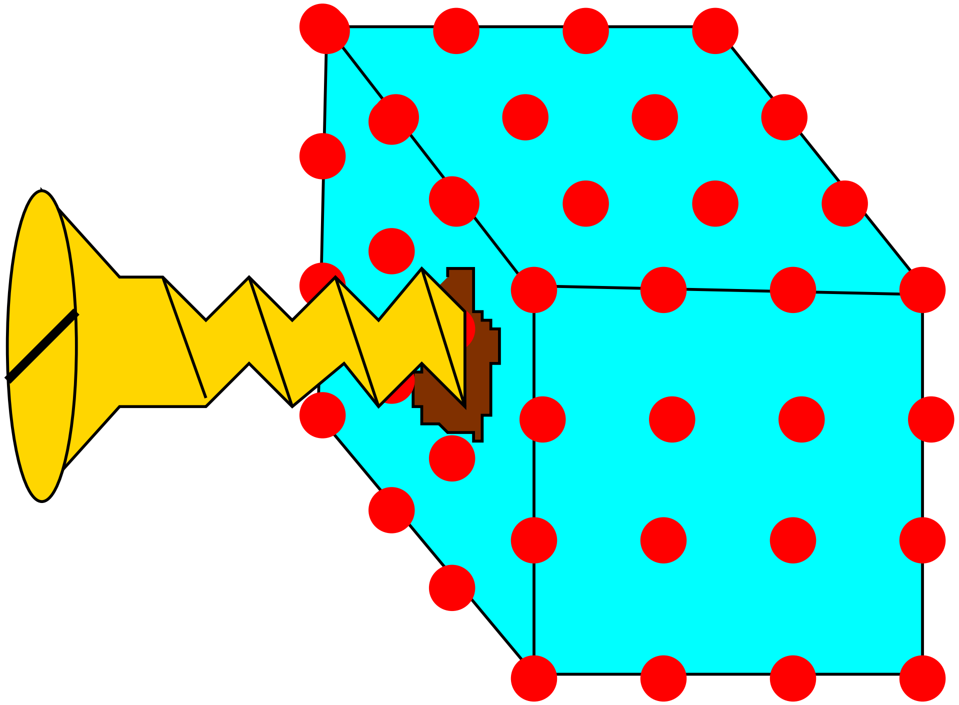
Lattice Fermions

Chiral symmetry

- reveals complications with lattice fermions



Consider some lattice Dirac operator D

- assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$
- most operators in practice satisfy this

Divide D into hermitean and antihermitean parts

- $D = K + M$
  - $K = (D - D^\dagger)/2$
  - $M = (D + D^\dagger)/2$

Then by construction

- $[K, \gamma_5]_+ = 0$
- $[M, \gamma_5]_- = 0$

On a lattice everything is finite; so $\mathrm{Tr}\gamma_5 = 0$

$M \to e^{i\theta\gamma_5} M$ is an exact symmetry of the determinant

- $|K + M| = |e^{i\gamma_5\theta/2}(K + M)e^{i\gamma_5\theta/2}| = |K + e^{i\theta\gamma_5} M|$

Where did the anomaly hide?

This must be a flavored chiral symmetry

All lattice actions must bring in extra structure

Naive and staggered fermions have doublers

- half use γ_5 and half $-\gamma_5$

Wilson and overlap fermions

- M is not a constant
- heavy states cancel the anomaly

Continuum free fermion action density

- $\overline{\psi} D \psi = \overline{\psi}(\not\partial + m)\psi$
- in momentum space $\overline{\psi}(i\not p + m)\psi$.

