Quantum Field Theory on the Lattice

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Two sets of lattice field theory talks

Michael Creutz: three talks
Zoltan Fodor: four talks

"computational details ... might be better for Zoltan to cover, i.e. things like hybrid monte carlo, the hadron spectrum ... g-2" and QCD thermodynamics.

- Scalar theory, Higgs bound & Monte Carlo
- QCD and hadron spectrum (Wilson)
- QCD thermodynamics (staggered & overlap)
- g-2 of the muon (staggered)

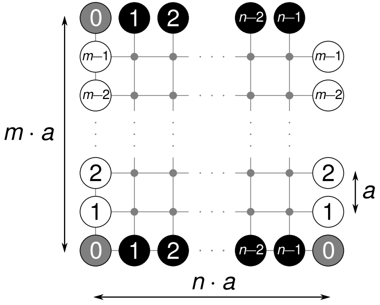
Outline

1. Lattice Regularization
2. Scalar field theory
3. Algorithms

Lattice regularization

"Most sytematic" nonperturbative approach:
lattice QFT

I. Take a finite segment of spacetime,
put fields at vertices of hypercubic lattice with lattice spacing a:



Bosons:

Periodic in all directions

Fermions:

Time direction: antiperiodic

Space directions: periodic

Lattice regularization

II. Path integral quantization $Z = \int \mathcal{D}\phi \exp(iS)$ (oscillates).

III. Use t$\rightarrow$it: Euclidean action gives Boltzmann factors exp(-S)

integral over spacetime	$\int \mathrm{d}^4 x$	$\longrightarrow$	sum over sites $a^4 \sum_x$
derivatives	∂_μ	$\longrightarrow$	finite differences
momentum $p \leq \frac{\pi}{a}$		$\Longrightarrow$	natural UV cutoff.

At finite "a" results differ from the continuum value.
E.g. for some dimensionless quantity R.

$$R^{\text{latt.}} = R^{\text{cont.}} + O(a^\nu)$$

Many ways to discretize (∂): Symanzik improvement to increase ν

To get physical results, need to perform:

IV. Infinite volume limit $(V \rightarrow \infty)$
V. Continuum limit $(a \rightarrow 0)$; CPU costs naively a^{-4} or a^{-5}.
Reality: far worse & frozen topology (0.05 fm) $\Rightarrow$ open boundary.

Example: one component real scalar field

Continuum action:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{g_0}{4!}\phi^4$$

Simplest lattice action with $\hat{\mu}$: unit vector in direction μ:

$$S = \sum_x a^4 \left(\frac{1}{2} \sum_{\mu=1}^{4} \left[\frac{\phi_{x+\hat{\mu}} - \phi_x}{a} \right]^2 + \frac{1}{2} m_0^2 \phi_x^2 + \frac{g_0}{4!} \phi_x^4 \right)$$

Path integral quantization $Z = \int \mathcal{D}\phi \exp(-S)$ with Euclidean S.

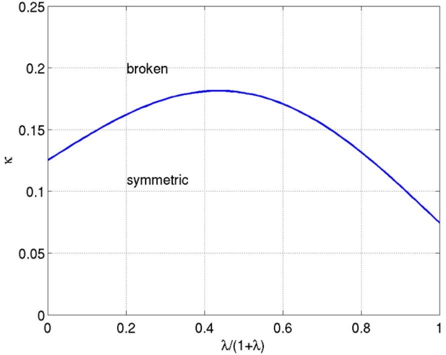
VI. Introduce: $\sqrt{2\kappa}\varphi = \phi a$, $(1-2\lambda)/\kappa - 8 = m_0^2 a^2$, $6\lambda/\kappa^2 = g_0$

$$S = \sum_x \left[\varphi_x^2 + \lambda \left(\varphi_x^2 - 1 \right)^2 \right] - 2\kappa \sum_{<xy>} \varphi_x \varphi_y = \sum_x u(\varphi_x) - 2\kappa \sum_{<xy>} \varphi_x \varphi_y$$

Lattice spacing "a" does not appear explicitly in the calculations.

