Schwinger Model (1962)

- Quantum Electrodynamics in 1+1 dimensions coupled to N_f Dirac fermions of mass m. Admits a theta-angle Coleman, Jackiw, Susskind (1975)

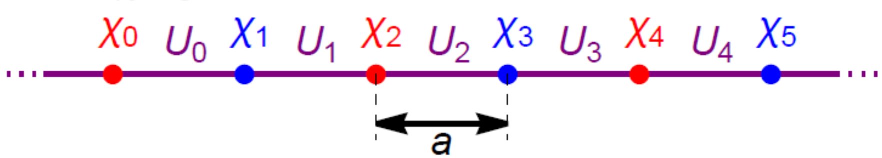
$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^2 - \frac{\theta}{4\pi}\epsilon^{\mu\nu} F_{\mu\nu} + \sum_{\alpha=1}^{N_f} \overline{\Psi}_\alpha (i\not{D} - m_\alpha)\Psi_\alpha$$

- One-flavor model exactly solvable for m=0 where it reduces to the free massive Schwinger boson.
- It is a tightly bound state of electron and positron.

Lattice Hamiltonian Approach

- Using the staggered fermions Banks, Susskind, Kogut (1975)

$$H = \frac{e^2 a}{2} \sum_{n=0}^{N-1} \left(L_n + \frac{\theta}{2\pi} \right)^2 + m_{\text{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_n^\dagger \chi_n$$
$$- \frac{i}{2a} \sum_{n=0}^{N-1} \left[\chi_n^\dagger U_n \chi_{n+1} - \chi_{n+1}^\dagger U_n^\dagger \chi_n \right]$$



- The Gauss Law Constraints Hamer, Zheng, Oitmaa (1997)

$$L_n - L_{n-1} = Q_n \,, \qquad Q_n \equiv \chi_n^\dagger \chi_n - \frac{1 - (-1)^n}{2}$$

$$[L_n, U_m] = \delta_{nm} U_n \qquad \{\chi_n, \chi_m^\dagger\} = \delta_{nm}$$

- The lattice approach revisited with the surprising result Dempsey, IRK, Pufu, Zan, arXiv: 2206.05308

$$m_{\text{lat}} = m - \frac{1}{8}e^2 a$$

- At m=0 the Hamiltonian is preserved by a "discrete chiral symmetry:" shift by one lattice unit accompanied by $\theta \to \theta + \pi$
- The mass shift greatly improves the extrapolation of strong coupling expansions

$$\omega_1 - \omega_0 = 1 + 2\mu + \frac{2x^2}{1+2\mu} - \frac{2(5+2\mu)x^4}{(1+2\mu)^3} + \frac{4(59+68\mu+24\mu^2+4\mu^3)x^6}{(1+2\mu)^5(3+2\mu)}$$

$$\mu = 2m_{\text{lat}}/(e^2 a) \qquad\qquad x^2 = y = 1/(ea)^4$$

- In earlier work the massless Schwinger model was assumed to be described by $\mu=0$, and extrapolation to large x did not seem to give good results.
- We instead set $\mu = -1/4$ to obtain

