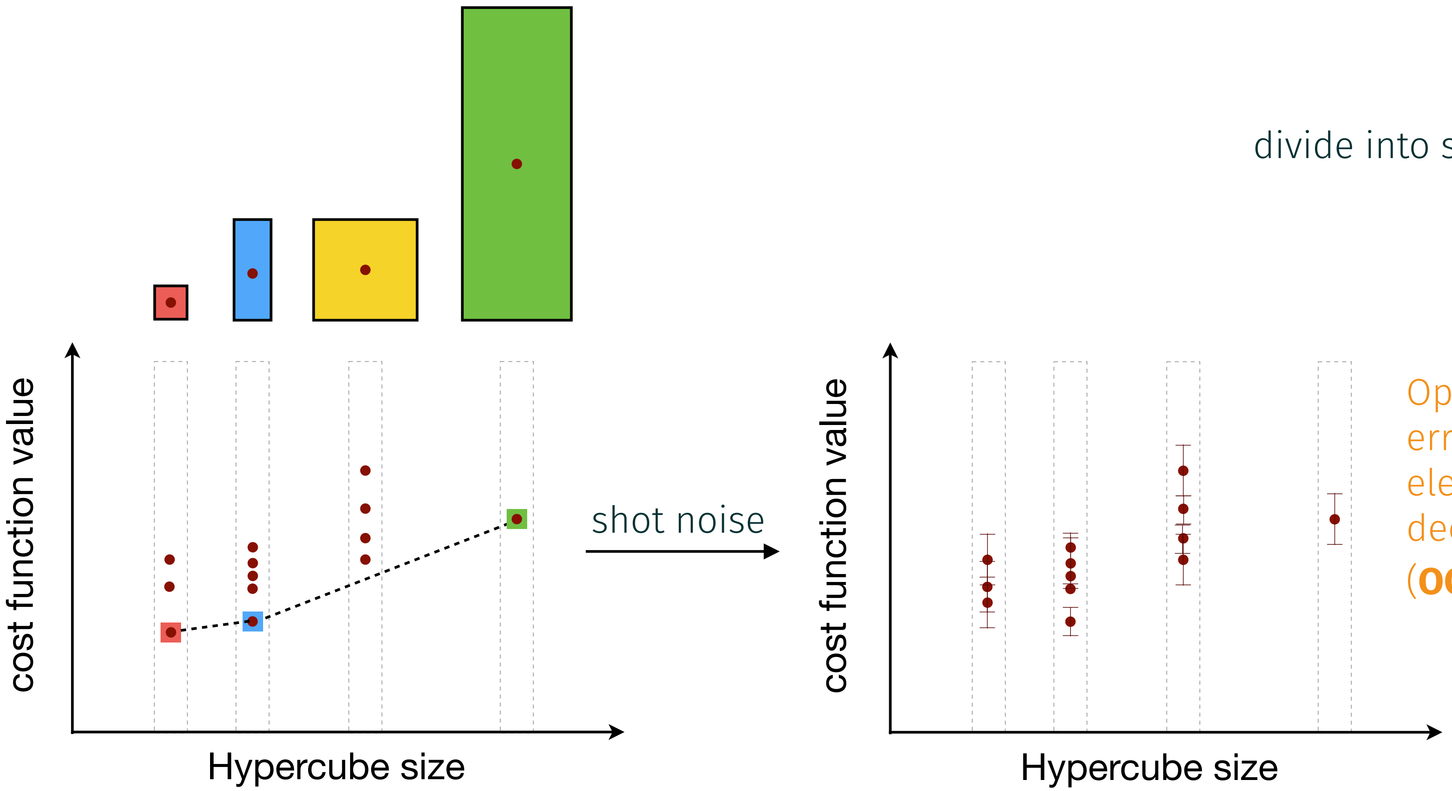


divide into smaller hyper-cubes

Identifying potentially optimal cubes:



Optimisation with error bars requires elements from decision theory (**OCBA**)

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Appendix:

Details on the 2-copy Quantum Protocols
to measure Rényi entropies

Quantum Information Perspective

- Nonlinear functionals of a quantum state can be measured by measuring the expectation value of the unitary operator $V^{(n)}$ on the n-fold copy of the state

$$\mathrm{Tr}\{\rho^n\} = \mathrm{Tr}\{V^{(n)}\rho^{\otimes N}\} \equiv \langle V^{(n)} \rangle$$

Renyi entropy

Measurement of expectation value

$$V^{(n)}\,|\psi_1\rangle \ldots |\psi_n\rangle = |\psi_n\rangle\,|\psi_1\rangle \ldots |\psi_{n-1}\rangle$$