D has both Hermitean and anti-Hermitean parts

- Hermitian mass term is a constant
- this won't be the case on the lattice

Lattice replaces p with trigonometric functions

- Fourier transform
- $\tilde{\phi}_q = \sum_j e^{-iqj} \phi_j$ $\qquad\qquad$ $\phi_j = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqj} \tilde{\phi}_q$

Naive lattice fermions

- replace ∂_μ with nearest neighbor differences
- $\overline{\psi}(i\not{p} + m)\psi \rightarrow \overline{\psi}\left(\frac{i}{a}\sum_\mu \gamma_\mu \sin(p_\mu a) + m\right)\psi$
- at small p goes to desired $\overline{\psi}(i\gamma_\mu p_\mu + m)\psi$

But this also has low energy excitations for $p_\mu \sim \pi/a$

- we have $2^4 = 16$ "doublers"

Wilson: Make the mass momentum dependent

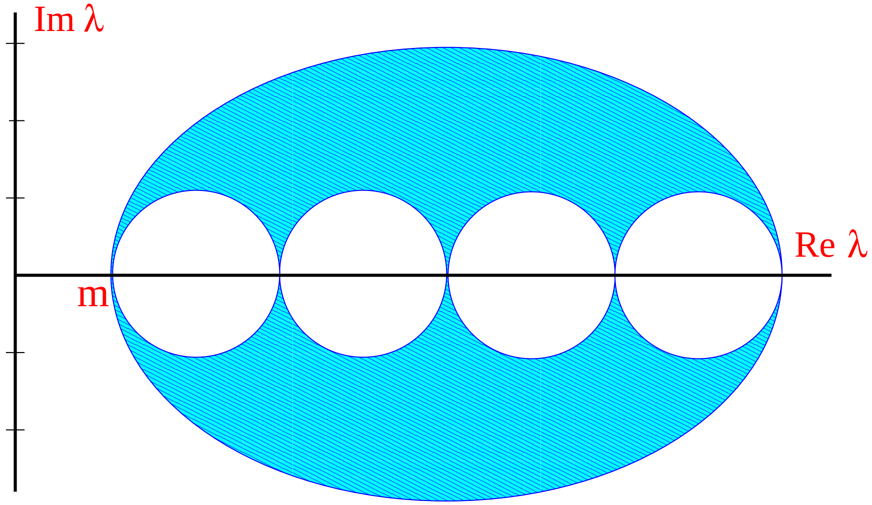
$$\overline{\psi} D_W \psi = \overline{\psi} \left(\frac{1}{a} \sum_\mu i\gamma_\mu \sin(p_\mu a) + \frac{r}{a}(1 - \cos(p_\mu a)) + m \right) \psi$$

- for small momentum $\frac{r}{a}(1 - \cos(p_\mu a)) = O(a)$
- for $p \sim \pi$ the eigenvalues are of order $1/a$

Eigenvalues for the free Wilson theory at

- $\lambda = \pm \frac{i}{a}\sqrt{\sum_\mu \sin^2(p_\mu a)} + \frac{r}{a}\sum_\mu (1 - \cos(p_\mu a)) + m$
- both real and imaginary parts even at $m = 0$

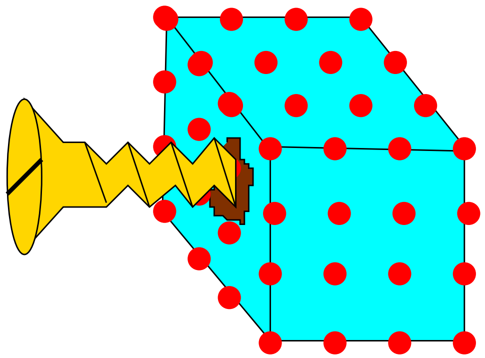
Eigenvalues form a set of "nested circles"



Notes

- $m \leftrightarrow -m$ not a symmetry
- essential for quantum theory with N_f odd
- chiral symmetry broken: $[D, \gamma_5]_+ \neq 0$
- m can get an additive renormalization

The Nielsen-Ninomiya theorem (1981)



Doubling closely tied to topology in momentum space

Consider the gamma matrix convention

- $\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$
- $\gamma_5 = \sigma_3 \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_0 = \sigma_2 \otimes I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Suppose a Dirac operator anti-commutes with γ_5

