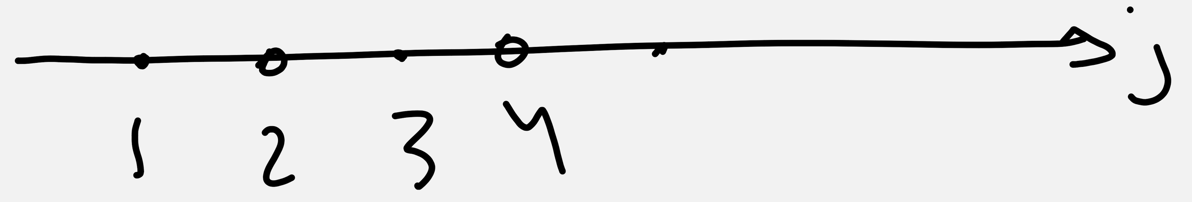
More Lattice Fermions:

Schwinger Model $\to$ QCD $\to$ AdS/CFT $\to$?

$$t = \frac{x}{2a} \; ; \quad a \equiv \text{lattice spacing}$$



$N \equiv$ # of sites is even

$$H_{hop} = -it \sum_j \left(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j \right)$$

Jordan Wigner

$$H_{XY} = t \sum_j \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \right)$$

$$c_\ell = \sum_{j<\ell} (i\sigma^z_j)\, \sigma_\ell^- \; ; \quad c_\ell^\dagger = \prod_{j<\ell} (-i\sigma^z_j)\, \sigma_\ell^+ \; ; \quad \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$$

Massless Dirac $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in 1+1 d

$$S = \int d^2x \; i\bar{\Psi} \gamma^\mu \partial_\mu \Psi \, , \quad \bar{\Psi} = \Psi^\dagger \gamma^0 \; ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} ,$$

$$\gamma^0 = \sigma^3 , \quad \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

$$\gamma_5 = \gamma^0\gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathcal{L} = i\Psi^\dagger \dot{\Psi} + i\Psi^\dagger \sigma^1 \partial_x \Psi,$$

$$\dot{\Psi} = -\sigma^1 \partial_x \Psi,$$

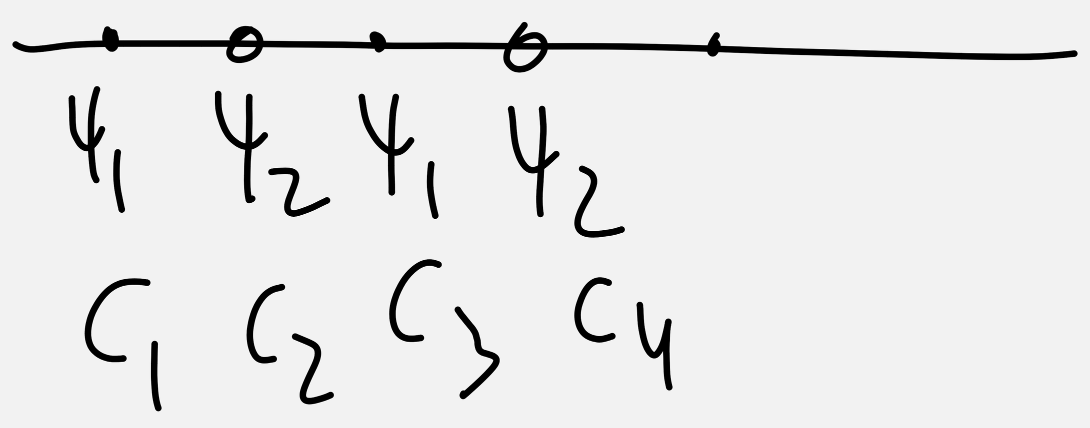
$$H = i\int dx \left(\psi_1^\dagger \partial_x \psi_2 + \psi_2^\dagger \partial_x \psi_1\right)$$

$$H_{mass} = m\int dx\, \Psi^\dagger \sigma_3 \Psi$$

$$= -m\int dx \left(\psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_1\right)$$

$$\to m\sum_j (-1)^j c_j^\dagger c_j;$$

Kogut-Susskind $\quad \psi_1 \to \dfrac{c_{2n-1}}{2\sqrt{a}}, \quad \psi_2 \to \dfrac{c_{2n}}{2\sqrt{a}},$



$$\partial_x \psi_1 = \frac{1}{2\sqrt{a}} \frac{1}{2a} \left(c_3 - c_1\right)$$

Chiral symmetry $\Psi \to e^{i\alpha\gamma_5}\Psi$; $\gamma_5 = \sigma_1$

$\alpha = \frac{\pi}{2}$; $e^{i\frac{\pi}{2}\sigma_1} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\sigma_1 = i\sigma_1$

$\psi_1 \to i\psi_2$; $\psi_2 \to i\psi_1$

$c_j \to c_{j+1}$ (discrete chiral symm) OK for $m=0$ but $m \to -m$ for massive.