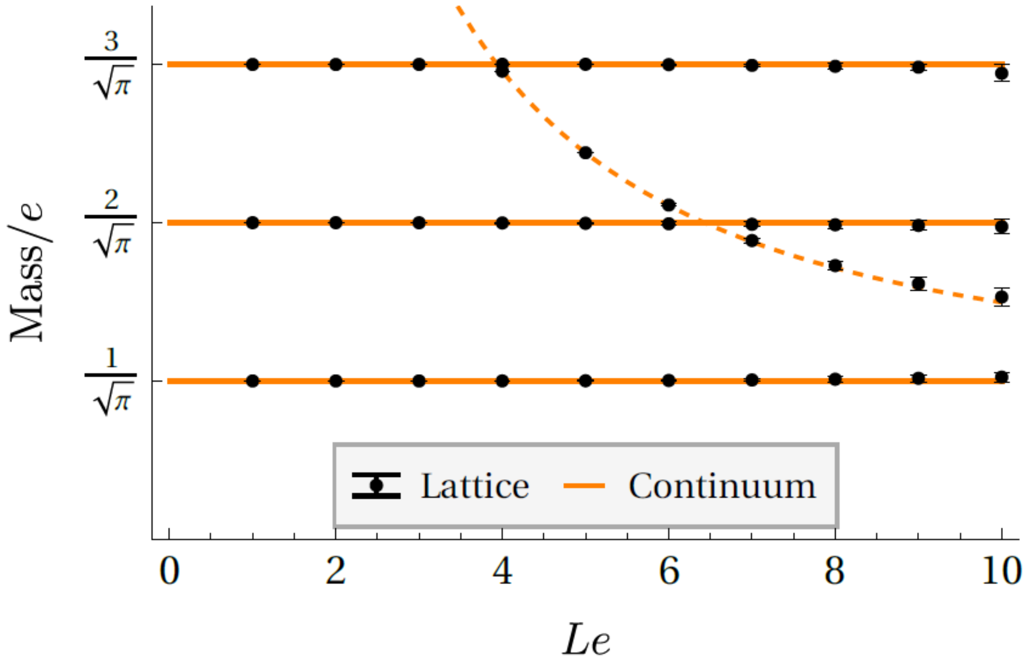
$$\delta\omega = \frac{1}{2} + 4y - 72y^2 + 2224y^3 + O(y^4)$$

- Pade extrapolating to weak coupling we find

$$E_1 - E_0 \approx \left(\frac{19}{188}\right)^{1/4} e \approx 0.56383e$$

- This reproduces the mass of Schwinger boson with error < 0.1 %.

- Exact diagonalizations on lattices with periodic boundary conditions also produced excellent results



- The DMRG methods give very precise results at the shifted mass. Connections with cold atoms.

Two-Flavor Schwinger Model

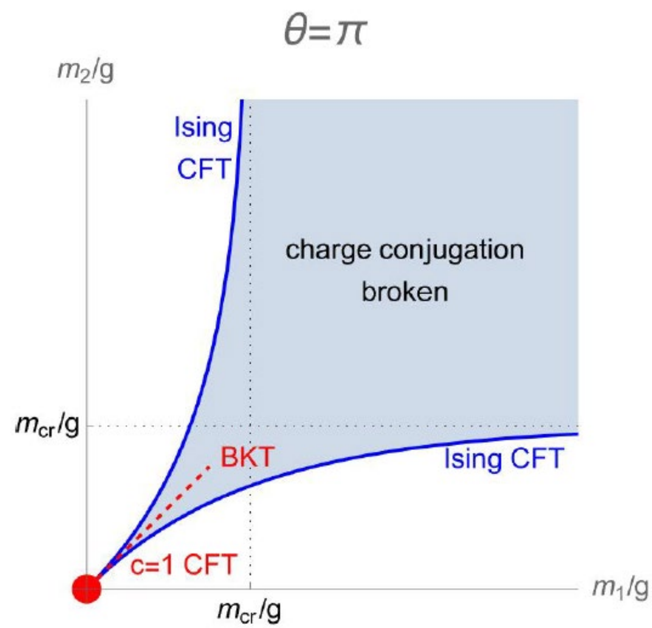
- For m=0 it is a conformal field theory coupled to a massive field. The CFT is a free massless boson at the self-dual radius where it has SU(2) x SU(2) symmetry.
- Charge conjugation symmetry

$$C: \qquad A_\mu \to -A_\mu \,, \qquad \Psi_\alpha \to \gamma^5 \Psi^*_\alpha$$

- Is preserved by the lagrangian for theta=0 or pi. At $\theta = \pi$ it can be broken spontaneously. This is reminiscent of spontaneous T breaking in QCD. Dashen; Gaiotto, Kapustin, Komargodski, Seiberg