- $D = -D^\dagger = -\gamma_5 D \gamma_5.$

In momentum space: D(p) a 4×4 matrix function of p_μ

- The most general form is

  - $D(p) = \begin{pmatrix} 0 & z(p) \\ -z^*(p) & 0 \end{pmatrix}$

- where $z(p)$ is a quaternion

  - $z(p) = z_0(p) + i\vec{\sigma} \cdot \vec{z}(p).$

Any chirally symmetric Dirac operator

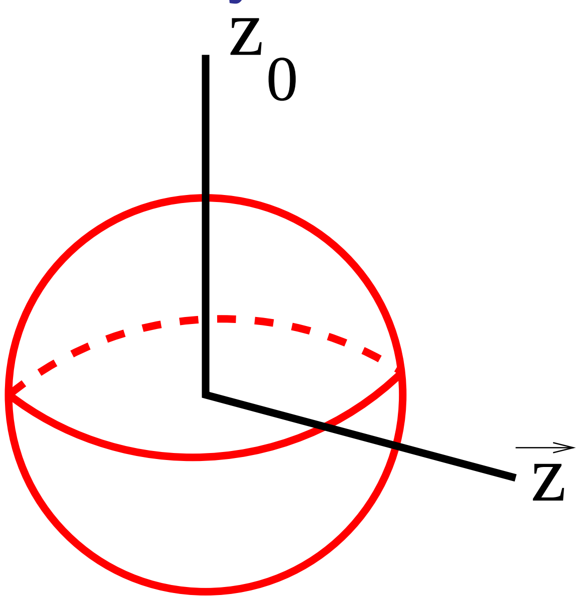
- maps momentum space into quaternions
  - $z(p) = z_0(p) + i\vec{\sigma} \cdot \vec{z}(p).$

Massless Dirac equation expands $D(p)$ around a zero

- $D(p) \simeq i\not{p} = i\gamma_\mu p_\mu$

Consider sphere of constant $|p|$ surrounding a zero

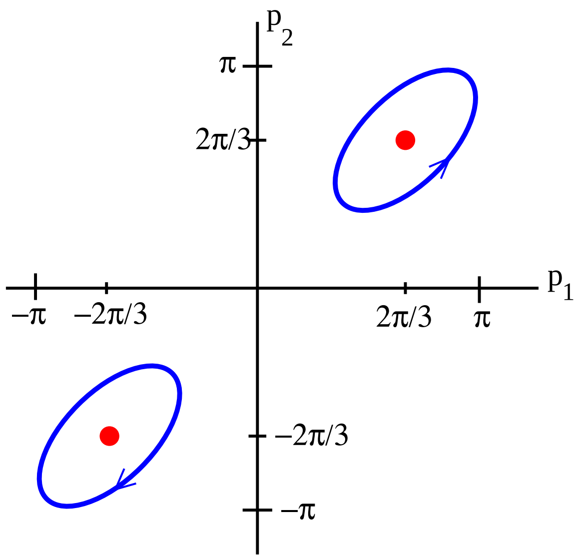
- wraps $z(p)$ non-trivially around the origin



Momentum space periodic over a Brilloin zone

- must have $z(p) = z(p + 2\pi n)$

Any wrapping must unwrap elswhere



Naive 16 doublers divide into two groups of 8

- with opposite mappings

Any chiral lattice theory with $D = -D^\dagger = -\gamma_5 D \gamma_5$

- requires an even number of species
- minimal operators with 2 flavors known
  - require some extra tuning

