Quantum Field Theory on the Lattice

Z. Fodor

Two sets of lattice field theory talks

Michael Creutz: three talks
Zoltan Fodor: four talks

"computational details ... might be better for Zoltan to cover, i.e. things like hybrid monte carlo, the hadron spectrum ... g-2" and QCD thermodynamics.

- Scalar theory, Higgs bound & Monte Carlo
- QCD and hadron spectrum (Wilson)
- QCD thermodynamics (staggered & overlap)
- g-2 of the muon (staggered)

Outline

1. Yang–Mills & (Fermions)

2. Hadron spectrum

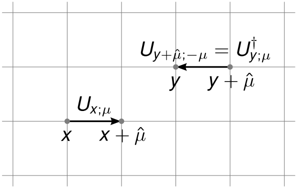
3. Isospin splitting

Yang–Mills theories on the lattice

Regularization has to maintain lattice version of gauge invariance.

Gauge fields $\longrightarrow$ on links connecting neighboring sites.

- Continuum: A_μ, .
- Lattice: $U_\mu = e^{\mathrm{i}agA_\mu}$, elements of group SU(3) itself.



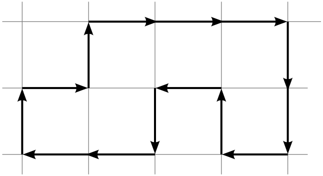
$$U_{x+\hat\mu;-\mu} = U^{-1}_{x;\mu} = U^\dagger_{x;\mu}$$

Lattice gauge transformation:

$$U'_{x;\mu} = G_x U_{x;\mu} G^\dagger_{x+\hat\mu}$$
$$\psi'_x = G_x \psi_x$$
$$\overline{\psi}'_x = \overline{\psi}_x G^\dagger_x$$

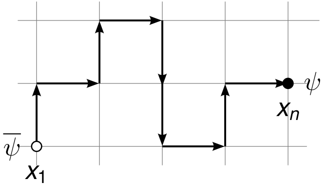
Gauge invariant quantities on the lattice

- Gluon loops



$$\mathrm{Tr}\left[U_{x_1;\mu}\, U_{x_1+\hat{\mu};\nu} \cdots U_{x_1-\hat{\epsilon};\epsilon}\right]$$

- Gluon lines connecting q and $\overline{q}$



$$\overline{\psi}_{x_1}\, U_{x_1;\mu}\, U_{x_1+\hat{\mu};\nu} \cdots U_{x_n-\hat{\epsilon};\epsilon} \psi_{x_n}$$

Gauge action

Continuum gauge action:

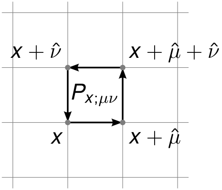
$$S_{\mathrm{g}}^{\mathrm{cont.}} = -\int \mathrm{d}^4 x \ \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$$

Simplest gauge invariant lattice action: Wilson action

$$S_{\mathrm{g}}^{\mathrm{Wilson}} = \beta \sum_{\substack{x \\ \nu<\mu}} \left(1 - \frac{1}{3} \mathrm{Re}\left[P_{x;\mu\nu}\right]\right), \quad \beta = \frac{6}{g^2}, \quad S_g^{latt.} = S_g^{cont} + O(a^2),$$

where $P_{x;\mu\nu}$ is the plaquette:

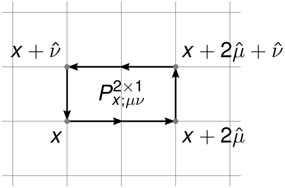
$$P_{x;\mu\nu} = \mathrm{Tr}\left[U_{x;\mu}\, U_{x+\hat{\mu};\nu}\, U^\dagger_{x+\hat{\nu};\mu}\, U^\dagger_{x;\nu}\right]$$



Gauge action – Symanzik improvement

Add 2×1 gluon loops to Wilson action:

$$S_{\text{g}}^{\text{Symanzik}} = \beta \sum_{\substack{x \\ \nu<\mu}} \left\{ 1 - \frac{1}{3} \left(c_0 \, \mathrm{Re}[P_{x;\mu\nu}] + c_1 \, \mathrm{Re}\left[P^{2\times 1}_{x;\mu\nu}\right] + c_1 \, \mathrm{Re}\left[P^{2\times 1}_{x;\nu\mu}\right] \right) \right\}$$



Consistency condition: $c_0 + 8c_1 = 1$.

