

# Textbook Quantum Measurements.

- Projective
  - Instantaneous
  - Irreversible.
- } —, Need new concepts.

Stern-Gerlach apparatus.

$$H = -\vec{\mu} \cdot \vec{B}$$

→ Force

$$F_z = \mu_z \frac{\partial B}{\partial z}$$

Q.M.  $\mu_z \rightarrow \pm \mu_B g / 2$   
takes 2 values.

$$\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$$
$$\hat{\mu} = \mu_z \hat{\sigma}_z$$

Measurements.  $\rightarrow$  projection operator.

$$\{ \hat{\Pi}_j \}$$

$j$  deliniates possible # of outcomes.

①  $\hat{\Pi}_i \hat{\Pi}_j = \delta_{ij} \hat{\Pi}_j$  " Given some state  $|\psi\rangle$

" Born rule "

②  $\sum_j \hat{\Pi}_j = \mathbb{1}$

$$P_j = \langle \psi | \hat{\Pi}_j | \psi \rangle$$

③  $\hat{\Pi}_j^2 = \hat{\Pi}_j$

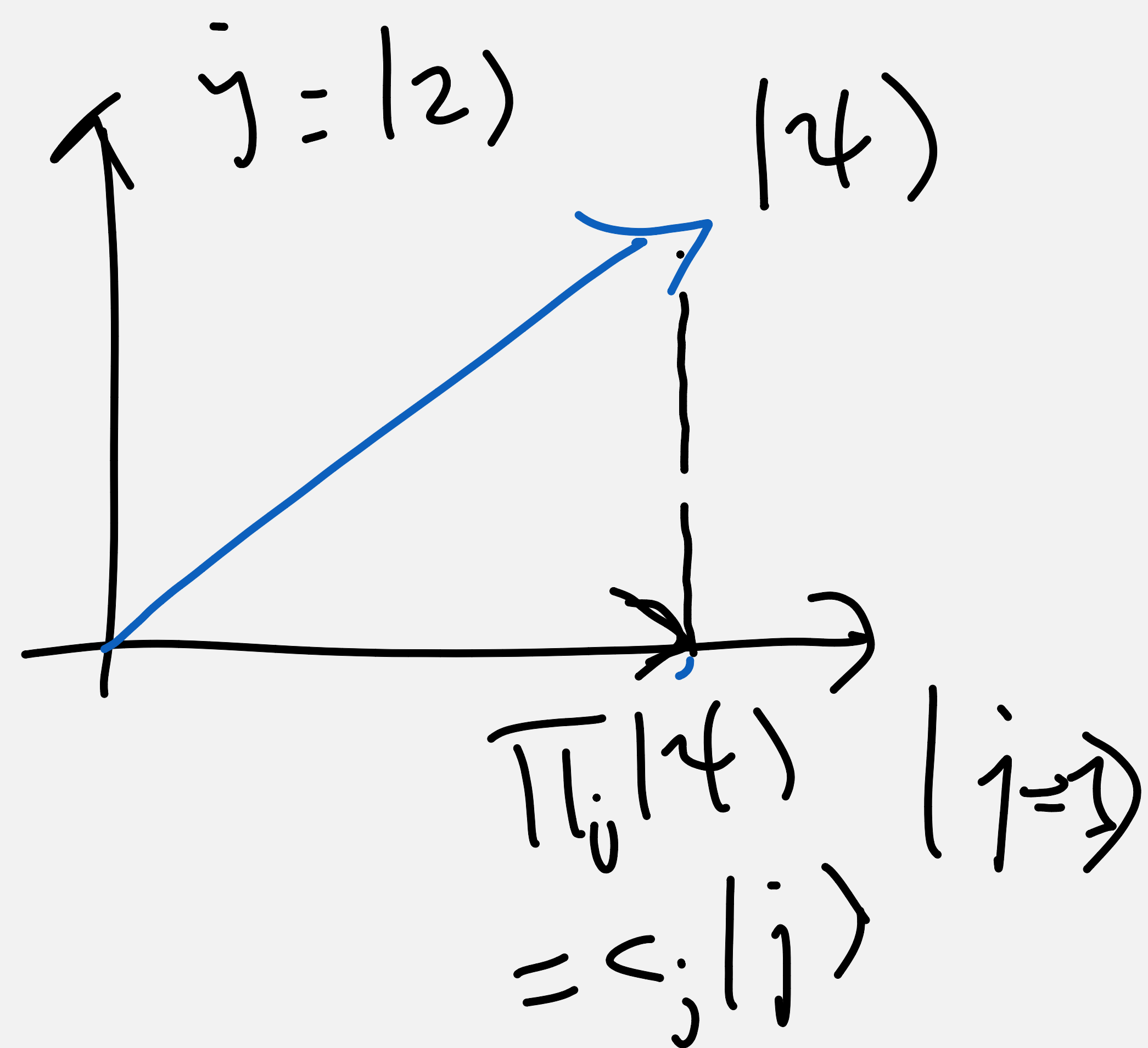
$$E_{\text{ker}} \rightarrow \hat{\Pi}_j \rightarrow |x\rangle\langle x|$$

$$P(x) = \langle \psi | x \rangle \langle x | \psi \rangle = |\psi(x)|^2$$

① + ② + ③  $\hat{\Pi}_j = \underline{|j\rangle\langle j|}$  new basis  $|j\rangle$  orthonormal.

$$\hat{\Pi}_j = |j\rangle\langle j| \quad ; \quad |4\rangle = \sum_k c_k |k\rangle \quad ; \quad \sum_k |c_k|^2 = 1$$

$$\hat{\Pi}_j |4\rangle = c_j |j\rangle$$



collapse of wave function.  
obtain result "j"

$$|4\rangle \xrightarrow{\hat{\Pi}_j} |j\rangle \quad (\text{normalized state})$$