$c_j \to c_{j+2}$ (spatial transl)

$m\bar{\Psi}\Psi \to m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$;

Bosonization: map to a periodic scalar ϕ

$\psi_L \to e^{i2\sqrt{\pi}\phi_L}$, $\psi_R \to e^{i2\sqrt{\pi}\phi_R}$,

$\epsilon^{01} = -1$

$\epsilon^{10} = +1$

Massless $\Psi \to S = \int d^2x \left(\frac{1}{2}(\partial\phi)^2\right)$; $\bar{\Psi}\gamma^1\Psi = \frac{\partial_0\phi}{\sqrt{\pi}}$

$\bar{\Psi}\gamma^0\Psi = -\frac{\partial_1\phi}{\sqrt{\pi}}$

$\bar{\Psi}\gamma^\mu\Psi = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi$

$\Uparrow$

$\partial_\mu J^\mu = 0$;

$J_5^\mu = \bar{\Psi}\gamma^5\gamma^\mu\Psi = \frac{1}{\sqrt{\pi}}\partial^\mu\phi$;

$$\partial_\mu J^\mu{}_5 = -\frac{\partial^2\phi}{\sqrt{\pi}} = 0 \text{ by EOM.}$$

$$\bar{\Psi}\Psi \to \sim \# \cos(2\sqrt{\pi}\,\phi)$$

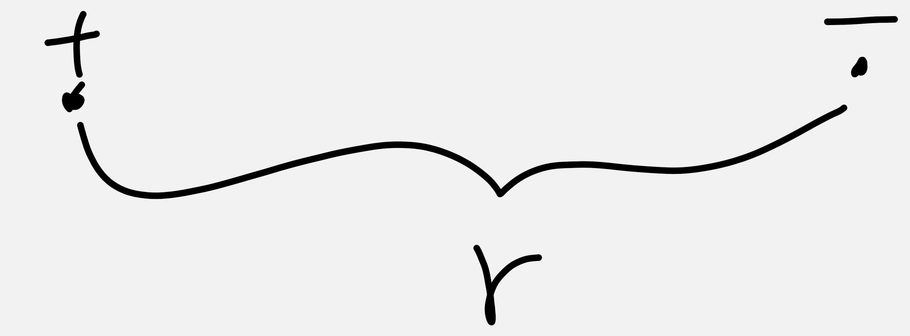
Schwinger (1962) ; A_μ is $U(1)$ gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \epsilon_{\mu\nu} F; \qquad F = F_{01};$$

$$\mathcal{L}_{\text{Schwinger}} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\cancel{\partial} - \cancel{A} - m)\Psi \;;\; \cancel{A} = \gamma^\mu A_\mu;$$

$g \equiv e$ has dims of mass. For $m=0$; bosonization gives free massive boson of $M_S = \frac{g}{\sqrt{\pi}}$,

Dimless coupling g/m ; $g/m \lessgtr 1$;



"capacitor"