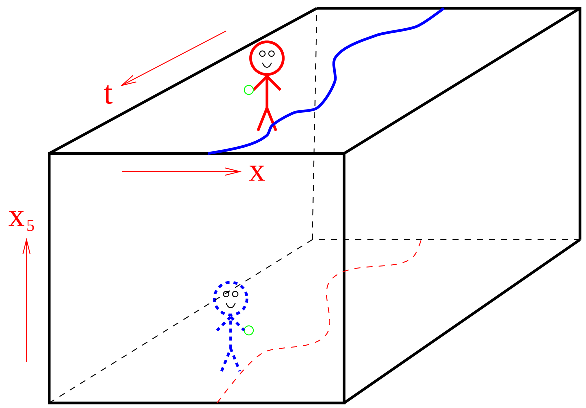
Domain wall and overlap fermions

Wilson fermions at $m < 0$ have surface modes

- naturally chiral
- mixing only through tunnelling between surfaces
- example of a "topological insulator"
  - robust conductivity only on surfaces

Use this for "domain wall" fermions

- work on a 5-d lattice of finite size $0 \leq x_5 < l_s$
- identify 4-d surface modes with physical quarks
- bulk modes go to infinite mass in continuum limit



Can be reformulated directly in four dimensions

- Require
  - $\gamma_5 D \gamma_5 = D^\dagger$
  - $D\gamma_5 = -\gamma_5 D + aD\gamma_5 D \equiv -\hat{\gamma}_5 D$
- "Ginsparg-Wilson relation" (1982)
- naive anticommutation with $O(a)$ correction

The GW relation is equivalent to the unitarity of

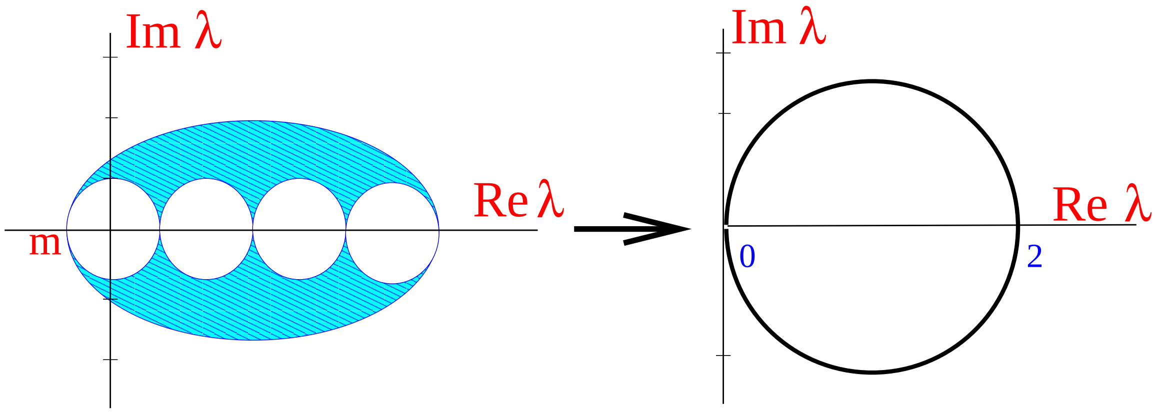
- $V = 1 - aD$
- GW $\longrightarrow$ $V^\dagger V = 1$

Neuberger: construct V via unitarization

- start with an undoubled Dirac operator, i.e. D_w
- project onto a unitary matrix
- $V = -D_w(D_w^\dagger D_w)^{-1/2}$.

To define $(D_w^\dagger D_w)^{-1/2}$

- 1) diagonalize $D_w^\dagger D_w$
- 2) take the square root of the eigenvalues
- 3) undo the diagonalizing unitary transformation.



Given V, the overlap operator is

- $D = (1 - V)/a.$

Properties

- computationally demanding
- "normal:" $[D^\dagger, D] = 0$, unlike Wilson
- $\gamma_5 D \gamma_5 = D^\dagger$
- a modified exact chiral symmetry
  - $\psi \to e^{i\theta\gamma_5}\psi \qquad \overline{\psi} \to \overline{\psi} e^{i\theta\hat{\gamma}_5}$
- where $\hat{\gamma}_5 = V\gamma_5$.