Observable

$$\hat{O} = \sum_j \lambda_j \hat{\Pi}_j$$

$$\hat{O} = \lambda_1 (\hat{\Pi}_1 + \hat{\Pi}_2) + \lambda_B \hat{\Pi}_{\text{rest}}$$

$$P_{\text{res}}(j) = \langle 4 | \hat{\Pi}_j | 4 \rangle = |\langle j | 4 \rangle|^2$$

$$\begin{aligned}
 (\overline{\Pi}_1 + \overline{\Pi}_2 | \psi \rangle &= (\overline{\Pi}_1 + \overline{\Pi}_2) \sum_j c_j |j\rangle \\
 &= \frac{c_1 |1\rangle + c_2 |2\rangle}{\sqrt{|c_1|^2 + |c_2|^2}}
 \end{aligned}$$

$$\hat{\rho} = \sum_j x_j \overline{\Pi}_j$$

$$\langle \psi | \hat{\rho} | \psi \rangle = \sum_j x_j \text{Prob}(j)$$

$$\langle x \rangle = (-d) \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}$$

divergent

distance between the beams

$$d = (n_o - n_e) T$$

↑ ↑ ↑  
 index of refraction difference. thickness.

before crystal

$$|\underline{\psi}\rangle = |\phi\rangle \otimes |\psi\rangle$$

↑ ↑  
 polarization transverse profile

$$|\phi\rangle = a|H\rangle + b|V\rangle$$

$$|\psi\rangle = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-\frac{x^2}{4\sigma^2}}$$

before screen

after crystal

$$\langle x | \underline{\psi}' \rangle = a |H\rangle \phi(x) + b |V\rangle \phi(x+d)$$

joint state Entangled

Q2

what is the probability (density) of finding the photon at position  $x$ ?

$$\begin{aligned} \text{prob}(x) dx &= \langle \Psi | \Pi_x | \Psi \rangle \\ &= |a|^2 \underbrace{|\psi(x)|^2}_{|\psi_u(x)|^2} + |b|^2 \underbrace{|\psi(x+d)|^2}_{|\psi_v(x)|^2} \end{aligned} \quad \leftarrow \text{Law of total probability.}$$

Abstract away the transverse degree of freedom

$$p(x) = (a^* \langle H + b^* \langle V |) \Omega_x^\dagger \Omega_x (a | H \rangle + b | V \rangle)$$

$$\Omega_x = \begin{pmatrix} \phi_H(x) & 0 \\ 0 & \phi_V(x) \end{pmatrix} = \begin{pmatrix} \phi(x) & 0 \\ 0 & \phi(x+d) \end{pmatrix} \quad \text{in } |H\rangle, |V\rangle \text{ basis.}$$

polarization rotation:

$$|H'\rangle = \frac{\Omega_x |V\rangle}{\|\Omega_x |V\rangle\|}$$

Simple example of  
General rule  
(formalism).



→ Kraus operator:

$$\{\Omega_j\}; \quad j=1, \dots, M.$$

obeys  $\sum_j \Omega_j^\dagger \Omega_j = \mathbb{1}$

$j$  denotes outcome.

Note  $\Omega_j \Omega_k \neq \delta_{jk} \Omega_j$

$\Omega_j^\dagger \Omega_j$  is a positive operator  
 $\langle \psi | \Omega_j^\dagger \Omega_j | \psi \rangle$

$$p_{\text{prob}}(j) = \langle \psi | \Omega_j^\dagger \Omega_j | \psi \rangle$$

Conditional state

$$|\psi'\rangle_j = \frac{\Omega_j |\psi\rangle}{\|\Omega_j |\psi\rangle\|}$$

Generalizes the Born rule

$$|4\rangle \longrightarrow \rho = |4\rangle\langle 4|$$

$$P_j = \text{prob}(j) = \text{Tr}[\Omega_j \rho \Omega_j^\dagger]$$

$$f_{(j)}' = \frac{\Omega_j \rho \Omega_j^\dagger}{P_j}$$


Bayes rule

diagonal matrix elements

$$\langle l | f_{(j)}' | l \rangle = \frac{P(j|l) p_{ll}}{P_j}$$

$P(j|l)$  is just the conditional probability of finding result  $j$ , given state is in  $l$ .

of diagonal matrix elements.

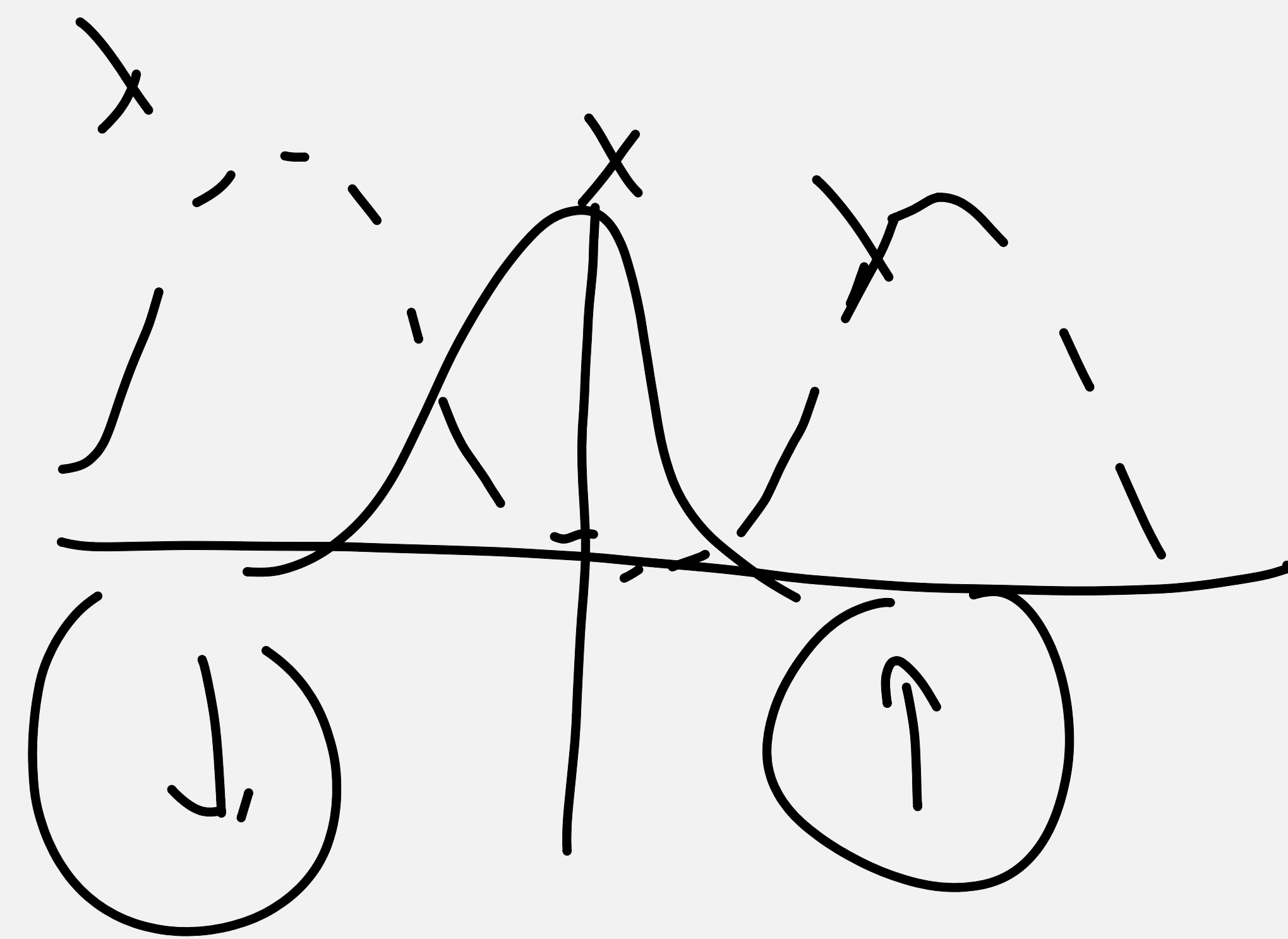
$$(f^{ij})_{e,e'} = \frac{\sqrt{p(j|e)} \sqrt{p(j|e')}}{P_j} f_{e,e'}$$