$$V(r) = \frac{g^2 r}{2};$$

$$H = \int \frac{g^2}{2} E^2 \, dx := \frac{g^2 r}{2}$$

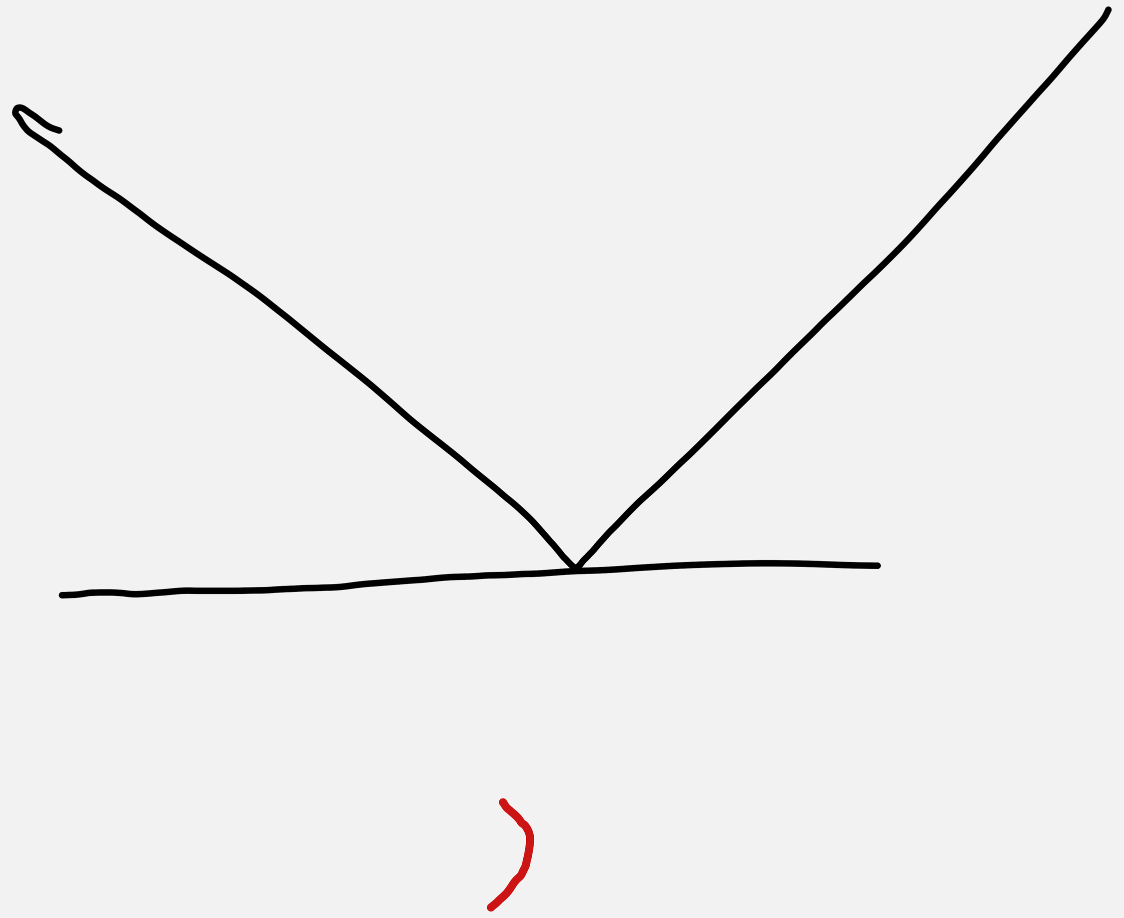
φ eqn of motion ⇒ anomaly eqn!

$$\partial^2 \phi = \frac{F}{\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}} F_{\mu\nu} \epsilon^{\mu\nu}$$

$$\partial_\mu J^\mu{}_5 = \frac{1}{\sqrt{\pi}} \partial^2 \phi;$$

$$\boxed{\partial_\mu J^\mu{}_5 = \frac{1}{2\pi} F_{\mu\nu} \epsilon^{\mu\nu}}$$

Bosonized:



$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2g^2} F^2 - A_\mu \epsilon^{\mu\nu} \partial_\nu \phi / \sqrt{\pi}$$

$$\to \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2g^2} F^2 + \frac{F\phi}{\sqrt{\pi}} \quad \Rightarrow F = -\frac{g^2 \phi}{\sqrt{\pi}}$$

$$\to \frac{1}{2}(\partial_\mu \phi)^2 - \underbrace{\frac{1}{2} \frac{g^2}{\pi}}_{\frac{1}{2} M_S^2} \phi^2$$

Proper mass gap!

Mass gap but fractional charge screened?