Phase Diagram at Zero Temperature

- Our proposal Dempsey, IRK, Pufu, Soegaard, Zan, arXiv: 2305.04437



- In particular, C is broken along the SU(2) invariant line $m_1 = m_2 = m$

Bosonized 2-flavor model

- Form two combinations of scalar fields

$$\phi_+ = 2^{-1/2}(\phi_1 + \phi_2 + \tfrac{1}{2}\pi^{-1/2}\theta)$$

$$\phi_- = 2^{-1/2}(\phi_1 - \phi_2)$$

$$\mathcal{L}_{\text{bos}} = -\frac{1}{4g^2}F_{\mu\nu}^2 - \frac{\phi_+}{\sqrt{\pi}}\epsilon^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\phi_+)^2 + \frac{1}{2}(\partial_\mu\phi_-)^2$$
$$+ \frac{e^\gamma}{\pi} m\mathcal{M}N_{\mathcal{M}}\cos\left[\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right] N_{\mathcal{M}}\cos\left[\sqrt{2\pi}\phi_-\right]$$

- Integrating out the gauge field, makes the plus field massive, but there is also a massless minus field.

- For generic theta, the effective Lagrangian is

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{2}(\partial_\mu \phi_-)^2 + m\mu \frac{e^\gamma}{\pi} \cos\frac{\theta}{2} N_\mu \cos\left[\sqrt{2\pi}\phi_-\right]$$

- The mass term is a relevant operator of dimension ½ which induces RG flow to a theory with mass gap $\sim |m\cos(\theta/2)|^{2/3} g^{1/3}$
- This vanishes for $\theta = \pi$. Could this theory be gapless?!
- No, but the energy gap is non-perturbatively small.

- When $\theta = \pi$

$$\mathcal{L}_{\rm bos} = \frac{1}{2}(\partial_\mu \phi_-)^2 + \frac{e^{3\gamma} I_s m^2}{8\pi^2 \mu^2}\Big(\mu^2 N_\mu \cos(\sqrt{8\pi}\phi_-) + 2\pi e^{-2\gamma}(\partial_\mu \phi_-)^2\Big) + O(m^4)\,,$$

- This was derived by Coleman in 1976, but he did not study the logarithmic RG flow of the two nearly marginal operators.
- This is the Berezinkii-Kosterlitz-Thouless flow in the sine-Gordon model

$$\mathcal{L} = \frac{1-\delta}{2}(\partial_\mu \phi)^2 + \frac{\alpha e^{2\gamma}}{32\pi}\mathcal{M}^2 N_{\mathcal{M}} \cos(\sqrt{8\pi}\phi)$$

- The beta functions are

$$\beta_{\overline{\alpha}} = 2\overline{\alpha}\overline{\delta}\,, \qquad \beta_{\overline{\delta}} = \frac{1}{32}\overline{\alpha}^2$$

- The SU(2) invariant RG flow is along the line

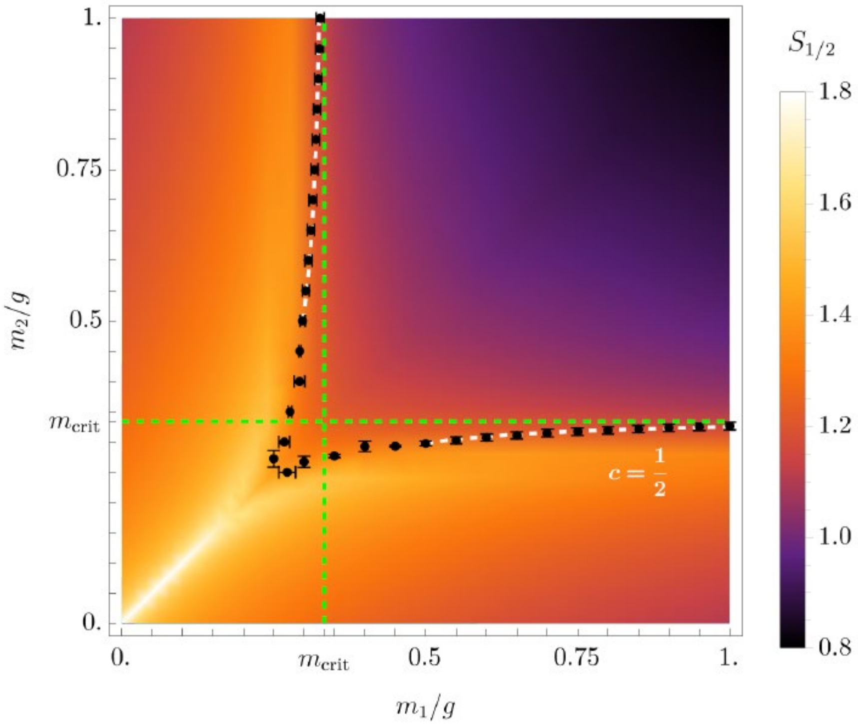
$$\beta_{\overline{m}} = M\frac{d\overline{m}}{dM} = -\frac{e^{\gamma} I_s}{4g^2}\overline{m}^3$$

