Quantum Amplification + Linear Response

→ take a quantum system, and faithfully register its properties,

→ classical output with many degrees of freedom.

$$X_{out} = \underline{\underline{G}}\, X_{in} \; ; \; G \geq 1$$

$$[a_{out}, a_{out}^{\dagger}] = \cancel{\underbrace{\sqrt{G}^{2}}_{G} \cdot \underbrace{[a_{in}, a_{in}^{\dagger}]}_{1}}$$

does not obey c.c.r.

Bosonic model

$$\hat{a}_{out} = \sqrt{G}\, \hat{a}_{in}$$

$$\hat{a}_{out}^{\dagger} = \sqrt{G}\, a_{in}^{\dagger}$$

Can salvage this:

Need to add another operator

$$\hat{a}_{out} = \sqrt{G}\,\hat{a}_{in} + \hat{F}$$

$$a_{out}^{\dagger} = \sqrt{G}\,a_{in}^{\dagger} + \hat{F}^{\dagger}$$

$$[a_{out}, a_{out}^{\dagger}] = 1$$

$$= G[a_{in}, a_{in}^{\dagger}] + \underbrace{\sqrt{G}[\hat{a}_{in}, \hat{F}^{\dagger}]}_{0} + \underbrace{\sqrt{G}[a_{in}^{\dagger}, \hat{F}]}_{0} + [\hat{F}, \hat{F}^{\dagger}]$$

assume no correlation between a + F

$$1 = G \cdot 1 + [\hat{F}, \hat{F}^{\dagger}]$$

$$\Rightarrow \boxed{[\hat{F}, \hat{F}^{\dagger}] = 1 - G}$$

How much noise is added?

$$\mathrm{Var}[\hat{a}_{out}] = ? \qquad : \mathrm{Var}(\hat{a}_{in}) = (\Delta a_{in})^2 = \left\langle \frac{a_{in}a_{in}^{+} + a_{in}^{+}a_{in}}{2} \right\rangle - |\langle a_{in} \rangle|^2$$

$$= \left\langle \left(\sqrt{G}\,\hat{a}_{in} + \hat{F}\right)\left(\sqrt{G}\,\hat{a}_{in}^{+} + \hat{F}^{+}\right) \right\rangle - \ldots$$

Simplify

$$\langle F \rangle = 0$$

$$= G\,(\Delta a_{in})^2 + \underbrace{\frac{1}{2}\left\langle \{\hat{F}, \hat{F}^{+}\} \right\rangle}_{\text{added noise.}}$$

$$\frac{1}{2}\left\langle \{\hat{F}, \hat{F}^{+}\} \right\rangle \geq \frac{1}{2}\underline{\left|\left\langle [\hat{F}, \hat{F}^{+}] \right\rangle\right|}$$

$$|u| + |v| \geq |u + v|.$$

$$\therefore \left(\Delta a_{out}\right)^2 \geq G\left(\Delta a_{in}\right)^2 + \frac{1}{2}\left|\langle[\hat{F}, \hat{F}^\dagger]\rangle\right|$$

$$\geq G\left(\Delta a_{in}\right)^2 + |G-1|/2$$

$\Rightarrow$ relative to input, must add at least $\frac{1}{2}$ a photon to the noise.

$G \gg 1$

Simple model to realize a quantum limited amplifier

Add an addition bosonic mode $\hat{b}$ $[\hat{b}, \hat{b}^{+}] = \underline{1}$

Single mode model! $\hat{F} = \sqrt{G-1}\, \hat{b}^{+}$

$\hat{F}^{+} = \sqrt{G-1}\, \hat{b}$

Linear response theory of detectors.

$$\hat{H} = \hat{H}_0 + \Theta(t - t_0)\, \delta H(t)$$

Consider the change in a system operator $\hat{A}$

interaction picture, $\hat{A}_I = e^{iH_0 t} A\, e^{-iH_0 t}$

$$\langle \hat{A}(t) \rangle = \langle \hat{A} \rangle_0 - \frac{i}{\hbar} \int_0^t dt' \langle [\hat{A}_I(t), \delta H(t')] \rangle_0$$

Kubo formula

Illustrate linear response theory

Voltage $V_{in} \longmapsto V_{out}$

$\hat{I}$ output operator. input variable X

$\longrightarrow$ output variable y.