$c_1 = -\dfrac{1}{12}$ gives tree level improvement $\Longrightarrow S_g^{latt.} = S_g^{cont.} + O(a^4)$

Fermion doubling

Continuum fermion action

$$S_{\mathrm{f}} = \int d^4x\, \overline{\psi}(\gamma^\mu \partial_\mu + m)\psi.$$

Naively discretized:

$$S_{\mathrm{f}}^{\mathrm{naive}} = a^4 \sum_x \left[\overline{\psi}_x \sum_{\mu=1}^{4} \gamma_\mu \frac{\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}}{2a} + m\, \overline{\psi}_x \psi_x \right]$$

Inverse propagator:

$$G_{\mathrm{naive}}^{-1}(p) = \mathrm{i}\gamma_\mu \frac{\sin p_\mu a}{a} + m.$$

Extra zeros at $p_\mu = 0, \pm\frac{\pi}{a}$ $\implies$ 16 zeros in 1st Brillouin zone.
In d dimensions 2^d fermions instead of 1 $\implies$ fermion doubling.
Wilson, staggered (domain wall, overlap) solves it “somehow”

List of most common fermionic actions

(Fermions will be discussed in detail by Michael Creutz)

From the cheapest to the most expensive ones:

• staggered: computationally the least demanding
"rooting" because $N_f = 4$ (use: thermodynamics, muon's g-2)

• Wilson: about 4-10 times more expensive than staggered
chiral symmetry is explicitely broken at a$>$0 (use: hadron spectrum, g-2)

• domain wall: about 20-50 times more expensive than Wilson
(use: for g-2 of the muon)

• overlap: similar to domain wall but even more expensive
most elegant (use: thermodynamics, g-2)

FLAG review of lattice results Colangelo et al. Eur.Phys.J. C71 (2011) 1695

Collaboration	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	$m_{ud,\overline{\mathrm{MS}}(2\mathrm{GeV})}$	$m_{s,\overline{\mathrm{MS}}(2\mathrm{GeV})}$
PACS-CS 10	P	★	■	■	★	*a*	2.78(27)	86.7(2.3)
MILC 10A	C	●	★	★	●	–	3.19(4)(5)(16)	–
HPQCD 10	A	●	★	★	★	–	3.39(6)*	92.2(1.3)
BMW 10AB	P	★	★	★	★	*b*	3.469(47)(48)	95.5(1.1)(1.5)
RBC/UKQCD	P	●	●	★	★	*c*	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum et al. 10	P	●	■	●	★	–	3.44(12)(22)	97.6(2.9)(5.5)

Importance sampling with fermions

Fermions can integrated out: determinant of the fermion matrix

$$Z= \int \prod_{n,\mu} [dU_\mu(n)] e^{-S_g} \det(M[U])$$

again: we do not take into account all possible gauge configurations each of them is generated with a probability $\propto$ its weight

Metropolis algorithm is the easiest importance sampling:
(all other algorithms are based on importance sampling)

$P(U \to U') = \min \left[1, \exp(-\Delta S_g) \det(M[U'])/\det(M[U])\right]$

gauge part: trace of 3×3 matrices (easy, without M: quenched)
fermionic part: determinant of $10^8 \times 10^8$ sparse matrices (hard)

determinant: represent it by a bosonic integral of pseudofermions
more efficient way than direct evaluation (inversion Mx=a), but still hard