- Starting with the bare values $\alpha = \frac{8e^{\gamma} I_s}{4}\frac{m^2}{g^2} = -8\delta$

$$\beta_{\overline{m}} = M\frac{d\overline{m}}{dM} = -\frac{e^{\gamma} I_s}{4g^2}\overline{m}^3$$

$$E_{\text{gap}} \sim e^{-A\frac{g^2}{m^2}}\,, \qquad A = \frac{2e^{-\gamma}}{I_s} \approx 0.111$$

- This exponentially small scale seems analogous to the appearance of Λ_{QCD}
- Numerical evidence in support of the exponentially small gap and our proposed phase diagram is provided by the Entanglement Entropy
- Calculated using



Lattice Hamiltonian calculation

- Analogously to the 1-flavor case, we adopt

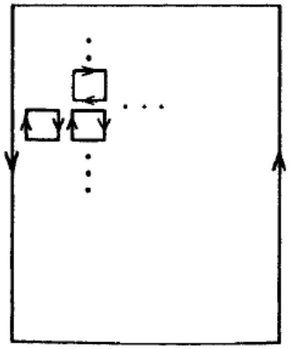
$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-1} \left(L_n + \frac{\theta}{2\pi} \right)^2 + \sum_{\alpha=1}^{N_f} m_{\text{lat},\alpha} \sum_{n=0}^{N-1} (-1)^n c_{n,\alpha}^\dagger c_{n,\alpha}$$
$$- \frac{i}{2a} \sum_{n=0}^{N-1} \sum_{\alpha=1}^{N_f} \left(c_{n,\alpha}^\dagger U_n c_{n+1,\alpha} - c_{n+1,\alpha}^\dagger U_n^\dagger c_{n,\alpha} \right) .$$

- The mass shift is $m_{\text{lat},\alpha} = m_\alpha - \frac{N_f g^2 a}{8}$
- Very important in the 2-flavor case. For m=0 a discrete chiral symmetry is preserved. It is the lattice translation by one site.

Euclidean Lattice SU(N) Theory

- The gauge field kinetic term is encoded in the plaquette terms.

$$S = -(1/2g^2) \sum_{n,\,\mu\nu} \mathrm{tr}\, U_\mu(n) U_\nu(n+\mu)\, U_{-\mu}(n+\mu+\nu) U_{-\nu}(n+\nu) + \mathrm{h.c.}$$



- In the strong coupling expansion where these terms are treated as perturbations, the Area Law of the Wilson loop is obvious.

$$\left\langle \prod_C \exp[iB_\mu(n)] \right\rangle \approx \exp[-F(g^{-2})A]$$

- To obtain the continuum limit, one needs to interpolate to the weak coupling limit on lattice scale due to Asymptotic Freedom

$$g^2(a) = \frac{g_0^2}{1 + (Cg_0^2/2\pi)\ln(a_0/a)}$$

- Can confinement disappear in this limit? Monte Carlo simulations strongly suggest that the answer is “No.” Lattice sizes beyond ~100^4 now.