$\hat{F}$ generalized force.

Let $H_{int} = \Sigma \, \underset{\sim}{\hat{X}} \, \underset{\sim}{\hat{F}}$

Apply Kubo formula.

$$\langle \hat{I}(t) \rangle = \langle \hat{I} \rangle_0 + \Sigma \, \lambda \, \langle \hat{X} \rangle_0$$

$$\lambda = -\frac{i}{\hbar} \int^{\infty} dt' \, \langle [\hat{I}(t'), \hat{F}(0)] \rangle_0$$

$$H_{int} = \tilde{\Sigma} \, \hat{y} \, \hat{I} \; ; \qquad \lambda' = -\frac{i}{\hbar} \int_0^{\infty} dt' \, \langle [\hat{F}(t'), \hat{I}(0)] \rangle_0$$

reverse gain

$$\lambda = \mathrm{Im}\,\frac{2}{\hbar}\int_0^{\infty} dt' \langle \hat{I}(t')\,\hat{F}(0)\rangle$$

Write correlators

$$2\int_{-\infty}^{\infty} dt' \langle \hat{I}(t')\,\hat{I}(0)\rangle = S_{II}$$

$$2\int_{-\infty}^{\infty} dt' \langle \hat{F}(t)\,\hat{F}(0)\rangle = S_{FF}$$

$$2\int_{-\infty}^{\infty} dt' \langle I(t')\,F(0)\rangle \leq S_{IF}$$

$$S_{FI}$$

$\langle I \rangle = \langle I_r \rangle_0 + \varepsilon \lambda \langle \hat{x} \rangle$

$\Rightarrow$ Measurement rate

$$\boxed{\Gamma_M = \frac{\varepsilon^2 \lambda^2}{S_{II}}}$$

recall $H_{int} = \varepsilon \hat{x} \cdot \hat{F}$

system variable

$\Rightarrow$ time-dependent phase to quantum system $\sim \frac{\varepsilon}{\hbar} \int_0^t dt' \, F(t')$

$\xrightarrow{\text{averaging}}$ dephasing rate

$$\boxed{\Gamma_\phi = \frac{\varepsilon^2}{\hbar^2} S_{FF}}$$

Inequality. $\Gamma_M / \Gamma_\varphi \leq 1$

Can not extract information faster than I disturb

Why?

$$\frac{\Gamma_M}{\Gamma_\varphi} = \frac{\hbar^2 \lambda^2}{S_{II} S_{FF}}$$

Cauchy-Schwartz

$$S_{II} S_{FF} \geq |S_{IF}|^2$$

$$= \hbar^2 (\lambda - \lambda')^2 + (\mathrm{Re}\, S_{IF})^2$$

$$\frac{\Gamma_M}{\Gamma_\varphi} = \frac{\hbar^2 \lambda^2}{\hbar^2 (\lambda - \lambda')^2 + (\mathrm{Re}\, S_{IF})^2} \leq 1$$