$\det M[U] \propto \int [d\bar\psi][d\psi] \exp(-\bar\psi M^{-1}[U]\psi)$

Hadron spectroscopy in lattice QCD

Determine the transition amplitude between:
having a "particle" at time 0 and the same "particle" at time t
$\Rightarrow$ Euclidean correlation function of a composite operator $\mathcal{O}$:

$$C(t) = \langle 0|\mathcal{O}(t)\mathcal{O}^\dagger(0)|0\rangle$$

insert a complete set of eigenvectors $|i\rangle$

$$= \sum_i \langle 0|e^{Ht}\,\mathcal{O}(0)\,e^{-Ht}|i\rangle\langle i|\mathcal{O}^\dagger(0)|0\rangle = \sum_i |\langle 0|\mathcal{O}^\dagger(0)|i\rangle|^2\,e^{-(E_i-E_0)t},$$

where $|i\rangle$: eigenvectors of the Hamiltonian with eigenvalue E_i.

and $$\mathcal{O}(t) = e^{Ht}\,\mathcal{O}(0)\,e^{-Ht}.$$

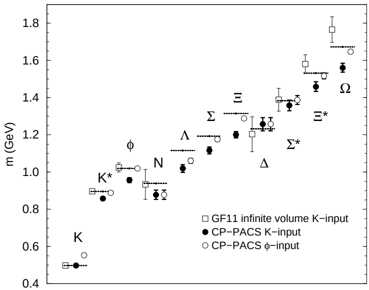
t large $\Rightarrow$ lightest states (created by $\mathcal{O}$) dominate: $C(t) \propto e^{-M\cdot t}$
$\Rightarrow$ exponential fits or mass plateaus M_t=log[C(t)/C(t+1)]

Quenched results

QCD is 50 years old ⇒ properties of hadrons (Rosenfeld table)

non-perturbative lattice formulation (Wilson) immediately appeared
needed 20 years even for quenched result of the spectrum (cheap)
instead of det(M) of a $10^6 \times 10^6$ matrix trace of 3×3 matrices

always at the frontiers of computer technology:



GF11: IBM "to verify QCD"
(10 Gflops = 1e10, '92)
CP-PACS: Hitachi QCD machine
(614 Gflops, '96)

the ≈10% discrepancy was believed to be a quenching effect

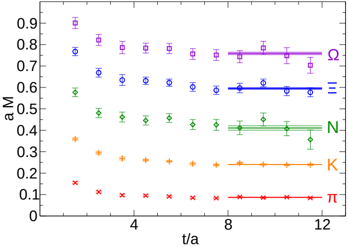
iPhone 14: 2000 Gflops (2e12)
Aurora supercomputer Argonne (2e18)

CPU is essentail but theory development is at least as important

Scale setting and masses in lattice QCD

in meteorology, aircraft industry etc. grid spacing is set by hand
in lattice QCD we use g,m_{ud} and m_s in the Lagrangian ('a' not)
measure e.g. the vacuum mass of a hadron in lattice units: $M_\Omega a$
since we know that M_Ω=1672 MeV we obtain 'a'

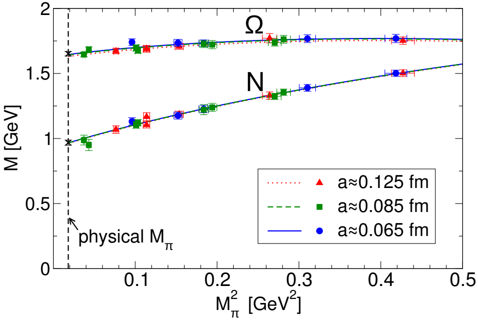
masses are obtained by correlated fits (choice of fitting ranges)
illustration: mass plateaus at the smallest $M_\pi \approx$190 MeV (noisiest)



volumes and masses for unstable particles: avoided level crossing
decay phenomena included: in finite V shifts of the energy levels
⇒ decay width (coupling) & masses of the heavy and light states

Dynamical N_f=2+1 QCD with continuum extrapolation

altogether 15 points for each hadrons



smooth extrapolation to the physical pion mass (or m_{ud})
small discretization effects (three lines barely distinguishable)

continuum extrapolation goes as $c \cdot a^n$ and it depends on the action
in principle many ways to discretize (derivative by 2,3... points)
goal: have large n and small c (in this case $n = 2$ and c is small)

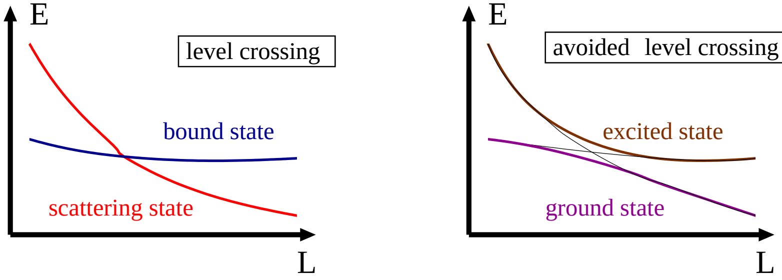
(Various) finite volume effects: resonance states

parameters, for which resonances would decay at V=∞

at V=∞ the lowest energy state is a two-particle scattering state

hypothetical case with no coupling $\Rightarrow$ level crossing as V increases

realistic case: non-vanishing decay width $\Rightarrow$ avoided level crossing

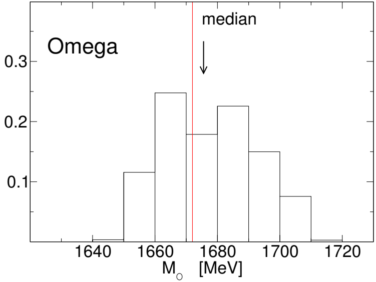


M. Luscher, Nucl. Phys. B364 (1991) 237

self-consistent analysis: width is an unknown quantity and we fit it

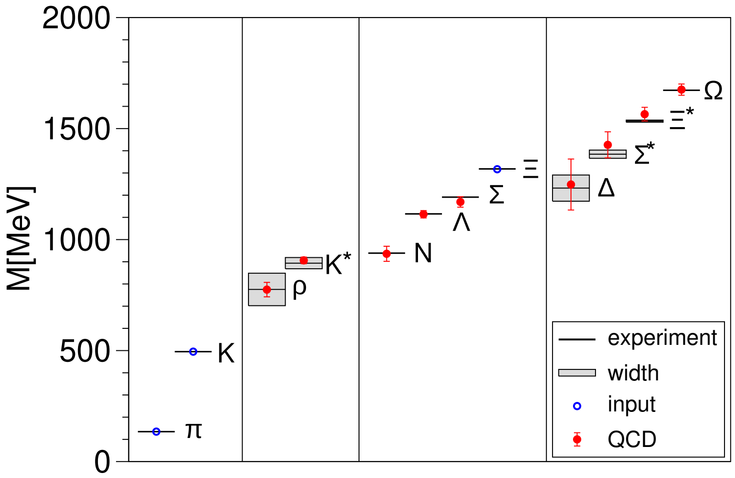
Analysis: avoid arbitrarinesses & include systematics

extended frequentist's method:
2 ways of scale setting, 2 strategies to extrapolate to $M_\pi(phys)$
3 pion mass ranges, 2 different continuum extrapolations
18 time intervals for the fits of two point functions
$2 \cdot 2 \cdot 3 \cdot 2 \cdot 18 = 432$ different results for the mass of each hadron



central value and systematic error is given by the mean and the width
statistical error: distribution of the means for 2000 bootstrap samples

Final result for the hadron spectrum 2008



Introduction to isospin symmetry

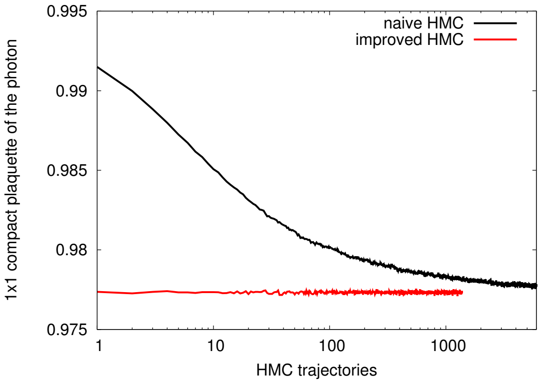
Isospin symmetry: 2+1 or 2+1+1 flavor frameworks
if 'up' and 'down' quarks had identical properties (mass,charge)
$M_n = M_p, \quad M_{\Sigma^+} = M_{\Sigma^0} = M_{\Sigma^-}, \quad$ etc.

