Adventures in Lattice Fermions

For N sites, have 2^N states

N is even so the lattice is bipartite

$$\{c_j^+, c_{j'}\} = \delta_{ij} \; ; \; \{c_j, c_{j'}\} = 0 ,$$

One site $j=1$; $c_1|0\rangle = 0$, $|1\rangle = c_1^+|0\rangle$; 2-state system

$$c_1 = \sigma_1^- ; \; c_1^+ = \sigma_1^+ \; ; \; \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\sigma^x + i\sigma^y\right)$$

$$q_1 = \frac{1}{2}\left[c_1^+, c_1\right] = \frac{1}{2}\sigma_1^z ,$$

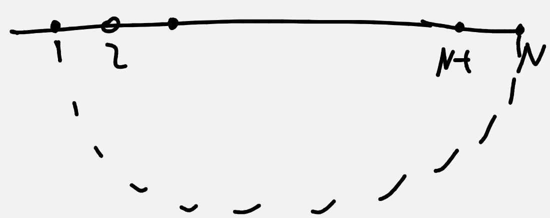
Hopping Hamiltonian:

PBC $\quad c_{N+1} = c_1$;

$$H_t = -t \sum_{j=1}^{N-1} \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) - t \left(c_N^\dagger c_1 + c_1^\dagger c_N \right)$$

$t > 0$; Can flip the sign of t by

$c_j \to (-1)^j c_j$;



Either open or closed

Fourier basis

$$c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k \,,$$

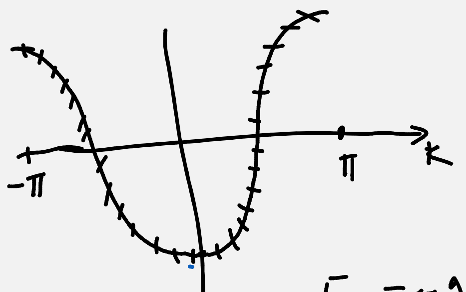
$k = \frac{2\pi n}{N}$, $k \sim k + 2\pi$; $k = -\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots \frac{N}{2}$,

$$H = -2t \sum_k (\cos k)\, c_k^\dagger c_k \, ;$$

Jordan-Wigner trafo.

$$c_j = \left(\prod_{n=1}^{j-1} \sigma_n^z\right) \sigma_j^- \, ; \quad \{c_j, c_k\} = 0 ;$$

$$c_j^\dagger = \left(\prod_{n=1}^{j-1} \sigma_n^z\right) \sigma_j^+ \, ,$$



Ground state

$$\prod_{k \in [-\frac{\pi}{2}, \frac{\pi}{2})} c_k^\dagger |0\rangle \; ; \quad \text{As } N \to \infty$$

$$\Delta k = \frac{2\pi}{N} \Delta n$$

Lieb, Schultz, Mattis

$$E_0 = -2t \frac{N}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos k \, dk = -\frac{2t}{\pi} N \, ,$$

XY model ;

$$H = -\frac{t}{2} \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right) =$$

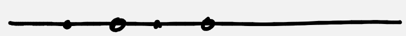
$$= -t \sum_j \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \right)$$

Acts on 2^N states of N qubits; $[\sigma_j^+, \sigma_{j+2}^-] = 0$;

Hubbard chain

$(c_{j\uparrow}, c_{j\downarrow}) \sim c_{j,\sigma}$

$\sigma = (\uparrow, \downarrow)$



$c_{j,\sigma} \to i^{j} c_{j,\sigma}$

change real to Im hopping parameter. $SU(2)_{spin}$ is manifest, also has hidden $SU(2)_{pseudospin}$

Free "staggered" lattice Dirac fermion

$$H = it \sum_j \left(c^{\dagger}_{j+1,\sigma} c_{j,\sigma} - c^{\dagger}_{j,\sigma} c_{j+1,\sigma} \right) = t \sum T_{j,j+1} + U \sum_j n_{j\uparrow} n_{j\downarrow}$$

$$R \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} ; \quad R \in SO(N) ; \quad R^T R = I_{N \times N},$$

"Scars are $SO(N)$ singlets"

Pseudospin generators:

$$SU(2)_{\text{pseudo}} \left\{ \begin{array}{l} \eta = \sum_j c_{j\uparrow} c_{j\downarrow} = \frac{1}{2} \sum_j c_{j\alpha} c_{j\beta} \epsilon_{\alpha\beta} \\ \eta^+ \\ Q = \frac{1}{2} \sum [c_{j\sigma}^+, c_{j\sigma}] \end{array} \right.$$

$$[\eta^+, \eta] = Q$$

In 1d: Yang

$(-1)^{j_x + j_y}$ removed by

$c_{j_x, j_y} \to i^{j_x + j_y} c_{j_x, j_y}$,

Yang's eta-pairing states

$(\eta^+)^k |0\rangle$; $k = 0, \dots, N$

Pseudospin $\frac{N}{2}$ multiplet

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