As with γ_5, $\hat{\gamma}_5$ is Hermitean and $\hat{\gamma}^2 = 1$

- all eigenvalues are ± 1
- defines an index
  - $\nu = \frac{1}{2}\mathrm{Tr}\hat{\gamma}_5 = \mathrm{Tr}\frac{\gamma_5+\hat{\gamma}_5}{2}$

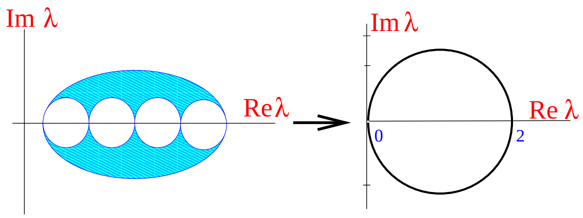
D has ν exact zero eigenvalues

- agrees with continuum index for smooth fields

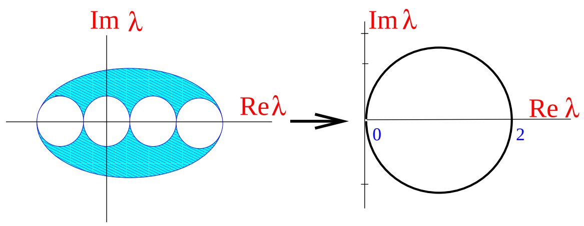
Some issues

- The "kernel" is somewhat arbitrary
  - with D_w, dependence on the hopping parameter
  - need $m < 0$ so one species projects out
  - need $|m|$ below doubler masses

- if $m > 0$ still satisfy GW but no massless particles

- 

  $N_F = 0$

- 

  $N_F = 5$

- GW does NOT require massless Goldstone bosons

Issues continued

- ν can depend on choice of m
  - depends on eigenvalue density in first circle
- how local is the overlap?
  - inversion destroys sparsity
  - is the non-locality exponential?

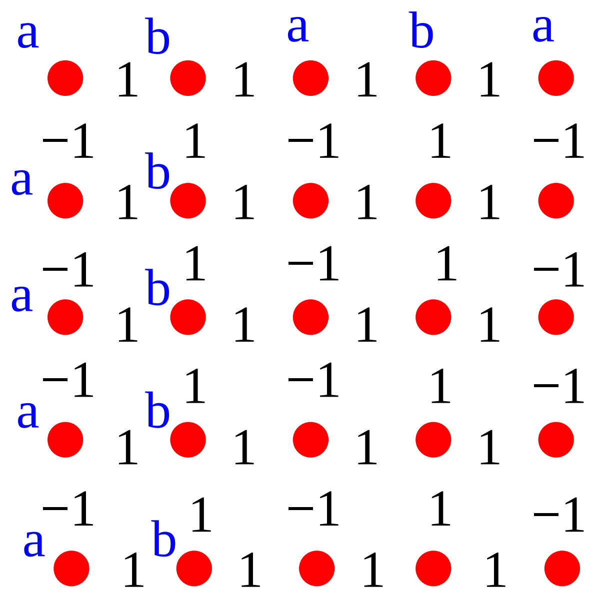
Staggered fermions

Start with a $Z_2 = \pm 1$ field on bonds of a 4D lattice

- Drive every Z_2 plaquette to -1
  - $\beta_{Z_2} = -\infty$
  - half integer magnetic flux through every plaquette

One choice puts phase factors on links as

- $Z_x = 1,\ Z_y = (-1)^x,\ Z_z = (-1)^{x+y},\ Z_t = (-1)^{x+y+z}$



Translation invariance

- by 1 in t direction
- by 2 in x, y, z directions
  - 8 site unit cell (2 in two dimensions)

Put a spinless fermion field on this lattice

- eigenvalues of D proportional to
- $\pm i\sqrt{\cos^2(p_x)+\cos^2(p_y)+\cos^2(p_z)+\cos^2(p_t)}$

Translation symmetry is by two in spatial directions

- restricts $\vec{p}$ components to "half" zones $0 \le p_j < \pi$
- temporal momentum has a full zone $-\pi \le p_t < \pi$