Phase diagram of the theory

Critical line in the λ versus κ plane.

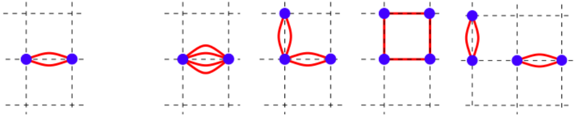


For λ=0 non-interaction (free theory).
The mass is $m_0^2 a^2 = 1/\kappa - 8$, thus $\kappa_c = 1/8$ is critical.
For $\lambda = \infty$ we recover the Ising model: $\kappa_c = 0.0748487...$ is critical.

High T (small β) expansion in 2d Ising

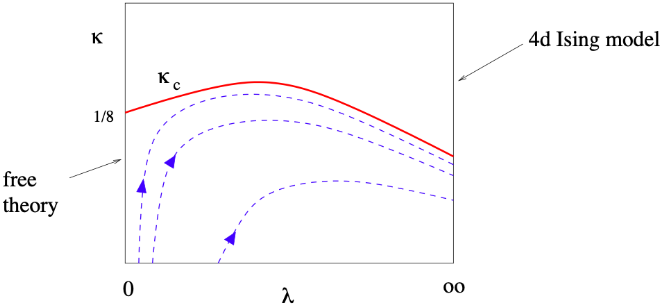
- Partition function at $\beta \ll 1$: expand in β, only terms which have s_i^{2n} survive!

$$
\begin{aligned}
Z &= \sum_{s_i} \prod_{<ij>} e^{\beta s_i s_j} \\
&= \sum_{s_i} \prod_{<ij>} \left(1 + \beta s_i s_j + \frac{1}{2!}\beta^2 (s_i s_j)^2 + \frac{1}{3!}\beta^3 (s_i s_j)^3 + \frac{1}{4!}\beta^4 (s_i s_j)^4 + \ldots\right) \\
&= 2^V \left[1 + \beta^2 \frac{2V}{2} + \beta^4 \left(\frac{2V}{4!} + \frac{6V}{2^2} + V + \frac{1}{2}\frac{2V(2V-7)}{2^2}\right) + O(\beta^6)\right]
\end{aligned}
$$



- 2^V number of possible configurations.
- Product: actually Hamiltonian sum in the Boltzmann exponents.
- Similarly can be done for the 4-dim ϕ^4 theory.

As a→0 keep λ_R constant, how to change λ



Blue curves: RG trajectories of constant λ_R; λ_R decreases to ↖; along κ_c we have $\lambda_R = 0$.

- Perturbatively $\beta_L(\lambda) = -a\frac{\partial\lambda}{\partial a} = \frac{3}{16\pi^2}\lambda^2 > 0$
- *No continuum limit!* RG trajectories hit the Ising limit $\lambda = \infty$ before the critical line $\kappa_c(\lambda)$ is reached; i.e. at finite a.

Triviality: how did we get there (summary)

• ϕ^4 theory is probably the simplest 'interactive' theory.

• Rewrite it on a Euclidean space-time grid.
$\Rightarrow$ use dimesionless quantities only: φ, κ and λ
• Any physical quantity will be given in lattice units 'a'
e.g. a characteristic length: 'how many times our lattice unit'.
Continuum limit: $a \rightarrow 0$ or physical lengths $\rightarrow \infty$ measured in 'a'.

• Using small/large κ (hopping parameter) expansion & RGE
correlation length (=inverse mass, see later) &
quartic coupling λ_R for all possible κ, λ values
$\Rightarrow$ critical line, correlation length ∞, continuum limit (is there any?)

Lines of constant physics (LCP): connecting points with the same λ_R.
LCPs always end on the Ising line and not on the critical line
we reach the maximal coupling before reaching the continuum limit

$m_R a$=0 defines the critical line (continuum limit)

construct lines of constant physics (LCP):

a$\to$0 but m_R=const., λ_R=const.

as "a" gets smaller along these LCPs the bare λ gets larger
actually before the LCPs reach the critical line one gets $\lambda = \infty$
(only the trivial theory $\lambda = 0$ reaches the critical line)

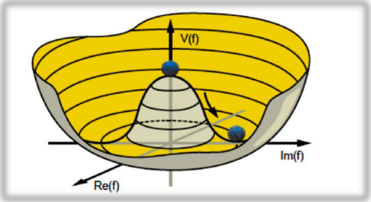
assume that the maximum momenta are a few times larger than M_H
maximal renormalized self-coupling, thus maximal Higgs mass
is obtained at the maximal bare coupling $\lambda = \infty$
using the Higgs vacuum expectation value (overall scale)
one obtains $M_H \lesssim$600 GeV

for even higher cutoffs (more than a few times) $\Rightarrow$ smaller M_H

Triviality of the ϕ^4 model: putting in numbers

we studied: 1 component case
similar: 4 component case (Higgs)

also trivial: renormalized quartic coupling $\lambda_R \rightarrow 0$ if $a \rightarrow 0$



quadratic term at minimum $\Rightarrow$ mass
m=$\sqrt{g/3}\phi_0$ with ϕ_0 vacuum expectation

ϕ_0=$(\sqrt{2}G_\mu)^{-1/2}$=246 GeV
Fermi constant G_μ from $\mu^- \Longrightarrow e^-\nu_\mu\bar{\nu}_e$

the larger the bare λ the larger λ_R

what is the maximal λ_R with meaningful physics?

Triviality for the Standard Model

cutoff must be well above the mass to have meaningful physics

let us say at least $>$ twice the mass $g_R < 41 \pm 6$ (with $\lambda = \infty$)
this leads to a mass of $\sqrt{41/3} \cdot 246$ GeV ≈ 900 GeV

using even larger cutoff (e.g. three times the mass)
brings us even closer to the critical line
reduces the largest possible renormalized coupling
this brings down the scalar (Higgs) mass

Triviality doesn't allow Higgs to be heavier than 600-900 GeV

other discretizations: qualitatively same, numerical values differ

Importance sampling

Monte Carlo simulation: calculate $\langle 0|\,\mathcal{O}\,|0\rangle$ stochastically.

Naive way: take random gauge configurations U_α according to the uniform distribution and calculate the weighed average:

$$\langle 0|\,\mathcal{O}\,|0\rangle = \frac{\sum_\alpha \mathcal{O}_\alpha\, e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}$$

S_α: value of $S_{\text{eff.}}$ at U_α,
$\mathcal{O}_\alpha$: value of $\mathcal{O}$ at U_α.

S_α large for most configurations $\longrightarrow$ small portion of configurations give significant contribution.

Importance sampling: generate configurations with probability based on their importance $\longrightarrow$ probability of U_α is proportional to e^{-S_α}.

Then $\quad \langle 0|\,\mathcal{O}\,|0\rangle = \frac{1}{N}\sum_{\alpha=1}^{N} \mathcal{O}_\alpha \quad$ with relative error $\frac{1}{\sqrt{N}}$.

Metropolis algorithm

Simplest method: Metropolis algorithm.
Choose an initial configuration U_0.

1. Generate U_{k+1} from U_k with a small random change.
2. Measure the change ΔS in the action.
3. If $\Delta S \leq 0$, keep U_{k+1}.
4. If $\Delta S > 0$, keep U_{k+1} with a probability of $e^{-\Delta S}$.

- U_0 is far from the region where e^{-S} is significant.
  $\Longrightarrow$ Many steps required to reach equilibrium distribution:
  Thermalization time.
- $U_k \longrightarrow U_{k+1}$ by small change.
  $\Longrightarrow$ Subsequent configurations are not independent.
  Number of steps required to reach next independent configuration:
  Autocorrelation time.