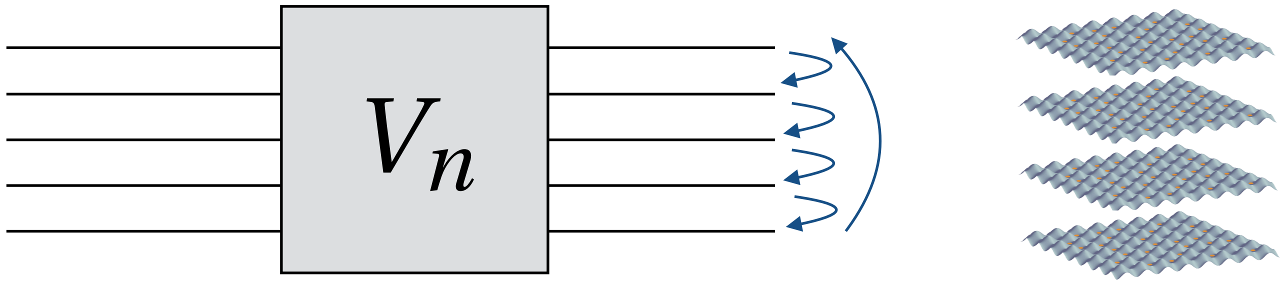
✓ unitary

✓ for n=2 swap is also hermitian

shift (or swap) operator

$\rho \otimes \rho \otimes \rho \otimes \rho \otimes \rho$

n copies



A. K. Ekert, C. Moura Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, PRL (2002).

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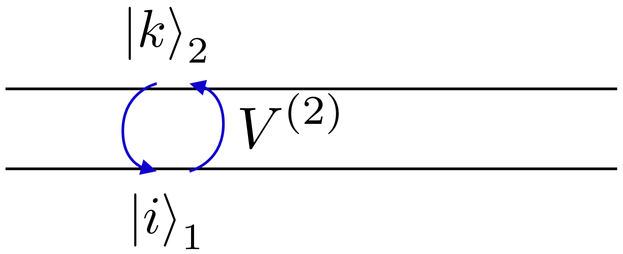
$$V^{(n)}|\psi_1\rangle \dots |\psi_n\rangle = |\psi_n\rangle|\psi_1\rangle \dots |\psi_{n-1}\rangle$$

✓ unitary
✓ for n=2 swap is also hermitian

Example n=2:

$\rho_1 \otimes \rho_1$



swap operator

$$V^{(2)} |i\rangle_1 \otimes |k\rangle_2 = |k\rangle_1 \otimes |i\rangle_2$$

$$\mathrm{tr}\{V^{(2)}\rho_1 \otimes \rho_2\} = \mathrm{tr}\left\{V^{(2)} \sum_{ijkl} \rho_{ij}^{(1)} \rho_{kl}^{(2)} |i\rangle\langle j| \otimes |k\rangle\langle \ell| \right\}$$

expectation value

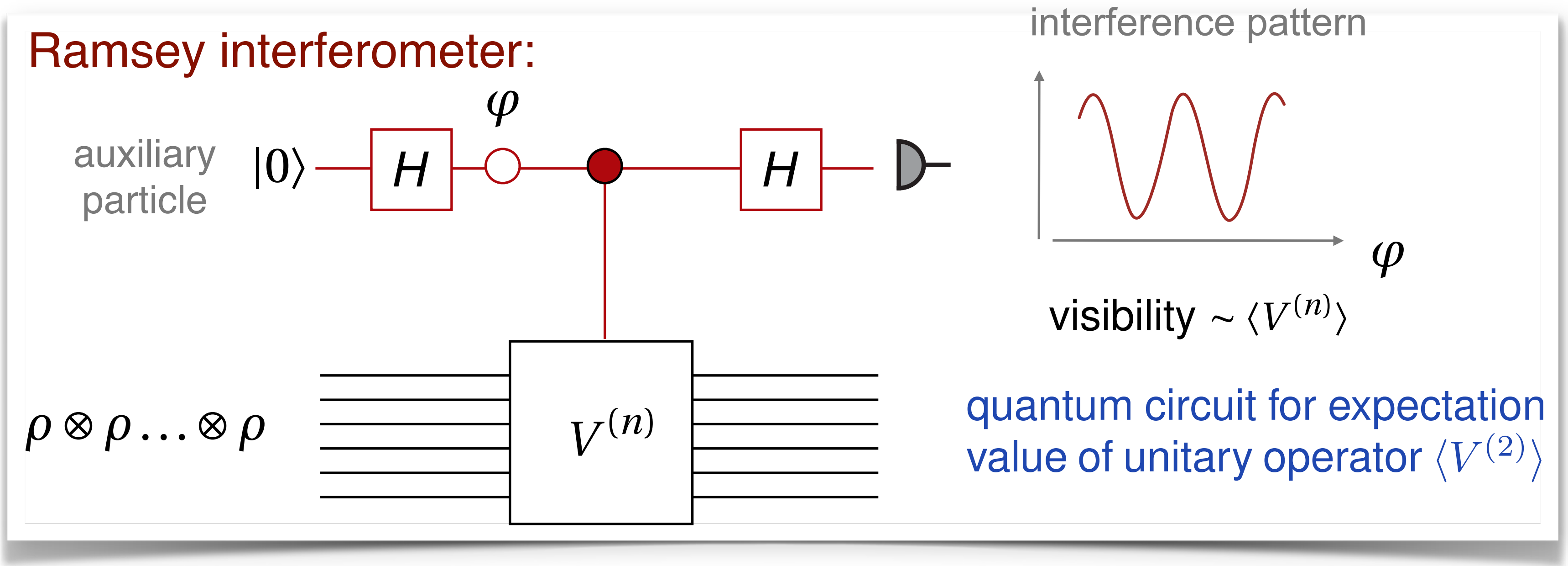
$$= \mathrm{tr}\{\rho_1\rho_2\}$$

Quantum Circuit

- Measurement via quantum network via ancilla qubit and controlled gate between ancilla and the copies of the system.