Dimensional Transmutation

- The QCD scale is exponentially small compared to inverse lattice spacing

$$\Lambda_{QCD} = a^{-1} e^{-2b_0/(4\pi)^2 g_0^2}$$

- This follows from the Asymptotic Freedom

$$\mu \frac{dg}{d\mu} = -b_0 \frac{g^3}{(4\pi)^2} + O(g^5) \ , \qquad b_0 = \frac{11}{3} N - \frac{2}{3} N_f$$

D. Gross, F. Wilczek; D. Politzer (1973)

The AdS/CFT Duality

Maldacena; Gubser, IRK, Polyakov; Witten

- Relates conformal gauge theory in 4 dimensions to string theory on 5-d Anti-de Sitter space times a 5-d compact space. For the $\mathcal{N}$=4 SYM theory this compact space is a 5-d sphere.
- The geometrical symmetry of the AdS_5 space realizes the conformal symmetry of the gauge theory.
- The AdS space-time is a generalized hyperboloid. It has negative curvature.
- Where does this come from?!

Stacking D3-branes.

Reviewed in hep-th/0009139

- When a gauge theory is strongly coupled, the radius of curvature of the dual AdS_5 and of the 5-d compact space becomes large: $\frac{L^2}{\alpha'} \sim \sqrt{g^2_{\mathrm{YM}} N}$

- String theory in such a weakly curved background can be studied in the effective (super)-gravity approximation, which allows for a host of explicit calculations. Corrections to it proceed in powers of

$$\frac{\alpha'}{L^2} \sim \lambda^{-1/2}$$

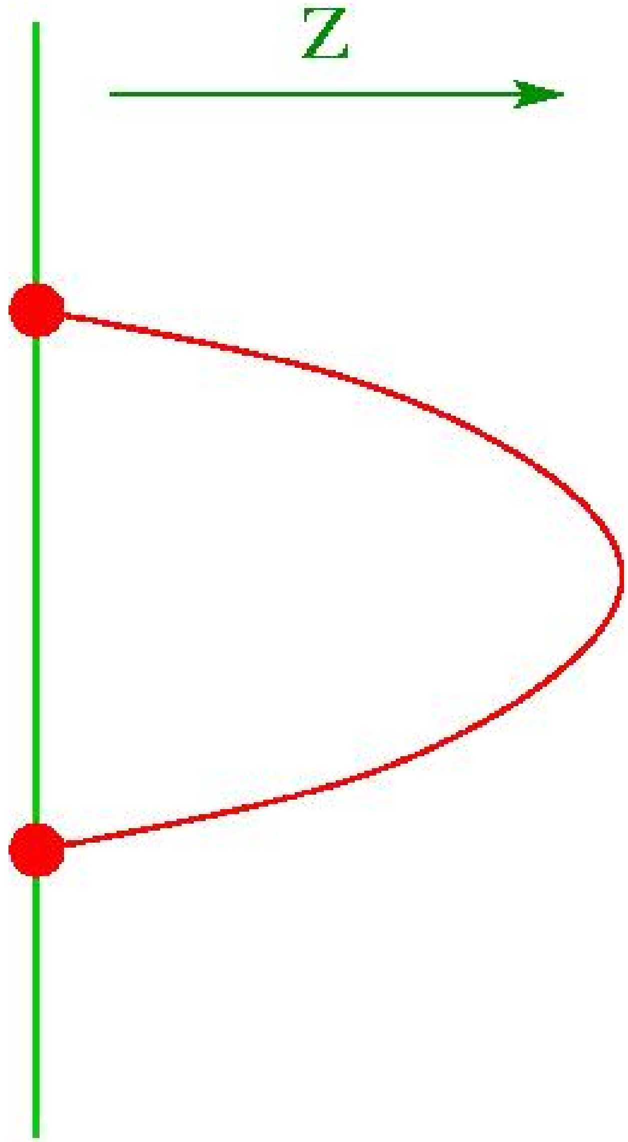
- Feynman graphs instead develop a weak coupling expansion in powers of λ. At weak coupling the dual string theory becomes difficult.