# Von Neumann

meter

free particle + spin  $\frac{1}{2}$

system



$$H_{SM} = g \delta(t) \hat{\sigma}_z \otimes \hat{p}$$

$$U_{SM} = \hat{T} e^{-i \int dt H_{SM}}$$

$$= e^{-\frac{i g}{\hbar} \hat{\sigma}_z \otimes \hat{p}}$$

$$|\uparrow\rangle \text{ or } |\downarrow\rangle$$

$$\sigma_z \rightarrow \pm 1$$

$$\langle x | e^{\pm \frac{i g}{\hbar} \hat{p}} | \psi \rangle = \psi(x \pm g)$$

poke a hole at position  $x$ .

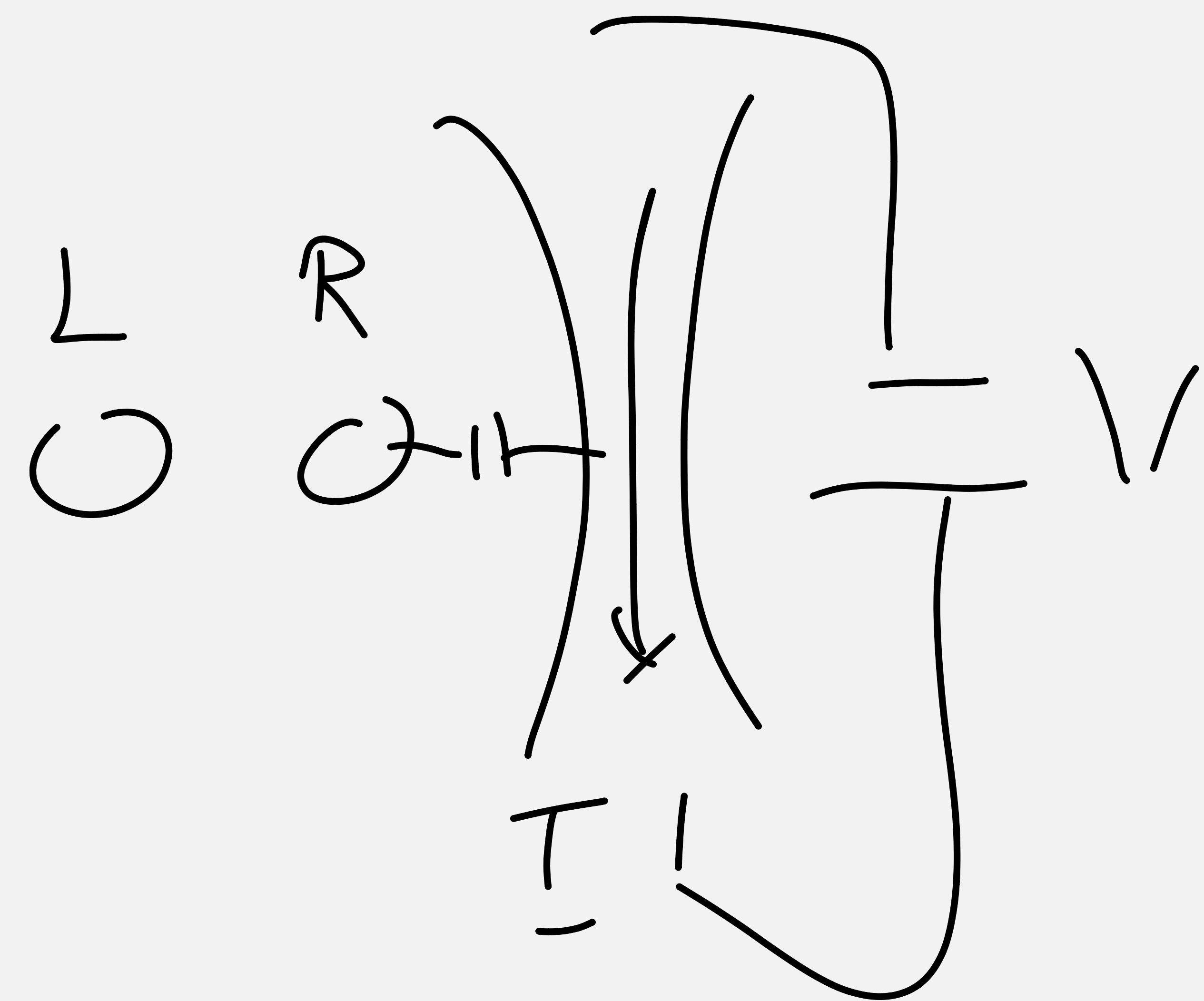
detection  $\rightarrow \Pi_x | \psi \rangle$

Question: Suppose the photon passes.  
What is polarization?

$$|\psi''\rangle = \left( c_v(x) |v\rangle + c_H(x) |H\rangle \right) |x\rangle$$

$$|\psi''\rangle = \frac{a \phi(x) |H\rangle + b \phi(x+d) |v\rangle}{\sqrt{|a|^2 |\phi(x)|^2 + |b|^2 |\phi(x+d)|^2}}$$

Quantum Dot



// Quantum point contact.