$$\mathrm{Tr}\{\rho^n\} = \mathrm{Tr}\{V^{(n)}\rho^{\otimes N}\} \equiv \langle V^{(n)} \rangle$$

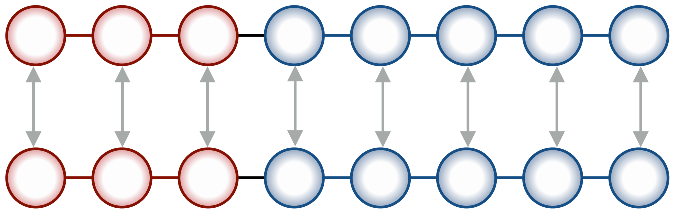
$$V^{(n)}|\psi_1\rangle \ldots |\psi_n\rangle = |\psi_n\rangle |\psi_1\rangle \ldots |\psi_{n-1}\rangle$$



A. K. Ekert, C. Moura Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, PRL (2002).

… we need a quantum computer (?)

$\rho \otimes \rho$



Protocols to measure Rényi entropies

- **a quantum information perspective**

  measuring nonlinear functionals of ρ

  quantum circuits / computers

  A. K. Ekert et al. PRL 2002

- **… and a much more practical protocol**

  bosons (& fermions) in 1D/2D optical lattices

  beamsplitter & microscope

  hard core bosons = spins in ion traps

  Bell state measurements

  A. Daley et al, PRL 2012

  C. Moura Alves, D. Jaksch, PRL 2004
  F. Mintert et al., PRL 2005

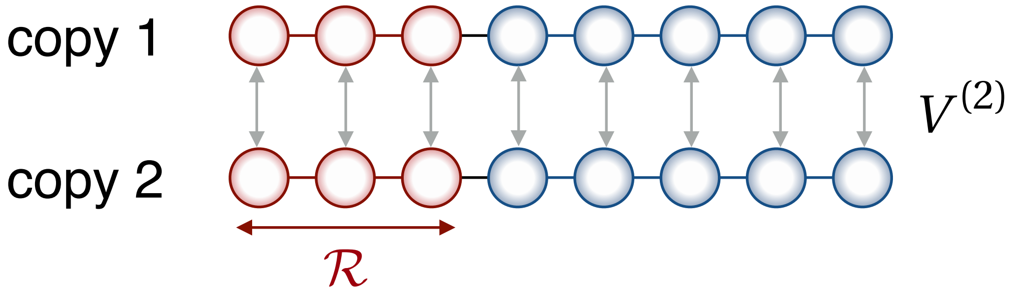
Measurement of Renyi Entropies (n=2)

- **SWAP operator**

$$\mathrm{Tr}\{\rho^2\} = \mathrm{Tr}\{V^{(2)}\rho\otimes\rho\} \equiv \langle V^{(2)}\rangle$$

$$V^{(2)}\,|\mathbf{n}_1\rangle\,|\mathbf{n}_2\rangle = |\mathbf{n}_2\rangle\,|\mathbf{n}_1\rangle$$

boson occupation numbers



Remarks:

- product of local operations $V^{(2)} = \prod_i V^{(2,i)}$
- $V^{(2)}$ hermitian & unitary: eigenvalues $\lambda = +1, -1$

$$V^{(2)} = (+1)\,P_+ + (-1)\,P_-$$

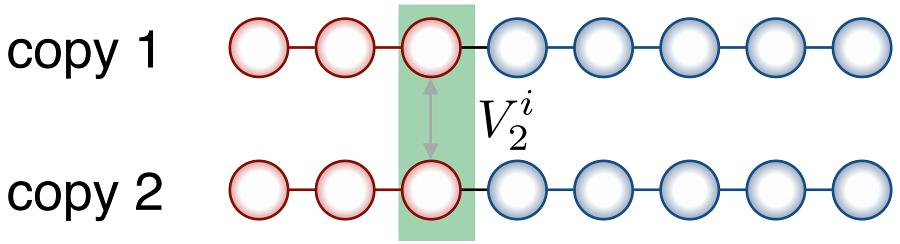
symmetric antisymmetric subspace (copy 1 ↔ 2)

Measure expectation values of projection operators onto (anti)symmetric subspace (with respect to exchange of copies)