$(-\partial_x^2 + m_s^2)A_0 =$

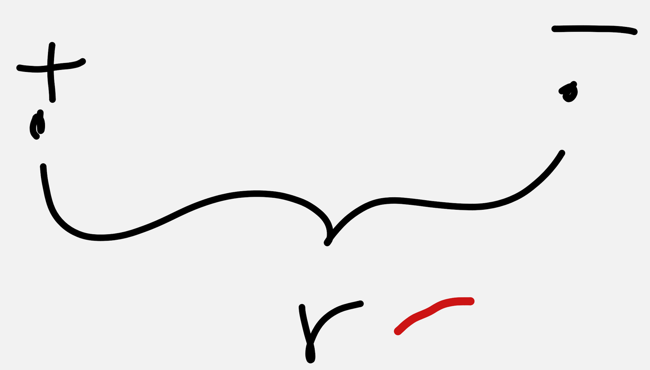
$\rho(x) = g'(\delta(x) - \delta(x+a))$

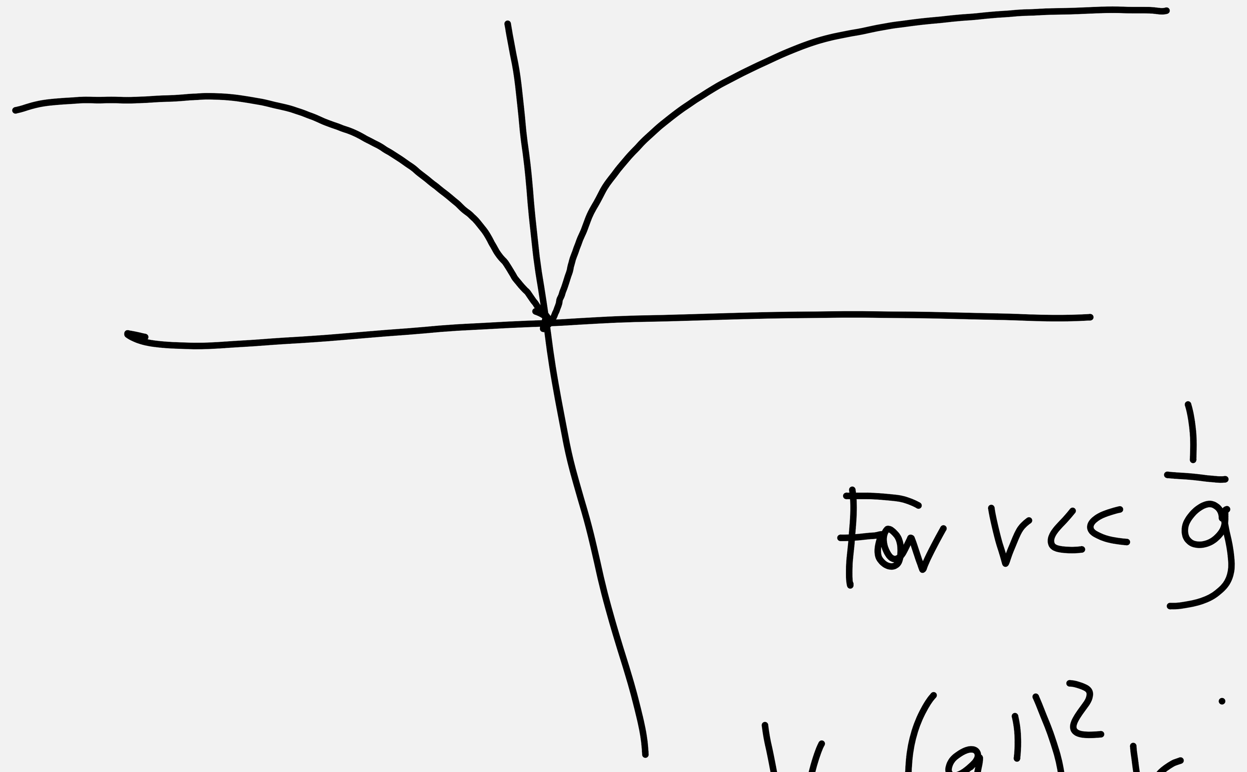
$A_1 = 0$ gauge.

$+\frac{1}{2}$ $\qquad$ $-\frac{1}{2}$

$$\mathcal{L} = \frac{1}{2g^2}F^2 + \frac{1}{2\pi}F\frac{1}{\partial_\mu^2}F \; ; \qquad F = -\partial_1 A_0 \; ; \qquad T_{\text{string}} = 0 \, ;$$

For $m = 0$:





$$\mathcal{L} = \frac{1}{2g^2}(\partial_1 A_0)^2 + \frac{1}{2\pi}A_0^2$$

Put in static charges $\pm\frac{1}{2}$.

$$A_0 \sim \left(e^{-M_s|x|} - e^{-M_s|x-r|}\right)$$

$$V(r) = \frac{(g')^2}{2M_s}\left(1 - e^{-M_s r}\right)$$

For $r \ll \frac{1}{g}$

$V \sim \frac{(g')^2 r}{2}$,

$m \ll g$;

$$\delta \mathcal{L} = -m\Sigma\left(\cos(2\sqrt{\pi}\phi) - 1\right)$$

$$\Sigma = \frac{g e^{\gamma}}{2\pi^{3/2}} = \langle \bar{\Psi}\Psi \rangle_{m=0}$$

$$\frac{1}{2g^2}F^2 + \frac{1}{2\pi} F \frac{1}{\partial^2 + m\Sigma} F$$

$$\frac{1}{g^2} \to \frac{1}{g^2} + \frac{1}{m\Sigma} \quad ; \quad V(r\to\infty) \sim m\Sigma r \, ;$$

Can add θ-term

$$\delta\mathcal{L}_\theta = -\int \frac{\theta}{4\pi} F_{\mu\nu}\epsilon^{\mu\nu} d^2x \,;$$

Introduce $E_\infty = \dfrac{\theta}{2\pi} g \,;$

$N_f = 2 \,;$ Coleman (1976)

$m \to 0 \,;$ gapless coupled to a massive field.

$m \ll g \,;$ $E_{gap} \sim m^{2/3} g^{1/3} \left(\cos\frac{\theta}{2}\right)^{2/3}$

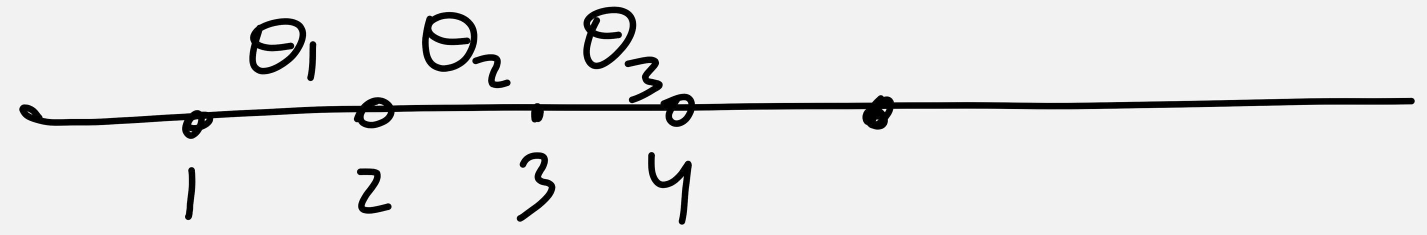
$\theta = \pi \,;$ $E_{gap} \sim e^{-A g^2/m^2}$ (May 2023)

QCD-like.

Lattice Hamiltonian Schwinger Model

Gauss' law

$\partial_x E \sim \rho(x) \sim \psi^\dagger \psi$



$L = -i\frac{\partial}{\partial \theta}$ takes integer values

Banks
Susskind
Kogut

To describe gauge field introduce $\theta \sim \theta + 2\pi$ on each link.

Hamer et al. 1997 ↓ explicit breaking of unit translation

$$H_{BSK} = -\frac{i}{2a}\sum_j \left(c_j^\dagger c_{j+1} e^{i\theta_j} - c_{j+1}^\dagger c_j e^{-i\theta_j} \right) + \frac{g^2 a}{2}\sum_j \left(L_j + \frac{\theta}{2\pi} \right)^2 + m_{lat} \sum_j (-1)^j c_j^\dagger c_j$$

$$L_n - L_{n-1} = c_n^\dagger c_n - \frac{1-(-1)^n}{2},$$

As $g^2 \to \infty$; $L_n = 0$; half-filling!

In XY basis
$$H = -\frac{i}{2a}\sum_j \left(\sigma_j^+ \sigma_{j+1}^- e^{i\theta_j} - \sigma_j^- \sigma_{j+1}^+ e^{-i\theta_j} \right) + m_{lat} \sum (-1)^j \frac{\sigma_j^z}{2} + \frac{g^2 a}{2}\sum_j \left(L_j + \frac{\theta}{2\pi} \right)^2$$

$m_{lat} = m - g^2 a/8$