The symmetry is explicitly broken by
• up, down quark electric charge difference (up: 2/3·e down:-1/3·e)
⇒ proton: uud=2/3+2/3-1/3=1 whereas neutron: udd=2/3-1/3-1/3=0
at this level (electric charge) the proton would be the heavier one
• up, down quark mass difference ($m_d/m_u \approx 2$): 1+1+1+1 flavor

The breaking is large on the quark's level ($m_d/m_u \approx 2$ or charges)
but small (typically sub-percent) compared to hadronic scales.

These two competing effects provide the tiny M_n-M_p mass difference
$\approx$ 0.14% is required to explain the universe as we observe it

Autocorrelation of the photon field

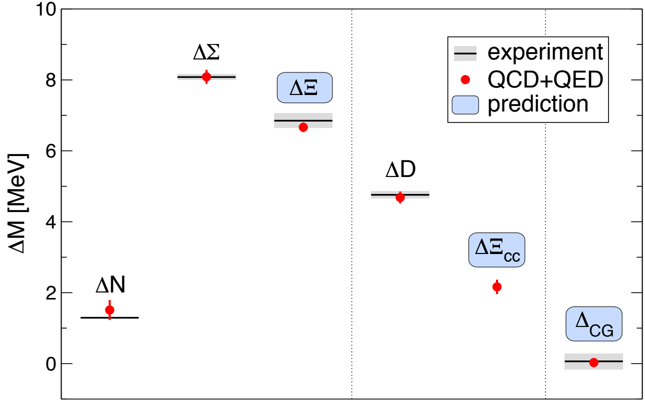


Standard HMC has $\mathcal{O}(1000)$ autocorrelation

Fourier transformed k-dependent mass terms to eliminate "knowledge"
Improved HMC has none (for the pure photon theory)
Small coupling to quarks introduces a small autocorrelation

Isospin splittings: 2015

splittings in channels that are stable under QCD and QED:



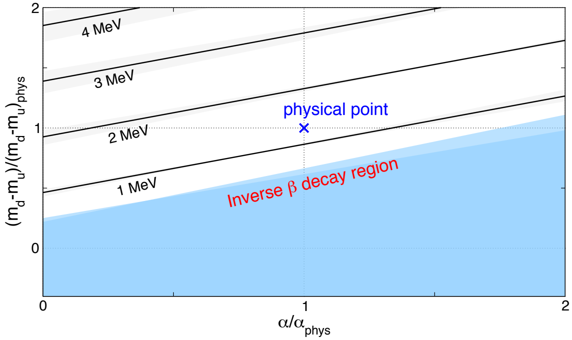
ΔM_N, ΔM_Σ and ΔM_D splittings: post-dictions

ΔM_Ξ, $\Delta M_{\Xi_{cc}}$ splittings and Δ_{CG}: predicitions

Quantitative anthropics

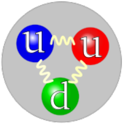
Precise scientific version of the great question:
Could things have been different (string landscape)?

eg. big bang nucleosynthesis & today's stars need $\Delta M_N \approx$ 1.3 MeV



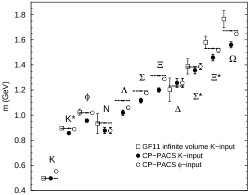
(lattice message: too large or small $m_d - m_u$ would shift α)

Summary: development within two decades

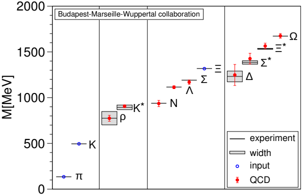


Strong + Higgs + Electro = Experiment

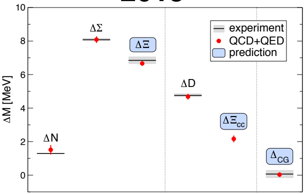
1996



2008



2015



high precision for non-perturbative questions (lattice formalism)