- **identify symmetric and antisymmetric subspaces of the SWAP operator**

V_2^i : swap operator → $+1$, -1

$a_{i,1} \leftrightarrow a_{i,2}$



eigenstates

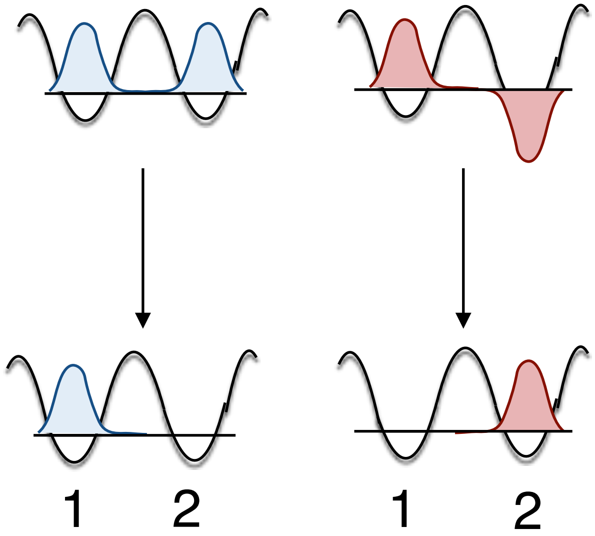
$$(a^\dagger_{i,1} + a^\dagger_{i,2})^n (a^\dagger_{i,1} - a^\dagger_{i,2})^m |\mathrm{vac}\rangle$$

eigenvalue

$$(-1)^m$$

- **50/50 beamsplitter**

$$(a^\dagger_{i,1})^n (a^\dagger_{i,2})^m |\mathrm{vac}\rangle$$



(quantum) measurement of V_2^i is simply a measurement of occupation numbers (modulo 2) after a 50/50 beam splitter.

This leads to a protocol, where beam splitter operations and a microscope are sufficient.

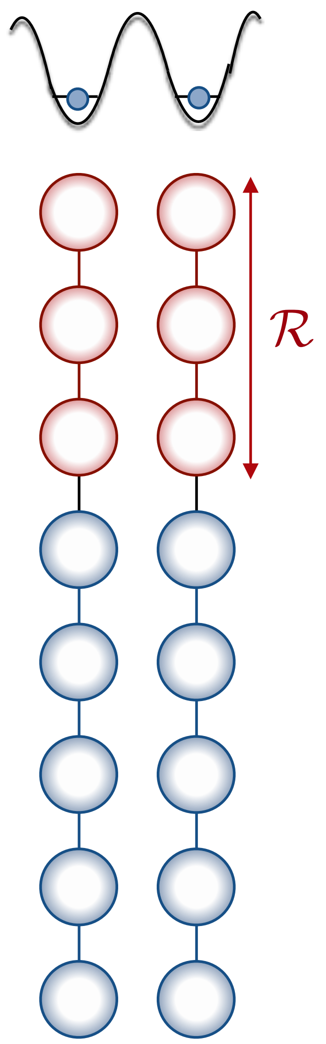
Note: protocol can be generalized to n

“The Recipe”: for n=2 (Bosons)

- Renyi entropy $\mathrm{Tr}\{\rho_{\mathcal{R}}^2\} = \mathrm{Tr}\{V_2^{\mathcal{R}} \rho_{\mathcal{R}} \otimes \rho_{\mathcal{R}}\} \equiv \langle V_2^{\mathcal{R}} \rangle$

2 copies

Eigenvalues: ± 1



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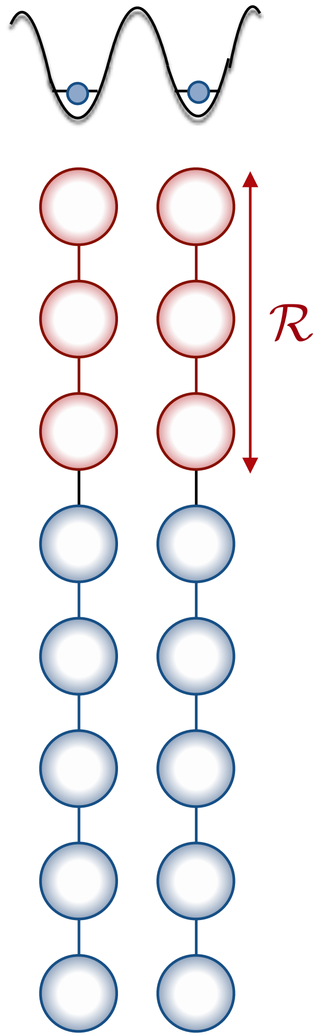
Bosons in 1D optical lattices:

- freeze the motion in the axial direction



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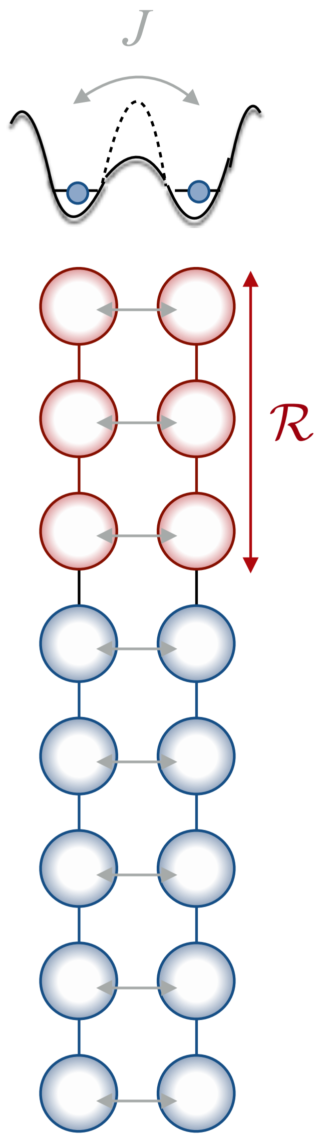
Bosons in 1D optical lattices:

- freeze the motion in the axial direction
- **tunneling** between the two copies using a **superlattice**

  (*turn interaction off!*)

$$a_{j,1} \rightarrow \frac{1}{\sqrt{2}}\left(a_{j,1} + a_{j,2}\right), \quad a_{j,2} \rightarrow \frac{1}{\sqrt{2}}\left(a_{j,2} - a_{j,1}\right)$$

single particle operations



"The Recipe": for n=2 (Bosons)

- Renyi entropy $\mathrm{Tr}\{\rho_{\mathcal{R}}^2\} = \mathrm{Tr}\{V_2^{\mathcal{R}} \rho_{\mathcal{R}} \otimes \rho_{\mathcal{R}}\} \equiv \langle V_2^{\mathcal{R}} \rangle$

Bosons in 1D optical lattices:

- freeze the motion in the axial direction
- **tunneling** between the two copies using a **superlattice**

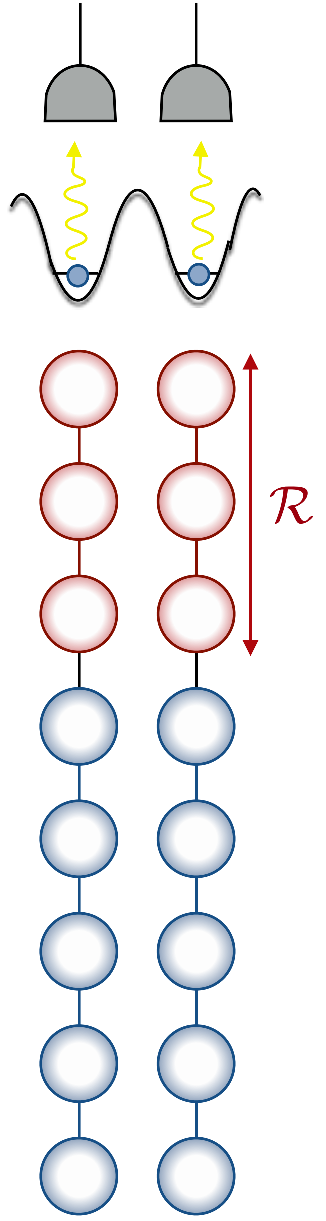
  (*turn interaction off!*)

$$a_{j,1} \to \frac{1}{\sqrt{2}}(a_{j,1} + a_{j,2}) , \quad a_{j,2} \to \frac{1}{\sqrt{2}}(a_{j,2} - a_{j,1})$$

- measure **site resolved** atom number

$\sum_{i\in\mathcal{R}} n_{i,2}$	$V_2^{\mathcal{R}}$
even	$+1$
odd	-1

quantum gas microscope



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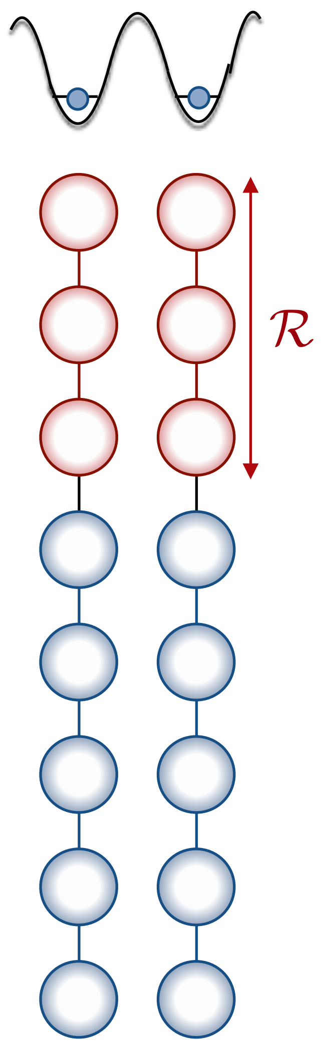
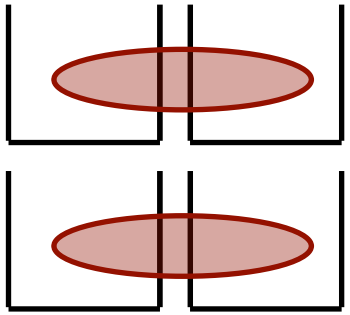
- measure **site resolved** atom number
- repeat

$$\mathrm{Tr}\{\rho_{\mathcal{R}}^2\} = \langle V_2^{\mathcal{R}} \rangle = \langle (-1)^{\sum_{i \in \mathcal{R}} n_{i,2}} \rangle_{\text{measure}}$$



Example: Detecting a Superfluid (two sites)



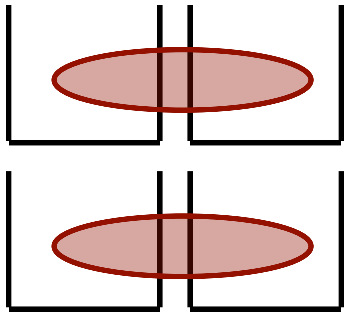
$$(a^\dagger_{1,1} + a^\dagger_{2,1})^2 (a^\dagger_{1,2} + a^\dagger_{2,2})^2 |\mathrm{vac}\rangle$$

SF 1 SF 2

Example: Detecting a Superfluid (two sites)

Copy 2

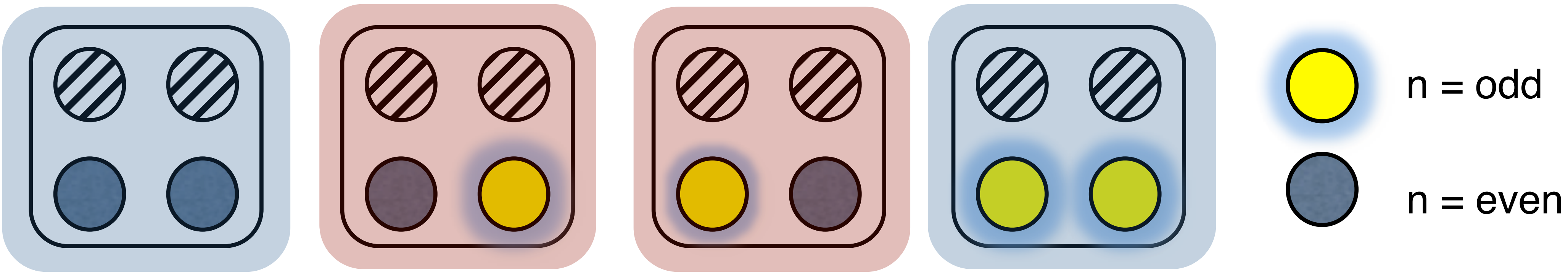
Copy 1



$$(a^\dagger_{1,1} + a^\dagger_{2,1})^2 (a^\dagger_{1,2} + a^\dagger_{2,2})^2 |\text{vac}\rangle$$

SF 1 SF 2

Possible read outs (after beam-splitter):



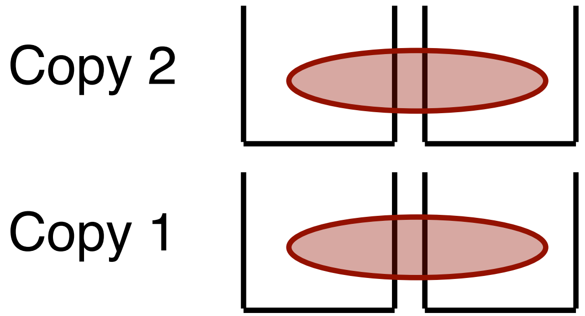
Probabilities:

$p = \frac{11}{16}$ $p = 0$ $p = 0$ $p = \frac{5}{16}$

$$\text{Tr}\{\rho^2\} = \langle V_2^{\{1,2\}} \rangle = +1 \times \left(\frac{11}{16} + \frac{5}{16} \right) - 1 \times (0 + 0) = 1$$

Pure

Example: Detecting a Superfluid (two sites)



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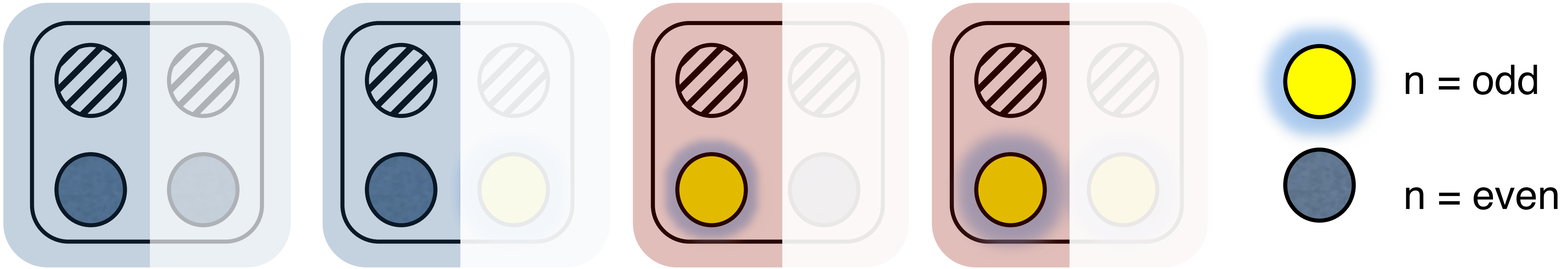
SF 1 SF 2

Possible read outs (after beam-splitter):



Probabilities:

$p = \frac{11}{16}$ $p = 0$ $p = 0$ $p = \frac{5}{16}$

$$\text{Tr}\{\rho_1^2\} = \langle V_2^{\{1\}} \rangle = +1 \times \left(\frac{11}{16} + 0\right) - 1 \times \left(0 + \frac{5}{16}\right) = \frac{3}{8}$$