Heatbath algorithm

Metropolis is not the only single field variable changing algorithm

For illustration purposes let us use the Ising model.

Impose the condition of detailed balance in another way:
Use the probabilities of the +1 spin (P_+) and that of the -1 spin (P_-).

These are proportional to the Boltzmann factors
$\exp(-\beta E_+)$ and $\exp(-\beta E_-)$

More specifically (their sum is 1):

$$P_+ = \exp(-\beta E_+)/[\exp(-\beta E_+) + \exp(-\beta E_-)]$$
$$P_- = \exp(-\beta E_-)/[\exp(-\beta E_+) + \exp(-\beta E_-)]$$

Take +1 with the probability of P_+ and -1 with the probability of P_-

Hybrid Monte Carlo: basic idea

Introduce momenta P_x for each field variable U_x as $P^2 = \sum_x P_x^2$.

$$\langle 0|\,\mathcal{O}[U]\,|0\rangle = \frac{\int [\mathrm{d}U]\ \mathcal{O}[U]\ e^{-S(U)}}{\int [\mathrm{d}U]\, e^{-S(U)}} = \frac{\int [\mathrm{d}U]\,[\mathrm{d}P]\ \mathcal{O}[U]\ e^{-P^2/2-S(U)}}{\int [\mathrm{d}U]\,[\mathrm{d}P]\, e^{-P^2/2-S(U)}}$$

Importance sampling for $-P^2/2 - S(U)$: it didn't change, we accept it
Create configurations through series of trajectories: hybrid algorithm.

1. Initial P based on its importance (m=1 but can be anything): random from Gaussian distribution $\exp(-P^2/2)$ (heatbath)
2. Keep $-P^2/2 - S(U)$ constant: obtain (P', U') via classical time evolution with Hamiltonian: $-P^2/2 - S(U)$
   Equations of motion: $\dot{U} = P, \quad \dot{P} = -\partial S(U)/\partial U$
   doesn't change the exponent, always accepted

Hybrid Monte Carlo: properties and accept/reject

Properties:

- Reversibility: start with $(P,U) \longrightarrow$ arrive at (P',U')
  start with $(-P',U') \longrightarrow$ arrive at $(-P,U)$
- Liouville's Theorem: measure $[\mathrm{d}U]\,[\mathrm{d}P]$ is preserved
- Energy $H(P,U)$ is preserved (Metropolis is always accepted)

In practice: integrate equations of motion with finite step size $\Delta\tau$

- Reversibility? ✓
- Area preserved? ✓
- $H(P,U)$ preserved? ✗ $\Delta H = H[P',U'] - H[P,U]$

$\longrightarrow$ Correct with accept–reject step at end of each trajectory:
if $\Delta H \leq 0 \longrightarrow$ accept
if $\Delta H > 0 \longrightarrow$ accept with probability $e^{-\Delta H}$
(computationally optimal if the accept rate is around 70%)

Importance sampling with fermions

Fermions can integrated out: determinant of the fermion matrix

$$Z = \int \prod_{n,\mu} [dU_\mu(n)] e^{-S_g} \det(M[U])$$

again: we do not take into account all possible gauge configurations each of them is generated with a probability $\propto$ its weight

Metropolis algorithm is the easiest importance sampling:
(all other algorithms are based on importance sampling)

$P(U \to U') = \min \left[1, \exp(-\Delta S_g) \det(M[U'])/\det(M[U])\right]$

gauge part: trace of 3$\times$3 matrices (easy, without M: quenched)
fermionic part: determinant of $10^8 \times 10^8$ sparse matrices (hard)

determinant: represent it by a bosonic integral of pseudofermions
more efficient way than direct evaluation (inversion Mx=a), but still hard

$\det M[U] \propto \int [d\bar{\psi}][d\psi] \exp(-\bar{\psi} M^{-1}[U]\psi)$

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