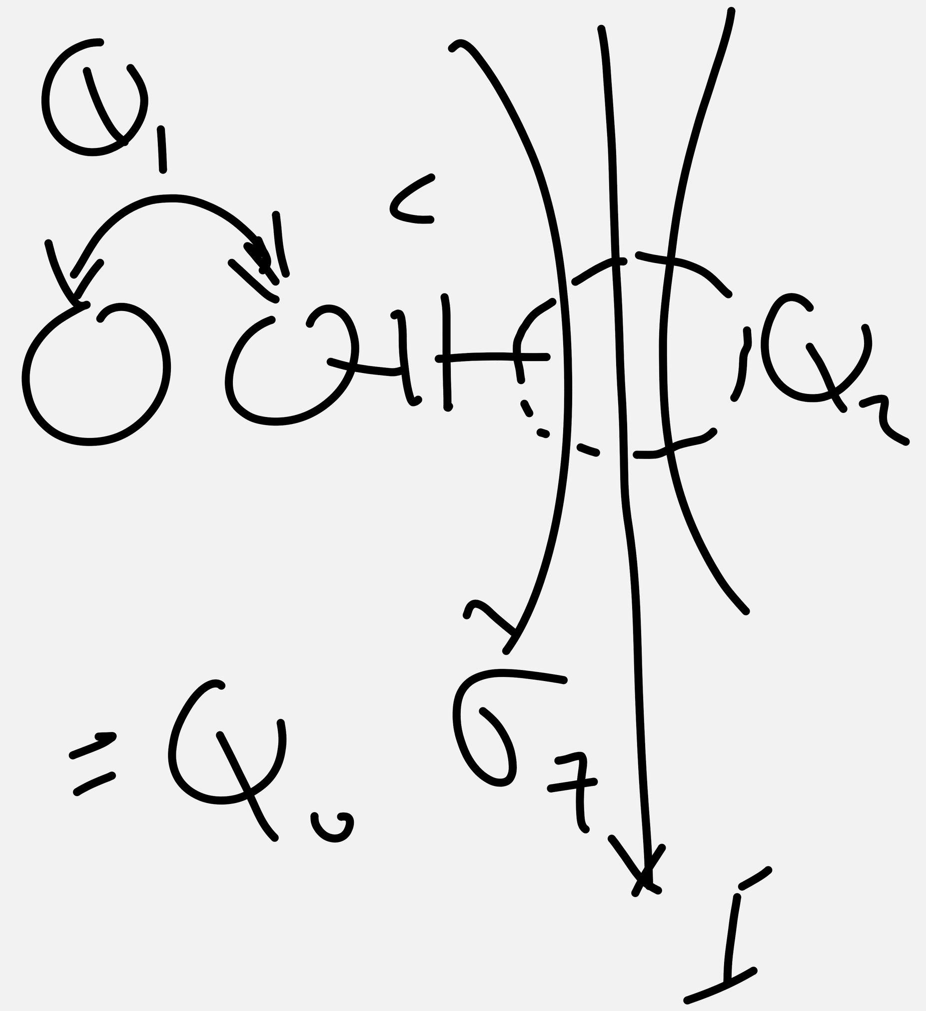
What conditions $=$?

$$\boxed{\mathrm{Re}\, S_{IF} = 0 \;;\; \lambda' = 0}$$

Minimal back action of detector on system

Physical Examples

Quantum double dot + Quantum point contact



$Q_1 = Q_0\,\hat{\sigma}_z$

$$H_{int} = \frac{Q_1 Q_2}{C}$$

$\hat{F} \longleftrightarrow \hat{Q}_2$ on the QPC

$I \longleftrightarrow$ electrical current

Cavity $\longleftrightarrow$ 2 level system

$\hat{F} \longleftrightarrow a^\dagger a$, # of photons

$I \longleftrightarrow$ optical quadrature measurement

Quantum amplifiers.

phase preserving

$$\hat{a}_{out} = \sqrt{G}\,\hat{a}_{in} + \sqrt{G-1}\,b^{\dagger}$$

b is different mode $\leftarrow$, different frequency.

$\longleftrightarrow$ "heterodyne".

Add $\frac{1}{2}$ photon of noise.

phase sensitive

Additional mode

$$\hat{a}_{out} = \sqrt{G}\,a_{in} + e^{i\phi}\sqrt{G-1}\,a_{in}^{\dagger} \rightarrow \hat{X} = \hat{a}e^{-i\phi} + a^{\dagger}e^{i\phi}$$

same frequency $\longrightarrow$ homodyne detection.

- Add power
- Nonlinearity.

parametric amplification.

2 modes of amplification

3 wave mixing

4 wave mixing.

large "pump" w/ some drive frequency ω_p

- Signal (input), $\hat{a}$
- idler , $\hat{b}$.

$\hat{E} = E_J \cos\phi$

$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \sim$

$\hat{\phi} \sim (\hat{a} + \hat{a}^{\dagger})$

3w $\omega_p \longrightarrow \omega_{signal} + \omega_{idler}$

4w $2\omega_p \longmapsto \omega_{signal} + \omega_{idler}$