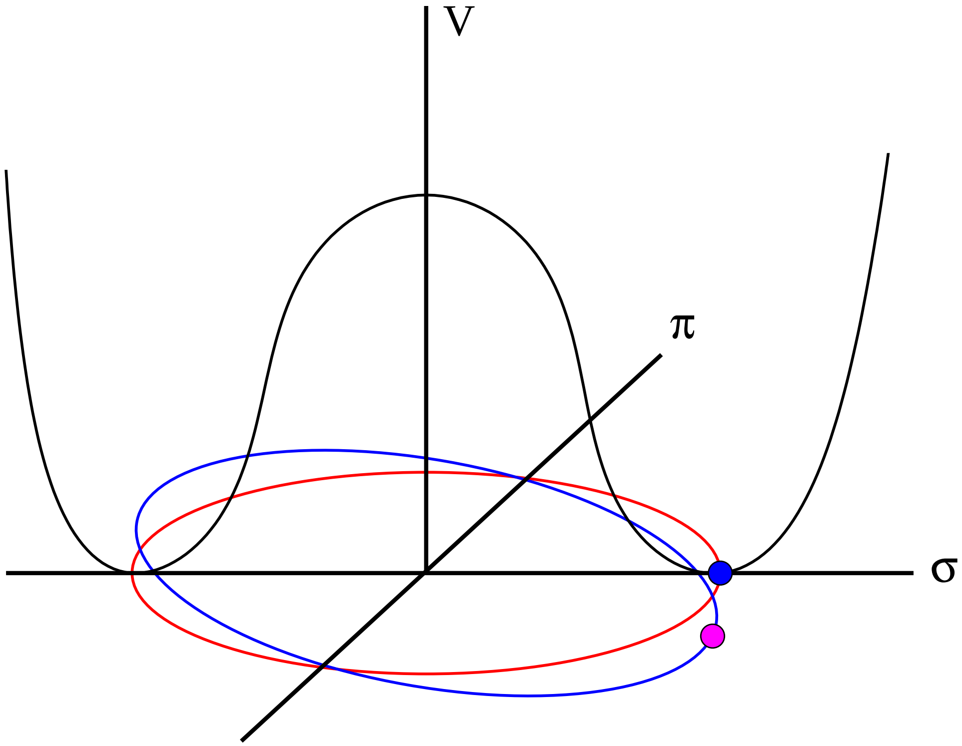
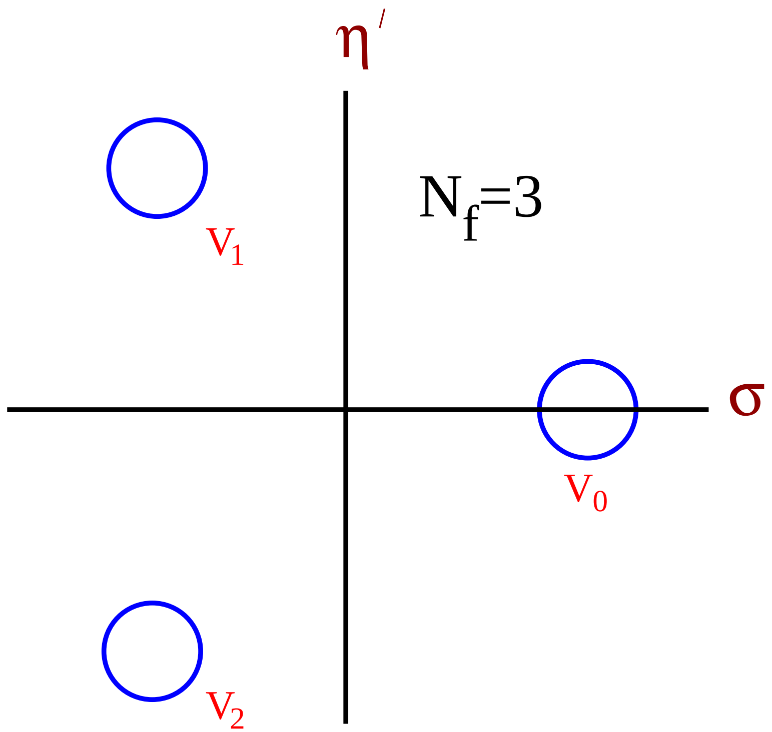
Chiral symmetry and the Theta parameter

Michael Creutz

BNL





Powerful tools for non-perturbative strong interactions

- the lattice
  - rigorous but computationally demanding
- chiral symmetry
  - expansion in quark masses and momenta
  - qualitative description of low energy physics
  - provides checks on lattice ideas

$$\psi_R = \frac{1+\gamma_5}{2}\psi$$
$$\psi_L = \frac{1-\gamma_5}{2}\psi$$
$$\overline{\psi} D\!\!\!\!/\, \psi = \overline{\psi}_R D\!\!\!\!/\, \psi_R + \overline{\psi}_L D\!\!\!\!/\, \psi_L$$
$$m\overline{\psi}\psi = m\overline{\psi}_L \psi_R + m\overline{\psi}_R \psi_L$$

Without a mass, ψ_L and ψ_R appear to be independent.

Three sources of chiral symmetry breaking in QCD

- spontaneous breaking $\langle\overline{\psi}\psi\rangle \neq 0$
  - explains lightness of pions
- implicit breaking of $U(1)$ by the anomaly
  - explains why η' is not so light
- explicit breaking from quark masses
  - pions are not exactly massless

Rich physics from the interplay of these effects

Venerable issues

- Dashen, 1971: possible spontaneous CP violation
  - before QCD!
- 't Hooft, 1976: anomaly and gauge field topology
- Fujikawa, 1979: fermion measure and the anomaly
- Witten, 1980: connection with chiral Lagrangians

Consider QCD with N_f light quarks and assume

- the field theory exists and confines
- spontaneous chiral symmetry breaking $\langle\overline{\psi}\psi\rangle \neq 0$
- $SU(N_f) \times SU(N_f)$ chiral perturbation theory
- anomaly gives η' a mass
- N_f small enough to avoid any conformal phase

Use continuum language

- assume a non-perturbative regulator (lattice?)
  - momentum space cutoff much larger than Λ_{QCD}
  - lattice spacing a much smaller than $1/\Lambda_{QCD}$

Construct effective potential V for meson fields

- V $=$ vacuum energy for a fixed field expectation
- formally via a Legendre transformation
- assume regulator allows defining composite fields

For simplicity initially consider

- degenerate quarks with small mass m
- N_f even
  - interesting subtleties for odd N_f

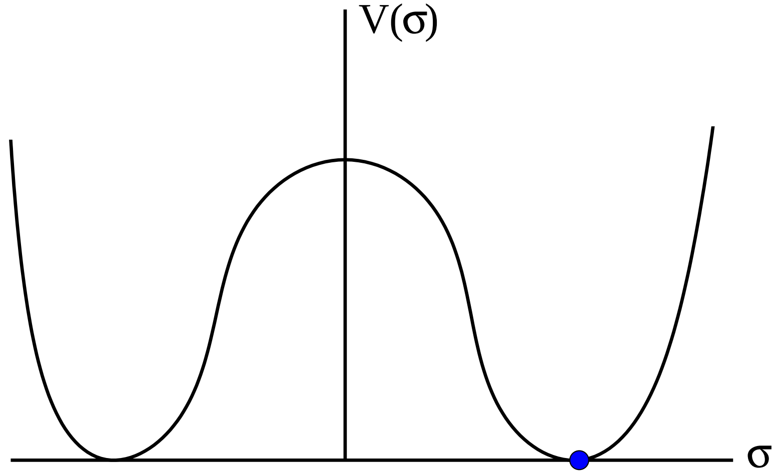
Work with composite fields

- $\sigma \sim \overline{\psi}\psi$
- $\pi_\alpha \sim i\overline{\psi}\lambda_\alpha\gamma_5\psi$
- $\eta' \sim i\overline{\psi}\gamma_5\psi$

λ_α: Gell-Mann matrices

Spontaneous symmetry breaking at $m = 0$

- $V(\sigma)$ has a double well structure



Ignore convexity issues

Massless theory symmetric under flavored chiral rotations

- $\psi \to e^{i\phi\gamma_5\lambda^\alpha}\psi$

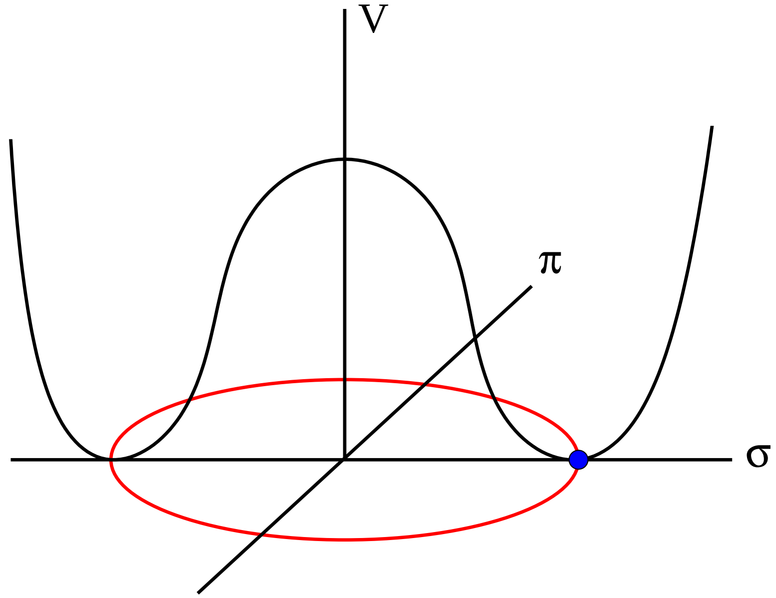
$N_f = 2$:

- $\sigma \to \cos(\phi)\sigma + \sin(\phi)\pi^\alpha$
- $\pi^\alpha \to \cos(\phi)\pi^\alpha - \sin(\phi)\sigma$

V should be symmetric under this rotation

Potential has $N_f^2 - 1$ “flat” directions

- one for each generator of $SU(N_f)$

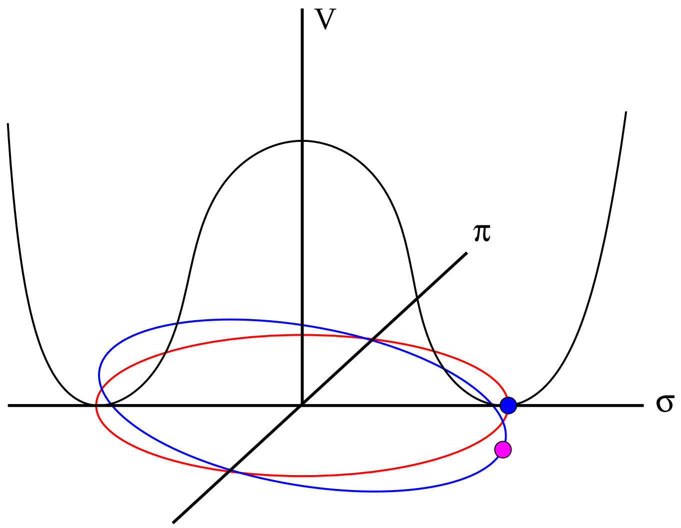


Excitations in the π direction don’t raise energy

- pions become massless

Small mass selects vacuum

- $V \to V - m\sigma$
- $\langle\sigma\rangle \sim +v \qquad \langle\pi\rangle = 0$
- Goldstones acquire mass $\sim \sqrt{m}$



The vacuum is not empty

- pions are waves on σ background

Anomaly: $V(\sigma,\eta')$ not symmetric under

- $\psi \to e^{i\phi\gamma_5}\psi$
- $\sigma \to \sigma\cos(\phi) + \eta'\sin(\phi)$
- $\eta' \to -\sigma\sin(\phi) + \eta'\cos(\phi)$

Near the vacuum state $\sigma \sim v$ and $\eta' \sim 0$

- $V(\sigma,\eta') = m_\sigma^2(\sigma - v)^2 + m_{\eta'}^2{\eta'}^2 + O((\sigma - v)^3, {\eta'}^4)$
  - both masses of order Λ_{QCD}

In quark language

Classical symmetry $[\gamma_5, \not{D}]_+ = 0$ implies

- $\psi \to e^{i\phi\gamma_5/2}\psi$
- $\overline{\psi} \to \overline{\psi}e^{i\phi\gamma_5/2}$

- mixes σ and η'
  - $\sigma \to \sigma\cos(\phi) + \eta'\sin(\phi)$
  - $\eta' \to -\sigma\sin(\phi) + \eta'\cos(\phi)$

This symmetry is "anomalous"

- any valid regulator must break it

Fujikawa: Variable change alters fermion measure

- $d\psi \to |e^{-i\phi\gamma_5/2}|\ d\psi = e^{-i\phi \mathrm{Tr}\gamma_5/2}\ d\psi$

But doesn't $\mathrm{Tr}\ \gamma_5 = 0$???

Not in the regulated theory!!!

- i.e. $\mathrm{Tr}\gamma_5 \equiv \lim_{\Lambda\to\infty} \mathrm{Tr}\left(\gamma_5\ e^{-D^2/\Lambda^2}\right) \neq 0$

Use eigenstates of D to define $\mathrm{Tr}\gamma_5$

- $D|\psi_i\rangle = \lambda_i|\psi_i\rangle$
- $\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle\ e^{-|\lambda_i|^2/\Lambda^2}$

Topology and the index theorem

Require $F_{\mu\nu} \to 0$ at spatial infinity

- A_μ goes to pure gauge $A_\mu(x) \to ih^*(x)\partial_\mu h(x)$

For $SU(2)$ $\qquad h = a_0 + i\vec{a}\cdot\vec{\sigma} \qquad$ with $a_\mu^2 = 1$

- infinity and the group space are both 3-spheres, S_3

$h(x=\infty)$ can wrap non-trivially about the group

- defines a "winding number 'ν' " for any gauge field

"Instanton": example of winding number 1 configuration

- $A_\mu = \frac{-ix^2}{g(x^2+\rho^2)} h^\dagger \partial_\mu h$
- where $h(x_\mu) = \frac{t + i\vec{\tau}\cdot\vec{x}}{\sqrt{x^2}}$
- group $SU(2)$ ρ is the instanton size

- minimizes action with winding 1
- a classical solution of the Yang-Mills equations

General $SU(N)$: use $SU(2)$ subgroups

Index theorem:

- with winding ν, D has ν zero modes $D|\psi_i\rangle = 0$
  - modes are chiral: $\gamma_5|\psi_i\rangle = \pm|\psi_i\rangle$
  - $\nu = n_+ - n_-$

Non-zero eigenvalues in chiral pairs

- $D|\psi\rangle = \lambda|\psi\rangle$
- $D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle = \lambda^*\gamma_5|\psi\rangle$

$|\psi\rangle$ and $|\gamma_5\psi\rangle$ cancel for $\mathrm{Tr}\gamma_5$

- only the zero modes count!

$$\boxed{\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle = \nu}$$

Where did the opposite chirality states go?

- continuum: lost at "infinity"
  - "above the cutoff"
- Wilson: real eigenvalues in doubler region
- overlap: modes across unitarity circle
  - $D\gamma_5 = -\hat{\gamma}_5 D \qquad \mathrm{Tr}\ \hat{\gamma}_5 = 2\nu$

This phenomenon involves both short and long distances

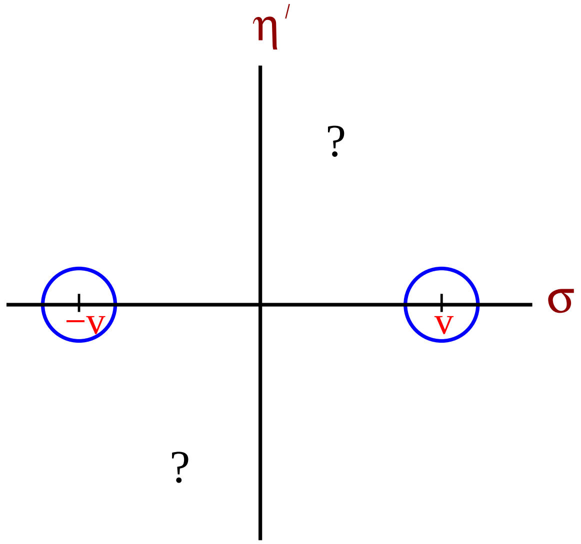
- zero modes pair with modes lost at the cutoff

Perturbative and non-perturbative effects entangled

- small instantons can “fall through the lattice”
- lattice scheme and scale dependent

Back to effective potential language

At least two minima in the σ, η' plane $\quad (\sigma, \eta') = (\pm v, 0)$



Question:

- do we know anything else about this potential?

Yes! there are actually N_f equivalent minima

Define $\psi_L = \frac{1+\gamma_5}{2}\psi \qquad \psi_R = \frac{1-\gamma_5}{2}\psi$

Singlet rotation of $\psi_L \to e^{i\phi}\psi_L$ alone

- not a good symmetry for generic ϕ

Flavored rotation $\psi_L \to g_L\psi_L = e^{i\phi_\alpha\lambda_\alpha}\psi_L$

- is a symmetry for $g_L \in SU(N_f)$
- $\mathrm{Tr}\lambda_\alpha = 0$

For special discrete values of ϕ these rotations can cross

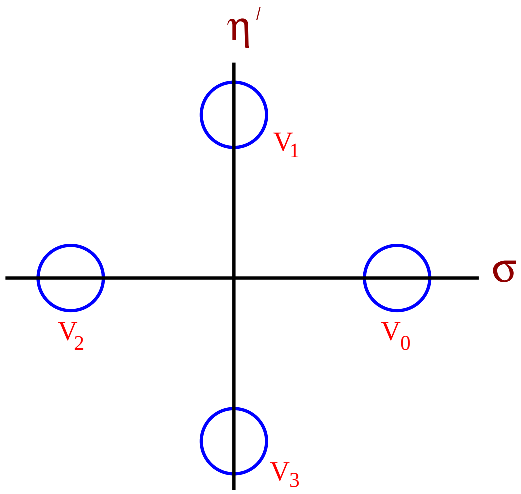
- $g = e^{2\pi i/N_f} \in Z_{N_f} \subset SU(N_f)$

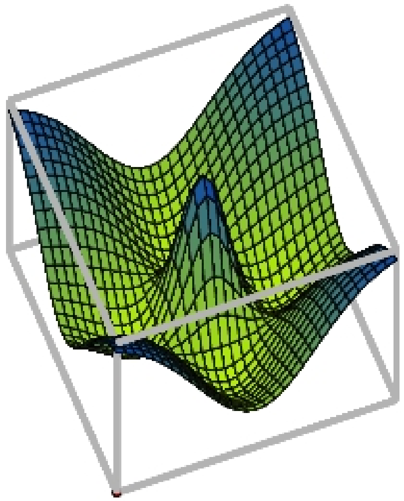
A valid discrete singlet symmetry:

$$\boxed{\begin{array}{l} \sigma \to +\sigma\cos(2\pi/N_f) + \eta'\sin(2\pi/N_f) \\ \eta' \to -\sigma\sin(2\pi/N_f) + \eta'\cos(2\pi/N_f) \end{array}}$$

$V(\sigma, \eta')$ has a Z_{N_f} symmetry

- N_f equivalent minima in the (σ, η') plane
- $N_f = 4$:



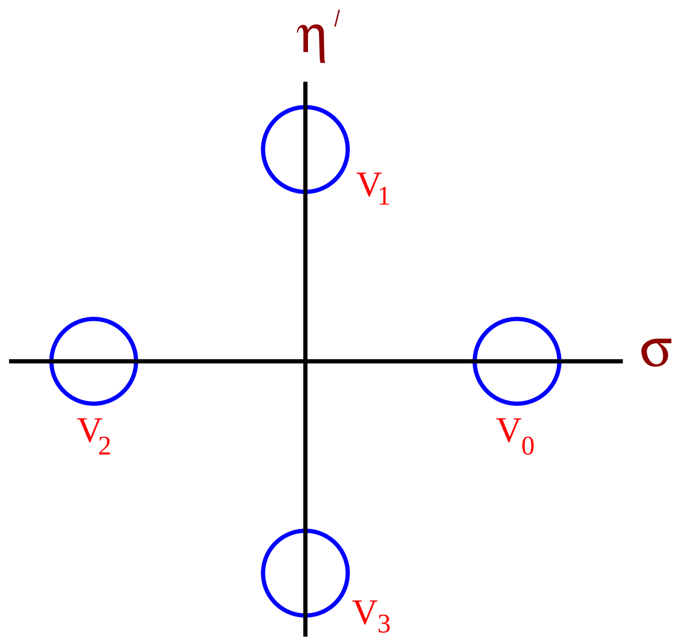


At the chiral Lagrangian level

- Z_N is a subgroup of both $SU(N)$ and $U(1)$

At the quark level

- measure gets a contribution from each flavor
- $\psi_L \to e^{2\pi i/N_f}\psi_L$ is a valid symmetry



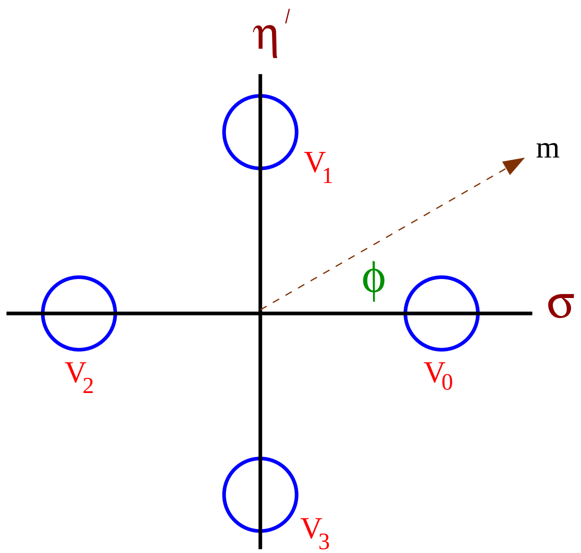
Mass term $m\overline{\psi}\psi$ tilts effective potential

- picks one vacuum (v_0) as the lowest

Anomalous rotation of the mass term

- $m\overline{\psi}\psi \to m\cos(\phi)\overline{\psi}\psi + im\sin(\phi)\overline{\psi}\gamma_5\psi$
- twists tilt away from the σ direction

An inequivalent theory!

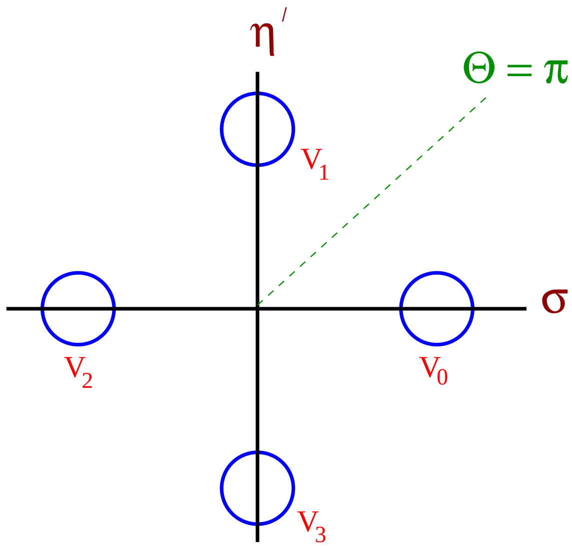


- as ϕ increases, vacuum jumps between minima

This rotation is the strong CP parameter Θ

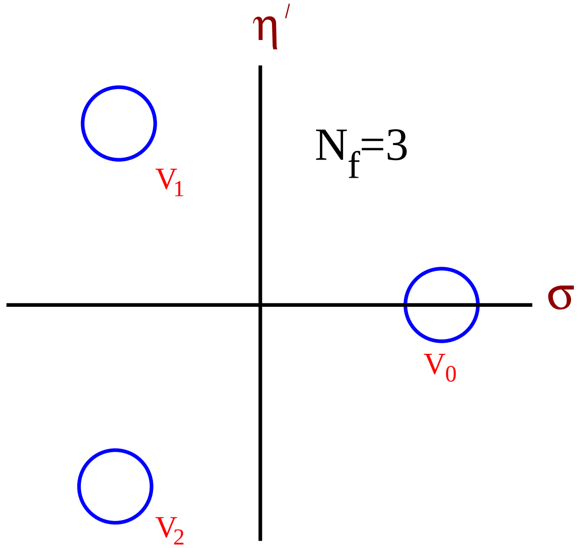
- here each flavor has been given the same phase
- conventional notation uses $\Theta = N_f \phi$
- the Z_{N_f} symmetry implies 2π periodicity in Θ

Degenerate light quarks $\Rightarrow$ first order transition at $\Theta = \pi$



Odd number of flavors, $N_f = 2N + 1$

- -1 is not in $SU(2N + 1)$



- $m > 0$ and $m < 0$ not equivalent!
- $m < 0$ represents $\Theta = \pi$
  - an inequivalent theory
  - spontaneous CP violation: $\langle \eta' \rangle \neq 0$

In perturbation theory m and $-m$ equivalent

- $\gamma_5 \ \gamma_\mu \frac{1}{\not{p}+m} \ \gamma_5 = \gamma_\mu \frac{1}{\not{p}-m}$

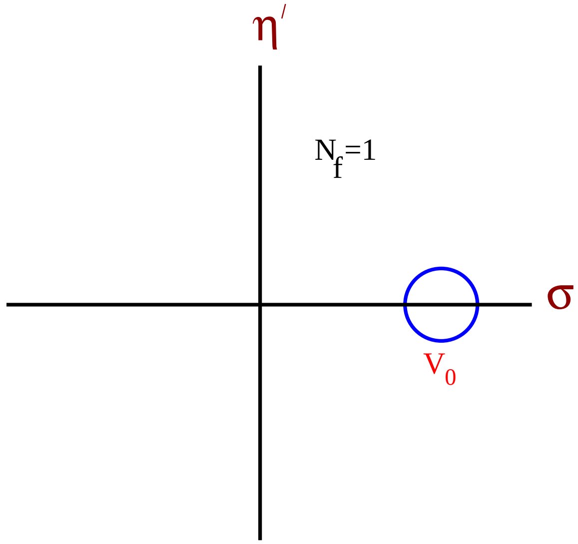
- Theta dependence invisible to perturbation theory!

Inequivalent theories have identical perturbative expansions!

$N_f = 1$

No chiral symmetry at all!

- one minimum, unique vacuum
- $\langle\overline{\psi}\psi\rangle \sim \langle\sigma\rangle \neq 0$ from 't Hooft vertex
- not a spontaneous symmetry breaking



No singularity at $m = 0$

- $m = 0$ not protected: "renormalon" ambiguity

For small mass

- no first order transition at $\Theta = \pi$

Summary

QCD with N_f massless flavors

- flavor singlet discrete Z_{N_f} chiral symmetry
- non-perturbative parameter Θ
- first order transition at $\Theta = \pi$ when $m \neq 0$

Sign of mass significant for N_f odd

- not seen in perturbation theory

No symmetry for $N_f = 1$ $\qquad$ $m = 0$ unprotected