Depending on dot position

$$\bar{I} = \bar{I}_L, \bar{I}_R$$

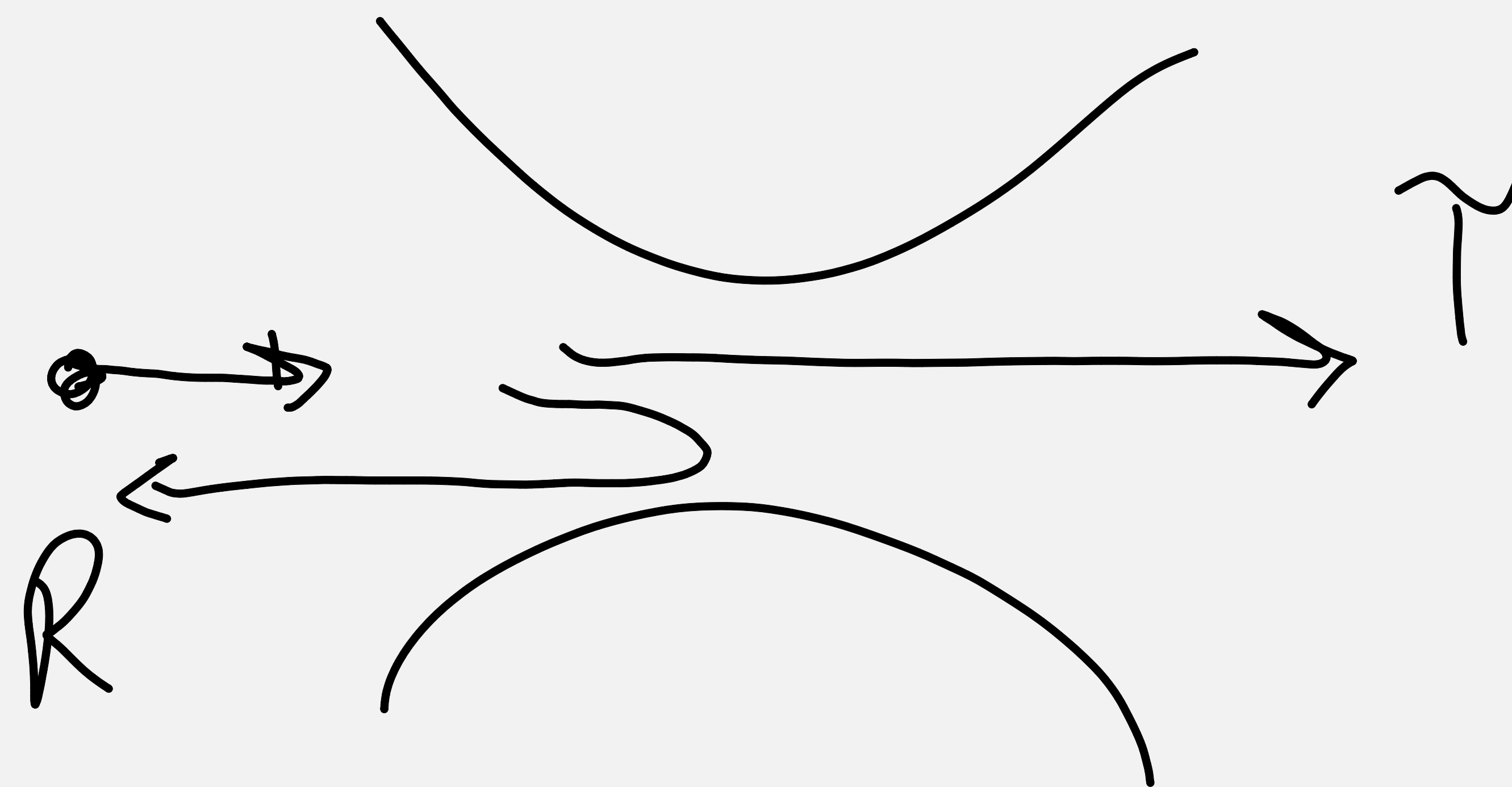
$$\bar{I}_L - \bar{I}_R \ll \frac{\bar{I}_L + \bar{I}_R}{2}$$

Assume weakly responding

Landauer  
- Büttiker

limit

$$\text{conductance} = \frac{e^2}{h} \approx 1$$



$$I(t) = I_0 + \xi$$

Apply voltage  $V$ :  $\tau_0 \sim \frac{h}{eV} \ll$  system time scales.

$$\int dt \langle \delta I(t) \delta I(0) \rangle = S_I \quad ; \quad \text{zero frequency noise power.}$$