The quark anti-quark potential

- The z-direction of AdS is dual to the energy scale of the gauge theory: small z is the UV; large z is the IR.

- The quark and anti-quark are placed at the boundary of Anti-de Sitter space (z=0), but the string connecting them bends into the interior (z>0). Due to the scaling symmetry of the AdS space, this gives Coulomb potential Maldacena; Rey, Yee

$$V(r) = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma\left(\frac{1}{4}\right)^4 r}$$



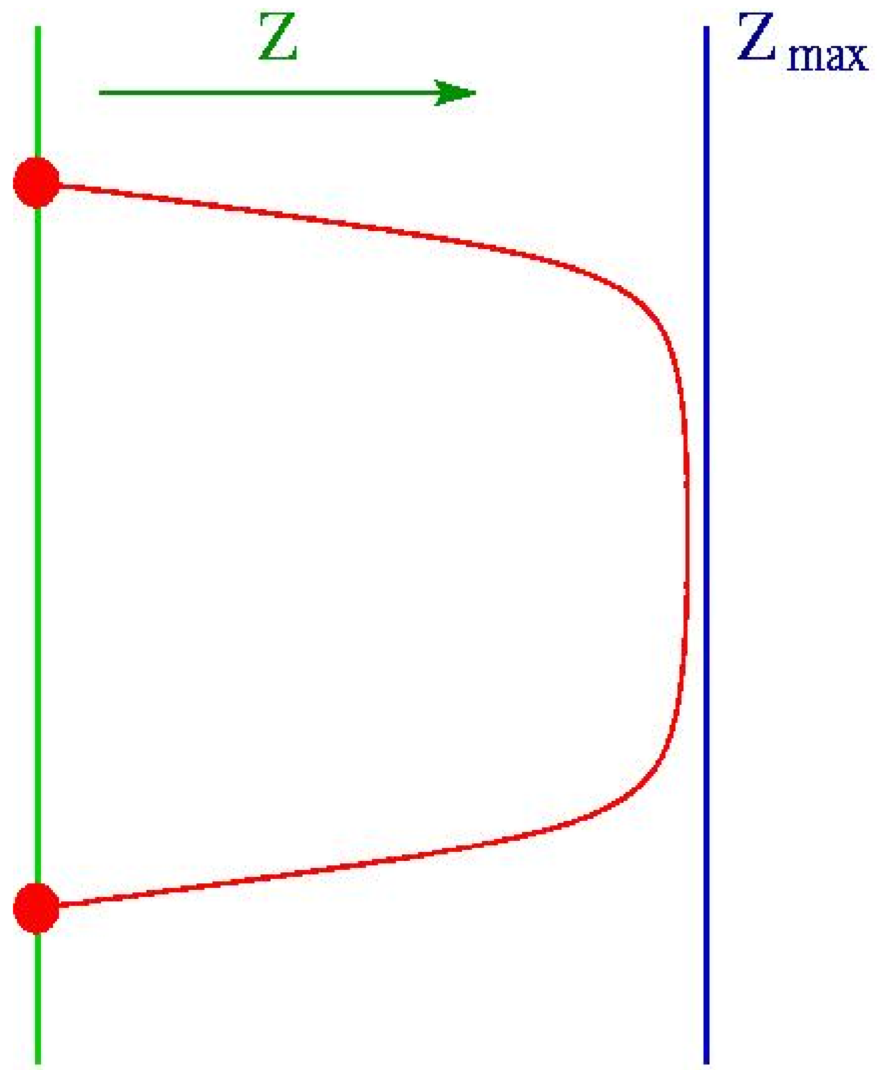
Confining = Fundamental

- The quark anti-quark potential is linear at large distances but nearly Coulombic at small distances.
- The 5-d metric should have a warped form Polyakov