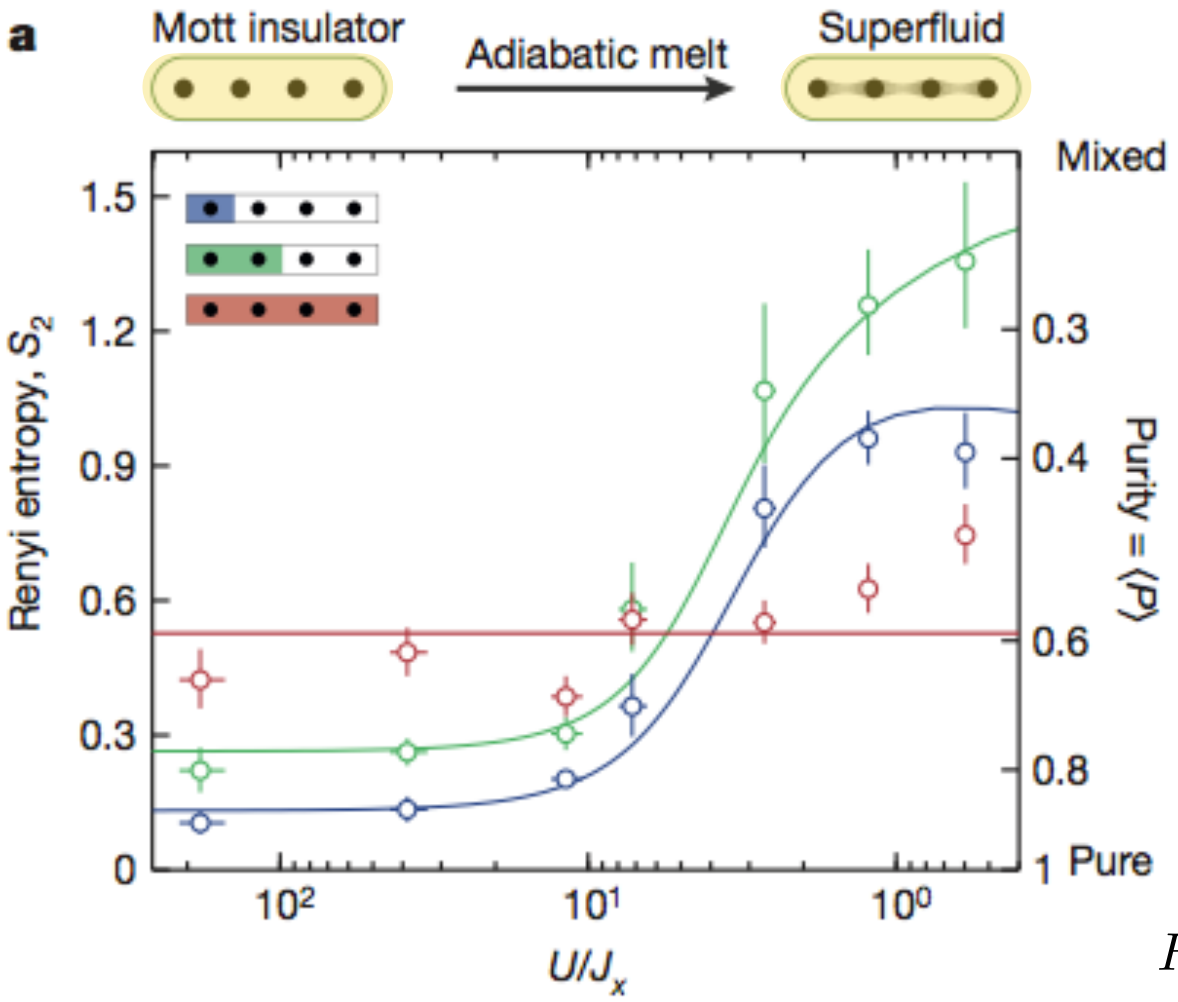
Mixed

Measuring entanglement entropy in a quantum many-body system

doi:10.1038/nature15750

Rajibul Islam[1], Ruichao Ma[1], Philipp M. Preiss[1], M. Eric Tai[1], Alexander Lukin[1], Matthew Rispoli[1] & Markus Greiner[1]

- **Entanglement in the ground state of the Bose-Hubbard model**



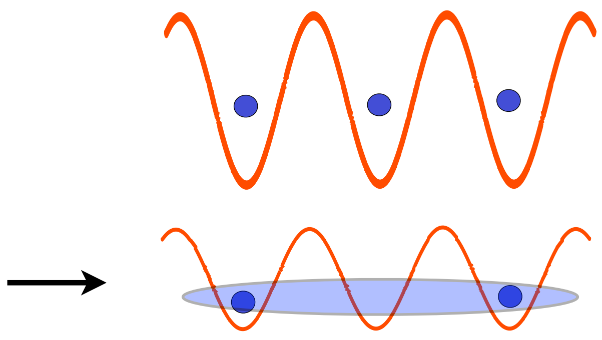
overall purity constant

$$H = -J \sum_{i,j} b_i^\dagger b_j + \frac{U}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i$$

Bose-Hubbard Hamiltonian

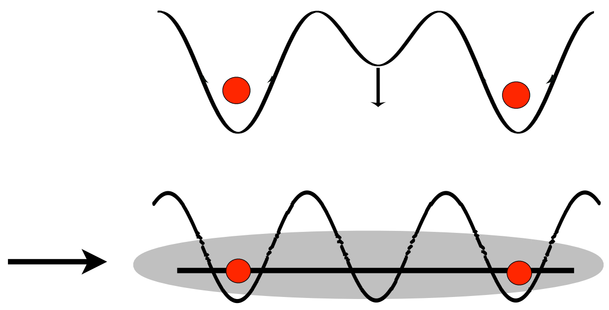
Theory: Quantum Quenches

Softcore bosons

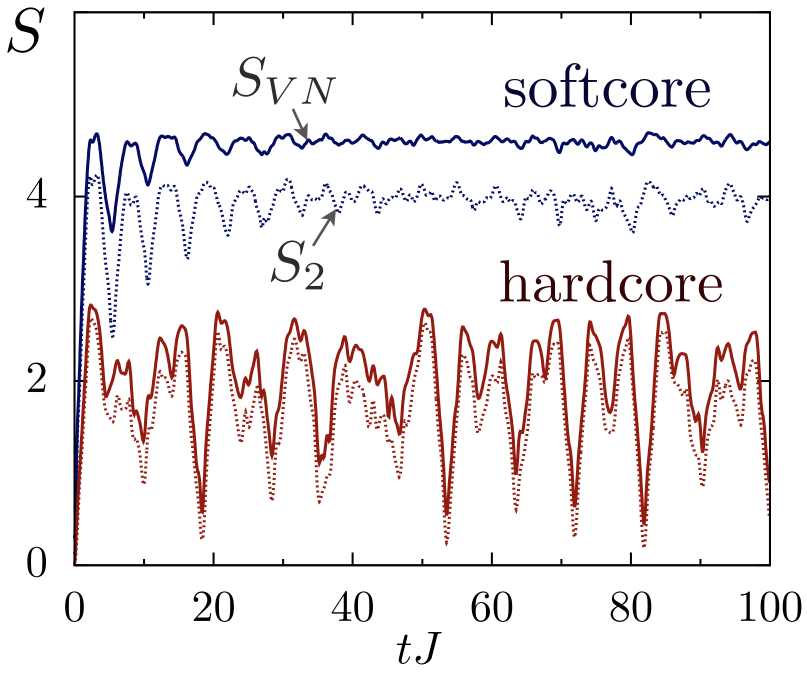


- Bose-Hubbard U/J=10 to U/J=1 quench

Hardcore bosons



- Odd sites initially filled



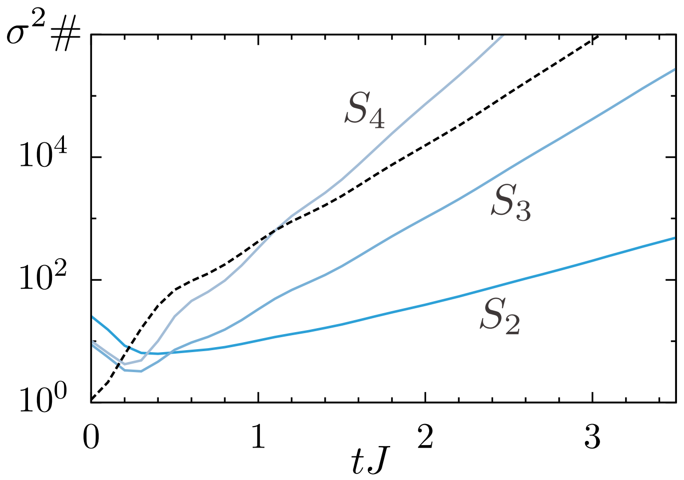
Softcore bosons, small system (N=M=8):
- Increasing entanglement, saturates (thermalization)

Hardcore bosons (M=8)
- Initial growth of entanglement, then oscillations (integrable system)

tDMRG calculations by
A Daley & J Schachenmayer

Remarks

- **number of measurements for a given precision**

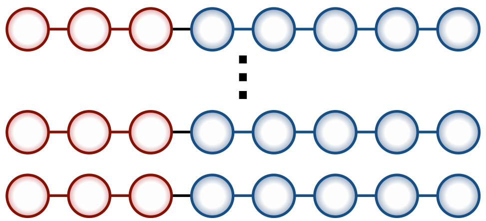


- Larger n requires more measurements for same precision
- However, combination of n=2 and higher n gives stronger bound on von Neumann entropy, e.g., (dashed line in measurement)

$$S_{VN}(\rho) \geq 2S_2(\rho) - S_3(\rho)$$

- **role of imperfections …**

Extensions



- **Higher order** Renyi entropies:

$$U_n^{FT} : a_{j,k} \rightarrow \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} a_{j,\ell} e^{i \frac{2\pi n}{n}(k-1)(\ell-1)}$$

- **Fermions**: same experimental procedure, different interpretation of measurement record