Fermions  $S_I = (eV) \frac{2e^2}{h} \tau (1 - \tau)$

$$I = G \cdot V \quad ; \quad G = \frac{e^2}{h} \tau_{L,R}$$

Consider  $Q = \int_0^T dt I(t) =$  random variable

Average

$$\langle Q_L - Q_R \rangle = T (\bar{I}_L - \bar{I}_R)$$

$$\langle \delta Q^2 \rangle = \int dt_1 dt_2 \langle \delta I(t_1) \delta I(t_2) \rangle$$

$$\stackrel{\circ}{=} T \sum_I \quad (\text{white noise approx.})$$

want to let  $T$  be

big enough, so that

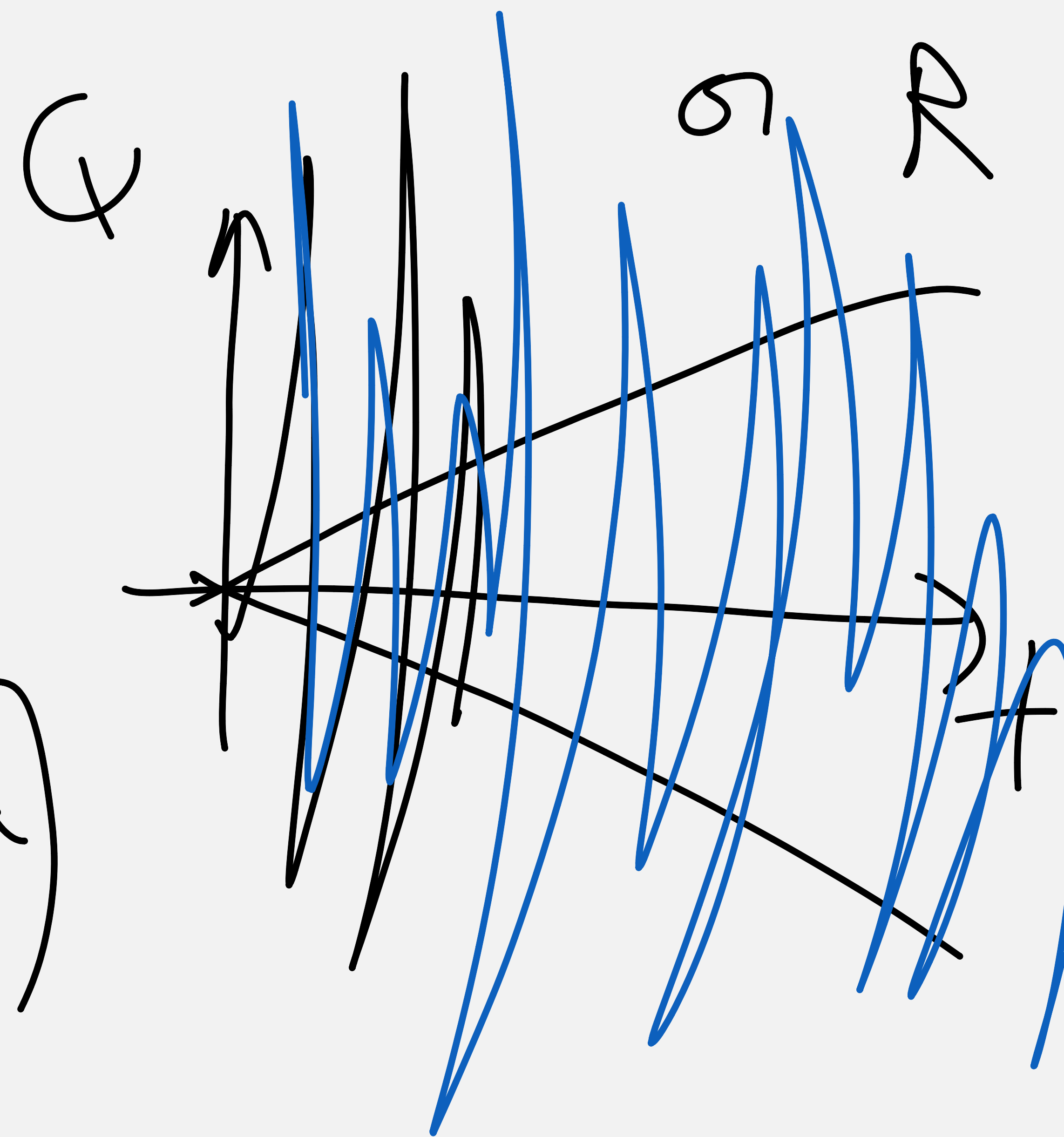
$$\bar{T} = T_M$$

$$\langle Q_L - Q_R \rangle \geq \sqrt{\langle \delta Q^2 \rangle}$$

$$T_M^2 (\bar{I}_L - \bar{I}_R)^2 \geq \sum_I T_M$$

Hypothesis

electron is  $L$

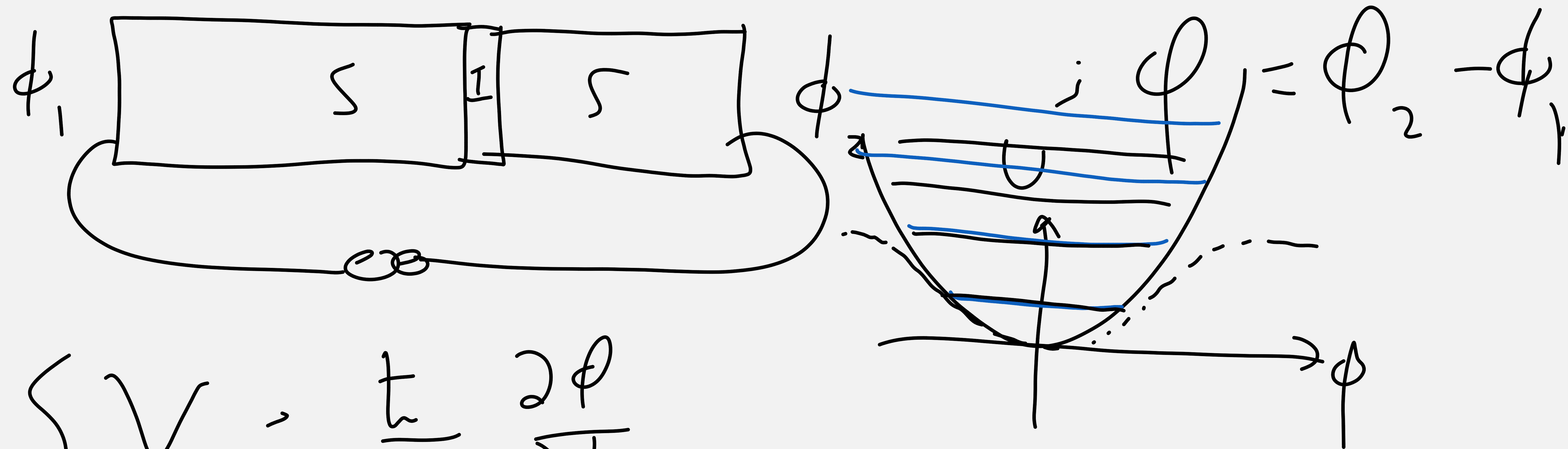




$\Rightarrow$  New time scale!

$$T_M \sim \frac{\sum_I}{(\bar{I}_2 - \bar{I}_a)^2} \quad \rightarrow \text{new ingredient in formalism.}$$

Josephson junction circuit qubits.



Josephson  
Eqns.

$$\begin{cases} V = \frac{\hbar}{2e} \frac{\partial \phi}{\partial t} \\ I = I_c \sin \phi \end{cases}$$

$$\text{Energy stored} = \int I V dt = -I_c \frac{\hbar}{2e} \cos \phi$$

Hamiltonian

2-level approx.

$$H = \hbar \omega_c a^\dagger a + \hbar \omega_q \frac{\hat{\sigma}_z}{2} + \hbar g (a^\dagger \sigma_- + a \sigma_+)$$

Jaynes-Cummings.

Dispersive limit  $\Delta = \omega_q - \omega_c$ ; detuning.

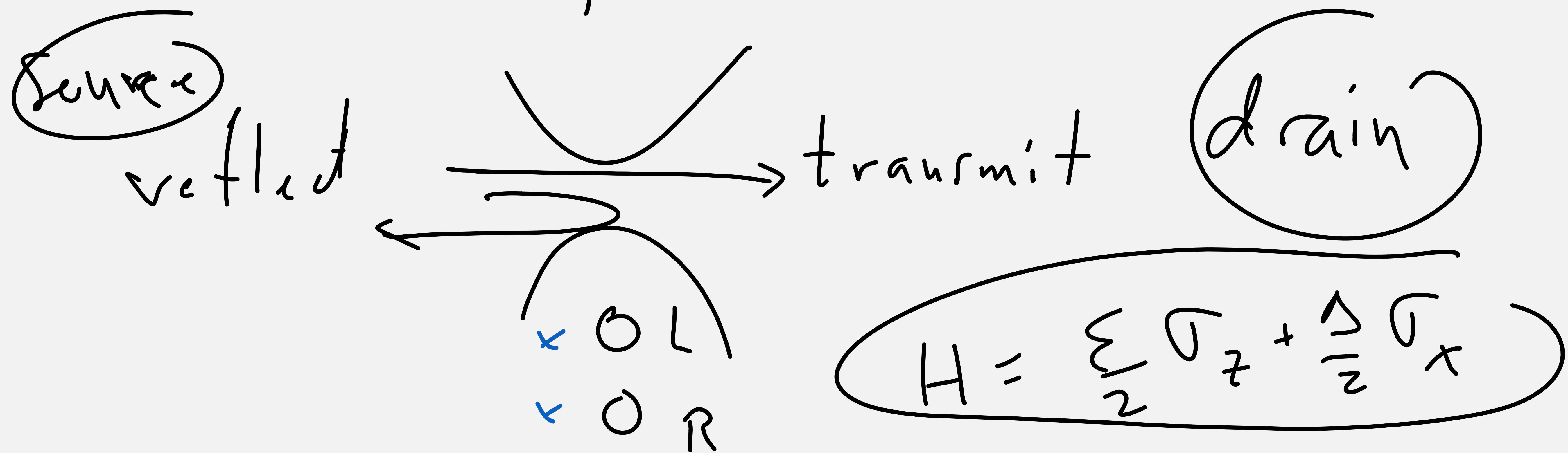
$$\Delta \gg g$$

$$H \approx \hbar \left( \omega_c + \frac{g^2}{\Delta} \sigma_z \right) \left( a^\dagger a + \frac{1}{2} \right) + \frac{1}{2} \hbar \omega_q \sigma_z$$

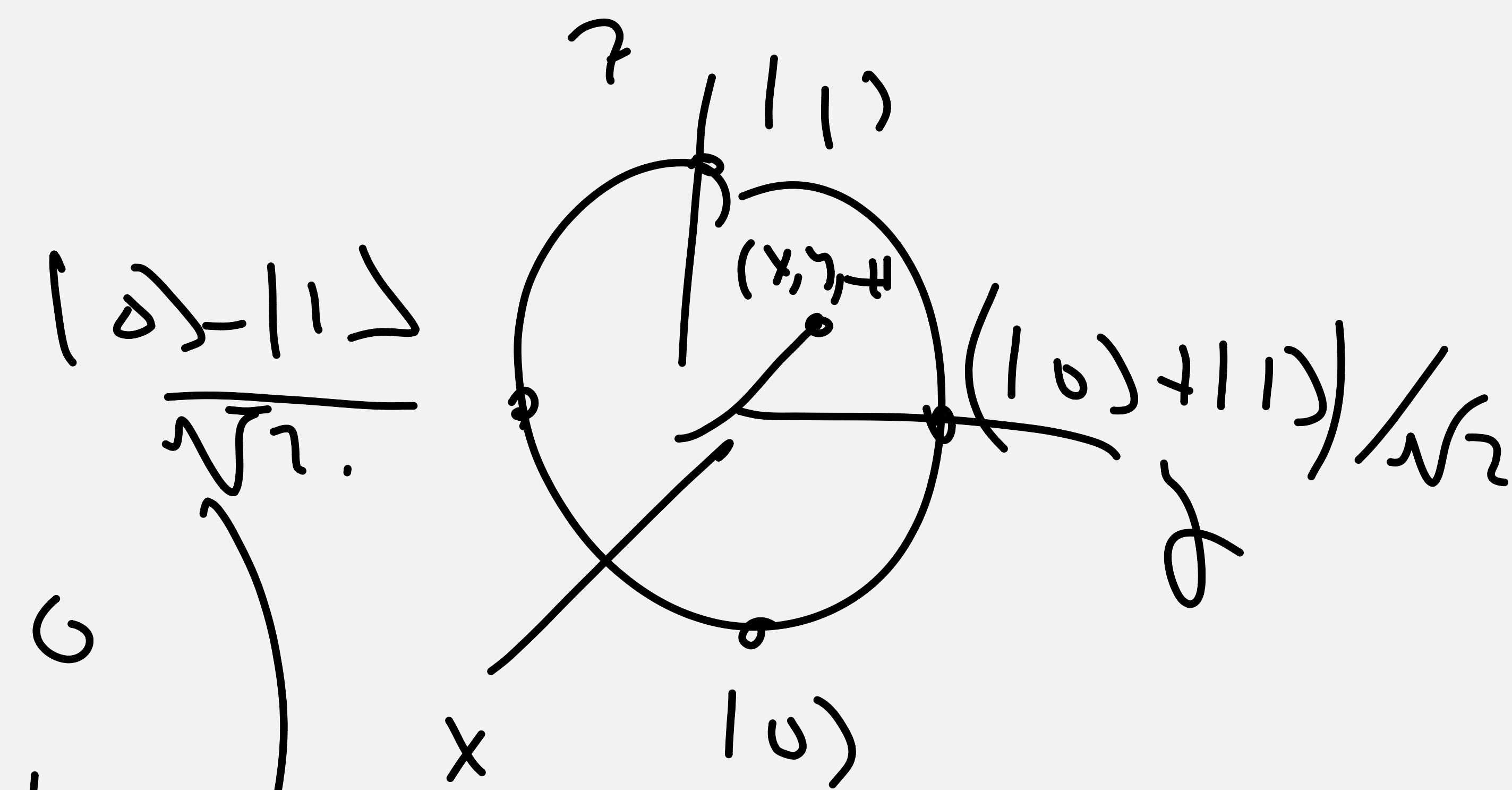
$$\omega_c \longrightarrow \omega_c + \frac{g^2}{\Delta} \sigma_z$$

Very similar to QPC readout.

Build up description from elementary events.



$(x, y, z)$  Bloch coordinates



Kraus operators

$$\mathcal{R}_S = \begin{pmatrix} r_L & 0 \\ 0 & r_R \end{pmatrix}; \quad \mathcal{R}_D = \begin{pmatrix} t_L & 0 \\ 0 & t_R \end{pmatrix}$$

Equations of motion

$$\dot{x} = -\frac{x}{2\tau_m} - \frac{xz}{\sqrt{\tau_m}} - \epsilon y$$

$$\dot{y} = -\frac{y}{2\tau_m} - \frac{yz}{\sqrt{\tau_m}} + \epsilon x - \Delta z$$

$$\dot{z} = \frac{1-z^2}{\sqrt{\tau_m}} + \epsilon y$$

Ito interpretation.

$$\mathcal{F} = \frac{I - I_0}{I - I_a} = z(t) + \sqrt{\tau_m}$$

$$\langle z(t) | z(0) \rangle = S(t)$$

Stochastic Path integral.

paths in Hilbert space.

$$(x, y, z) \rightarrow \vec{q}$$

$$(\vec{q}_0, \vec{q}_1, \vec{q}_2, \dots, \vec{q}_n)$$

$$(r_0, r_1, r_2, \dots, r_n)$$

$$\mathcal{P}(\{\vec{q}\}, \{r\} | t, \vec{q}_i, \vec{q}_f) = \int \mathcal{D}p e^{\mathcal{S}}$$

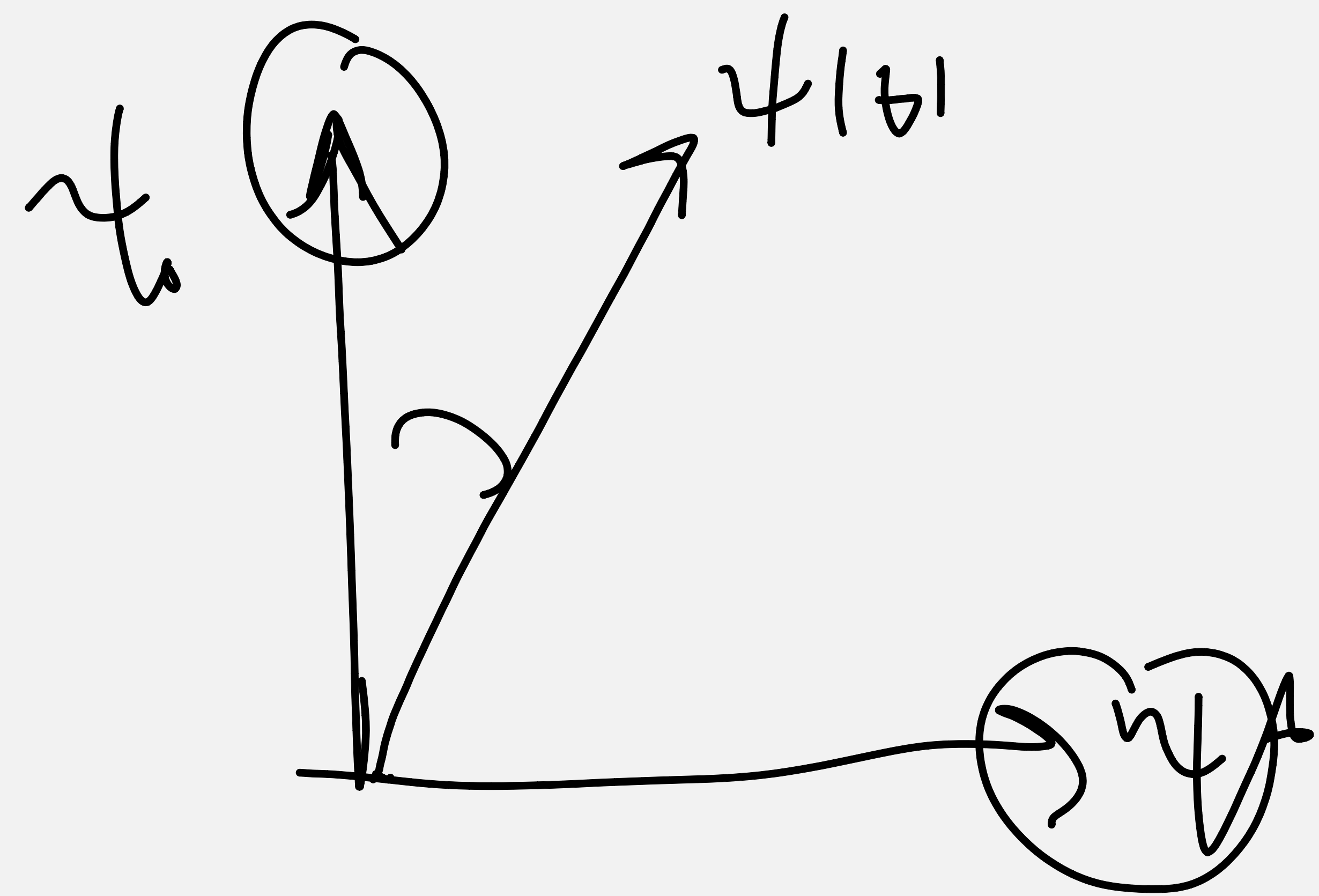
$$\mathcal{S} = \int_0^t dt' \left[ -i \vec{p} \cdot (\dot{\vec{q}} - \mathcal{L}(\vec{q}, \vec{r})) + \mathcal{F}(\vec{q}, \vec{r}) \right]$$

$\mathcal{L}, \mathcal{F}$   
 encode backaction  
 + probability  
 density.

Quantum Zeno effect  $\rightarrow$  Many fast measurements freeze state.

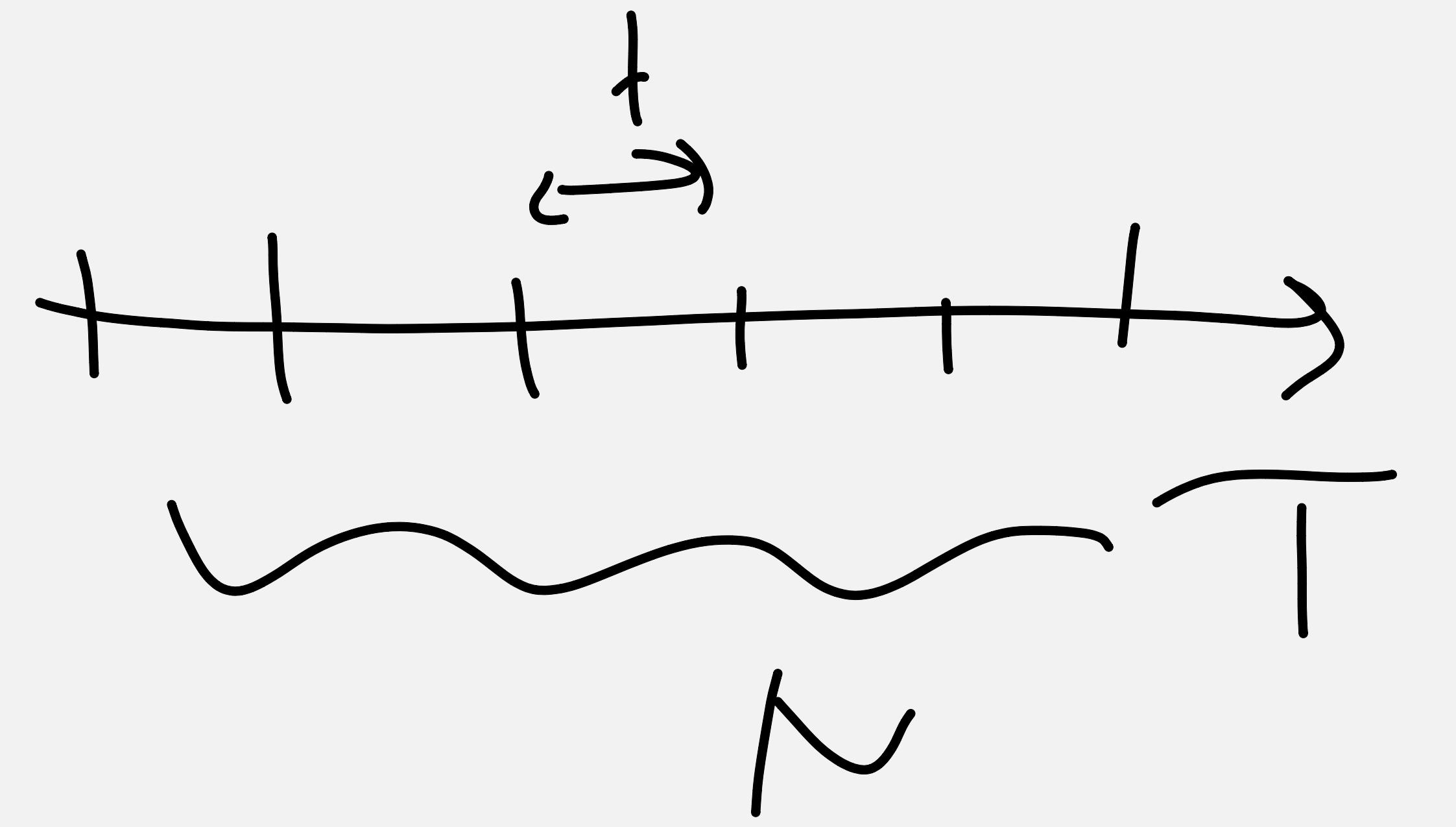
$$|\psi_t\rangle = e^{-iHt} |\psi_0\rangle$$

$$\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$$



$$P_Z = \left| \langle \psi_0 | e^{-iHt} | \psi_0 \rangle \right|^2$$

$$\approx 1 - t^2 \frac{(\Delta H)^2}{\hbar^2}$$



$$P_{Z,N} = \left( 1 - \frac{T^2 (\Delta H)^2}{\hbar^2 N} \right)^N \approx \exp\left[ -\frac{T^2 (\Delta H)^2}{\hbar^2 N} \right] \rightarrow 1$$

$\Delta H^2 = \text{var}[H]_{\psi_0}$