Interpret 8 site unit cell as

- two 4 component spinors
- two zeros in momentum space
  - $p = (\pi/2, \pi/2, \pi/2, \pm\pi/2)$

Net 4 effective Dirac species (tastes)

Chiral symmetry

- only nearest neighbor hopping
  - action anticommutes with $(-1)^{x+y+z+t} \sim$ "γ_5"
- Two Dirac "cones" of each chirality
- a four "flavor" theory with one exact chiral symmetry
  - actually a "flavored" chiral symmetry

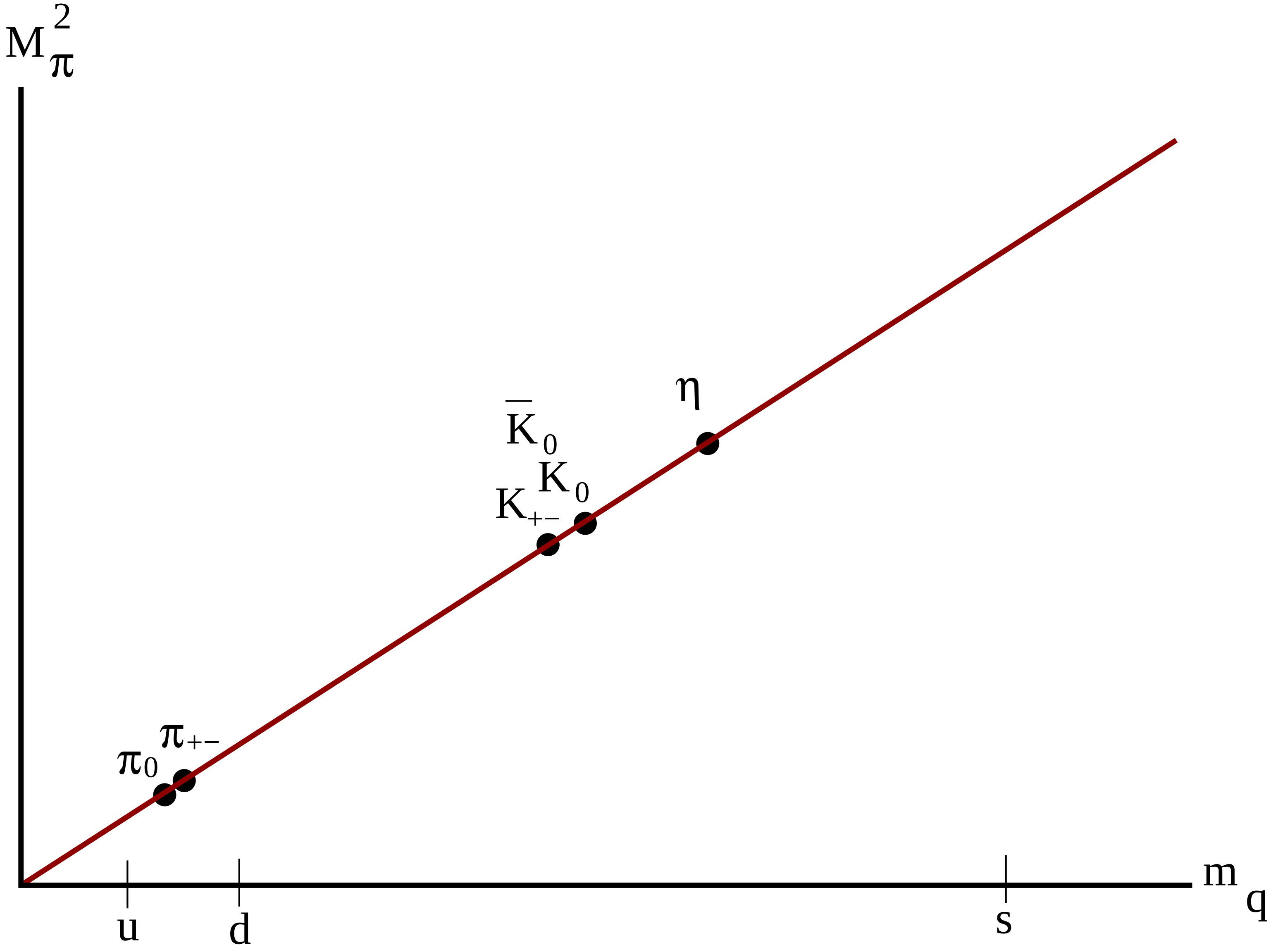
These are staggered fermions

- spin emerges dynamically for a one component field
  - Z_2 background field at "$\beta = -\infty$" gives sign factors
- inherently multiple degenerate species
  - required for chiral symmetry

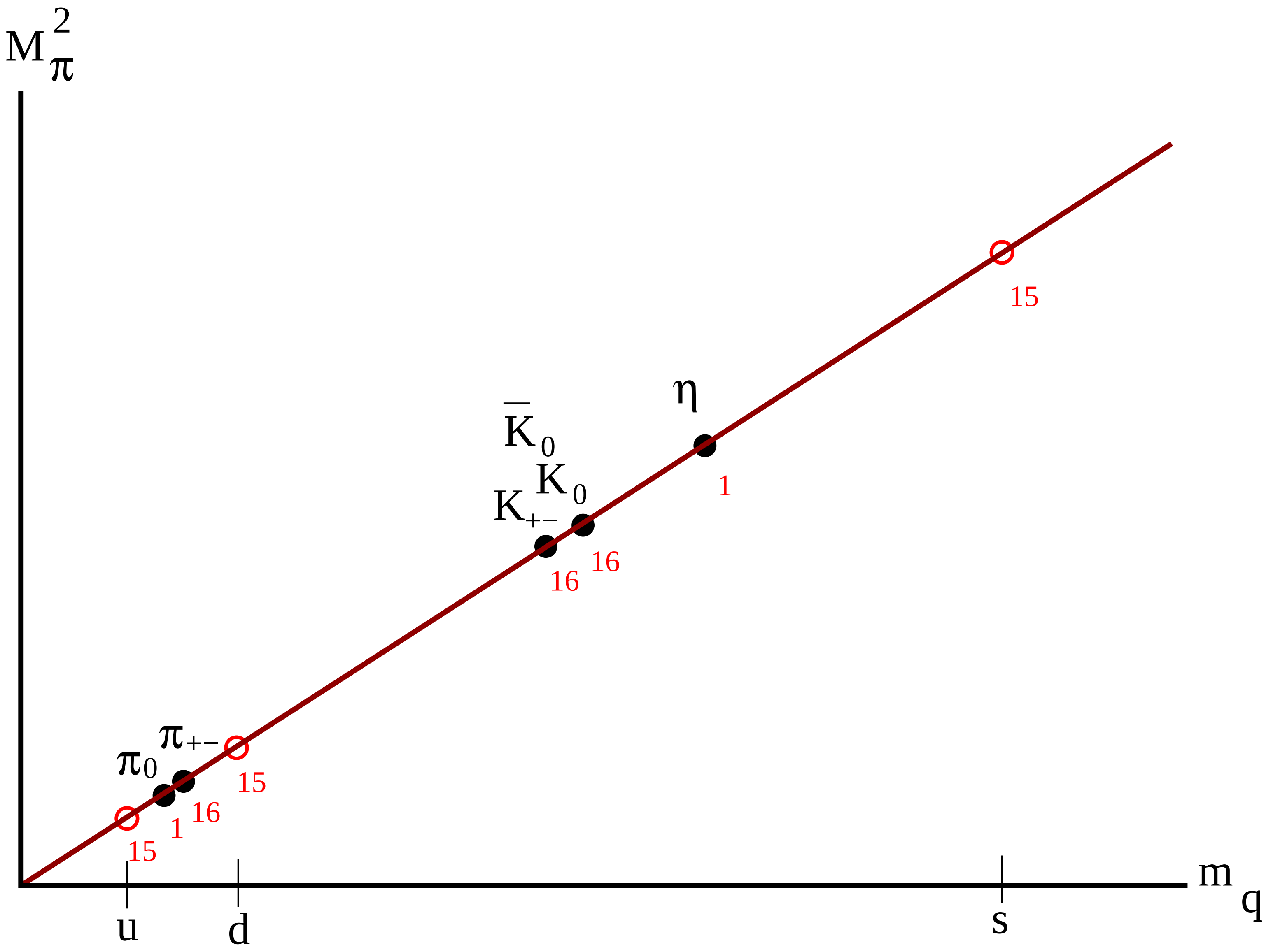
If 4 taste symmetry becomes exact SU(4) in continuum