$$ds^2 = \frac{dz^2}{z^2} + a^2(z)\big(-(dx^0)^2 + (dx^i)^2\big)$$

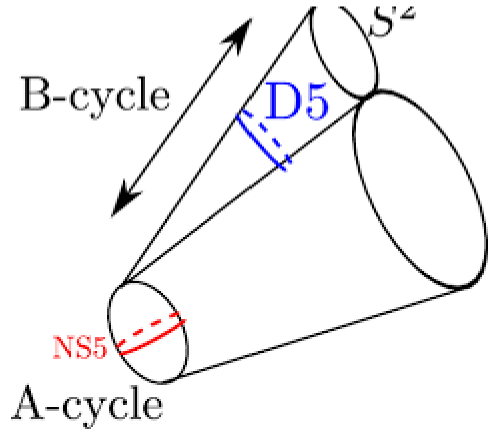
- The space ends at a maximum value of z where the warp factor is finite. Then the confining string tension is

$$\frac{a^2(z_{\max})}{2\pi\alpha'}$$



Confinement and Warped Throat

- To break conformal invariance, change the gauge theory: add to the N D3-branes M D5-branes wrapped over the sphere at the tip of the conifold.
- The 10-d geometry dual to the gauge theory on these branes is the warped deformed conifold (IRK, Strassler)



$$ds_{10}^2 = h^{-1/2}(y)\big(-(dx^0)^2 + (dx^i)^2\big) + h^{1/2}(y)ds_6^2$$

- ds_6^2 is the metric of the deformed conifold, a Calabi-Yau space defined by the following constraint on 4 complex variables:

$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2$$

- The quark anti-quark potential is qualitatively similar to that found in numerical simulations of QCD (graph shows lattice QCD results by G. Bali et al with r_0 ~ 0.5 fm).
- The dual gravity provides a `hyperbolic cow' approximation, i.e. a toy model, for QCD.

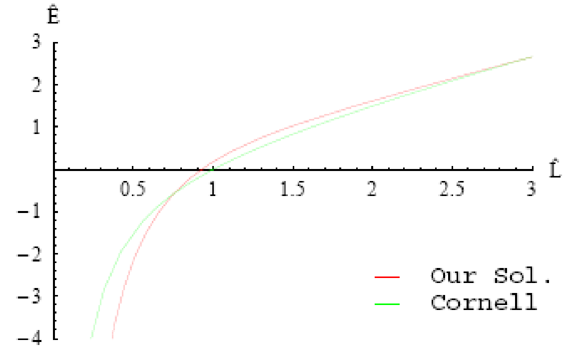
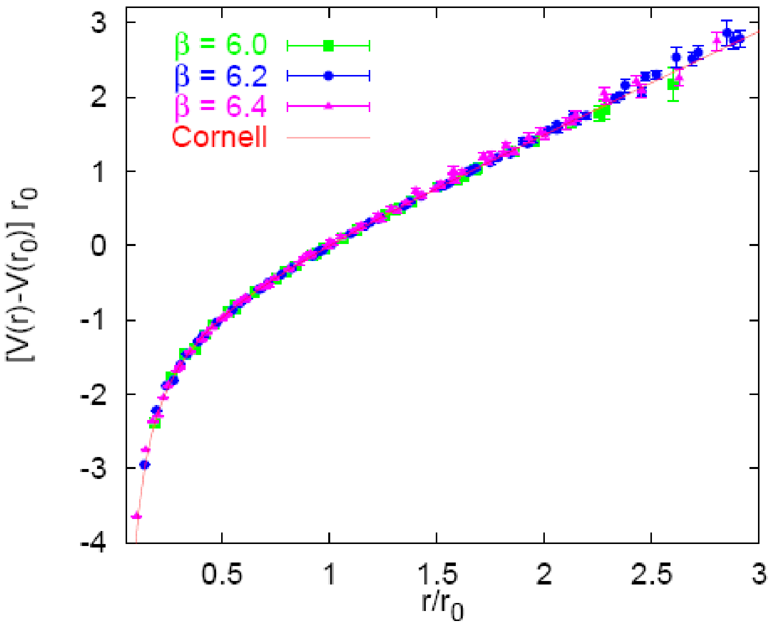


Figure 11: Comparison to the Cornell model



Thank you!