- SU(4): $4 \otimes \overline{4} = 15 \oplus 1$
- $15 \oplus 1$ neutral $\overline{u}u$
- $15 \oplus 1$ neutral $\overline{d}d$
- $15 \oplus 1$ neutral $\overline{s}s$
  - singlets combine into π_0, η, η'
  - SU(4) prevents 15's mixing

QCD light pseudoscalar spectrum



Staggered pseudoscalar spectrum



Twisted mass and lattice artifacts

For the two flavor continuum theory

- $\psi \to e^{i\theta\gamma_5\tau_3}$ is a valid symmetry
- $\overline{\psi}\psi \to \cos(\theta)\overline{\psi}\psi + i\sin(\theta)\overline{\psi}\gamma_5\tau_3\psi$

Generalized mass term $m\overline{\psi}\psi$

- $m\ \overline{\psi}\psi + \mu\ i\overline{\psi}\gamma_5\tau_3\psi$
  - equivalent to theory with mass $\sqrt{m^2+\mu^2}$

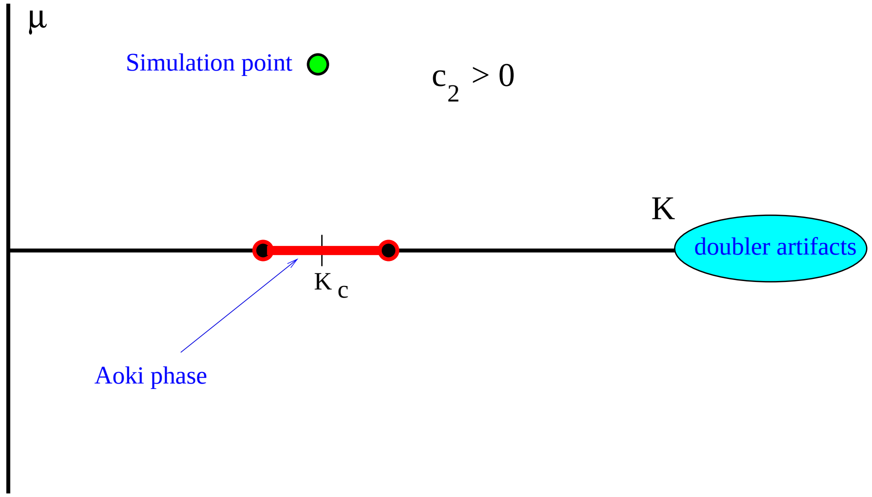
Lattice artifacts break this symmetry

With two degenerate Wilson fermions

- artifacts can generate flavor and CP violation
- Aoki (1983): pions interpreted as
  - two Goldstone bosons from flavor breaking
  - a third massless state at a second order boundary

μ: "Magnetic field" conjugate to Aoki phase

- allows continuation around Aoki phase



Motivations for twisted mass:

- $O(a)$ lattice artifacts can be tuned to cancel
- faster than overlap or domain wall

Summary

- Staggered: very fast, but spectrum not QCD
- Wilson: fast, but bad chiral properties
- Twisted mass: fast, needs fine tuning
- Domain wall: improved chiral symmetry at a cost
- Overlap: slow but elegant chiral properties

Which action is best?

All allow various “improvements” (smearing, . . .)

All are in current use; pick your favorite

More details: