T Symmetry and Its Violation

Part 2: Anomalies and Axions

(5) Hard Theory of the Chiral Anomaly

"History", Fermion Path Integral, Noether + Jacobian, Zero Modes, Result

"History" (Feynman graph calculation)

 $\pi^0 \rightarrow \gamma \gamma$: Discrepancy between formal symmetry and explicit quantum field theory calculation (and experiment)

For massless f, classically, $\partial_{\mu}(\bar{f}\gamma^{\mu}\gamma^{5}f) = 0$. This corresponds, according to Noether's theorem, to conservation of $f_{L} - f_{R}$ number.

A subtle calculation, following the strict rules of quantum field theory, gives a non-zero answer:



 $\Delta L \propto \eta \epsilon^{\rho \sigma \mu \nu} F_{\rho \sigma} F_{\mu \nu}$



$$p \int \frac{d^4q}{q^3} \rightarrow p \int \frac{d^4q \, q k_1 k_2}{(q^2)^3} = \text{finite}$$

BUT before claiming any cancellations, we need to regulate our integrals.

$$p \int \frac{d^4q}{q^3} \rightarrow p \int \frac{d^4q \, q k_1 k_2}{(q^2)^3} = \text{finite}$$

A convenient (gauge invariant) regularization was invented by Pauli and Villars. One introduces a very massive spin 1/2 boson bto echo f. It comes into the loop with the opposite sign!

This removes the divergence, but since $\partial_{\mu} \bar{b} \gamma^{\mu} \gamma^5 b \propto M$ we get



Remarkably, this (undressed) triangle graph gives the whole answer.

A More Profound Approach: Fermion Path Integral

$$Z = \int D\bar{\psi} D\psi \exp i \int d^4x L$$

Central result:
$$\int D\bar{\psi} D\psi \exp i \int d^4x \bar{\psi} \mathcal{O} \psi = \text{Det}\mathcal{O}$$

(Like many path integrals, this is a heuristic device. *Vigor first, rigor later*.)

Noether Procedure and Jacobian

$$\psi \rightarrow e^{i\alpha\gamma^5}\psi; \bar{\psi} \rightarrow \bar{\psi}e^{i\alpha\gamma^5}$$

$$L \rightarrow L + i\partial_{\mu}\alpha \,\bar{\psi}\gamma^{\mu}\gamma^{5}\psi$$

Invariance of Z, assuming invariance of the measure, $\Rightarrow \partial_{\mu} \bar{\psi} \gamma^{\mu} \gamma^{5} \psi = 0$, i.e. conservation law.

But there's also a Jacobian!

Det
$$e^{2i\alpha\gamma^5}$$
 (functional determinant)

$$= \exp \operatorname{Tr} \log e^{2i\gamma^5 \alpha} = 2i\alpha \exp \operatorname{Tr} \gamma^5$$

(Here we take $\alpha = \text{constant.}$)

Evaluation 1: Modes

$$\psi = \sum_{n} \psi_{n} a_{n} ; i \gamma^{\mu} \nabla_{\mu} \psi_{n} = \lambda_{n} \psi_{n}$$

$$i\gamma^{\mu}\nabla_{\mu}(\gamma^{5}\psi_{n}) = -\lambda_{n}(\gamma^{5}\psi_{n})$$

 \Rightarrow For $\lambda_n \neq 0 \ \psi_n$ and $\gamma_5 \psi_n$ are orthogonal, and thus do not contribute to the trace.

We get contributions (of opposite sign) from left- and right-handed zero modes, and that's all.

⇒ The determinant "counts" the difference between the number of 0-modes with L versus R chirality. **Evaluation 2: Singularity**

$$\lim_{M \to \infty} \lim_{x \to y} \int d^4 y \, \operatorname{tr} \bar{\psi}(y) \gamma^5 e^{(\nabla^{\mu} \gamma_{\mu})^2 / M^2} \psi(x)$$

$$\lim_{M \to \infty} \int d^4k \ e^{-ikx} \operatorname{tr} \gamma^5 e^{(\nabla^{\mu}\gamma_{\mu})^2/M^2} e^{ikx}$$

 $\lim_{M\to\infty} \int M^4 d^4 s \operatorname{tr} \gamma^5 e^{(\nabla^{\mu}\gamma_{\mu} + iMs^{\mu}\gamma_{\mu})^2/M^2}$

$$\lim_{M\to\infty} \int M^4 d^4 s \operatorname{tr} \gamma^5 e^{(\nabla^{\mu}\gamma_{\mu} + iMs^{\mu}\gamma_{\mu})^2/M^2}$$

$$(\nabla^{\mu}\gamma_{\mu} + iMs^{\mu}\gamma_{\mu})^2 = \nabla^2 + [\gamma_{\mu}, \gamma_{\nu}]F^{\mu\nu} + iMs_{\mu}\nabla^{\mu} - M^2s^2$$

⇒ The only way to get enough γ matrices without taking on too many factors 1/M is to use the F^2 term exactly twice! Thus, we reproduce (and justify) the "pure triangle" result —

and also connect field topology to zero modes!

(6) $U_A(1)$ in QCD

A Quantum Field Theory Showcase, and a Challenge

(6a) Why We Don't Want $U_A(1)$

The Missing NG Boson

In the approximation $m_u = m_d = 0$, QCD appears (classically) to exhibit chiral flavor $U(2)_L \otimes U(2)_R = U(1)_B \otimes U(1)_A \otimes SU(2)_L \otimes SU(2)_R$ symmetry, allowing independent unitary transformations of (u_L, d_L) and (u_R, d_R) .

The diagonal $U(1)_B$ - a common phase for all four - corresponds to baryon number conservation. A symmetry-reducing condensate $\langle \bar{u}_L u_R \rangle = \langle \bar{d}_L d_R \rangle = v \neq 0$ develops.

(Once that was an ingenious hypothesis; now it is a computed fact.)

The spontaneous breaking $SU_L(2) \otimes SU_R(2) \rightarrow SU_{\Delta}(2)$ results in three Nambu-Goldstone bosons. This is the basis for a very successful theory of pions. This condensate also breaks *axial baryon number* $U_A(1)$ symmetry, under which left handed and right handed quarks acquire opposite phases. However, in this case there is no suitable candidate for the (approximate) Nambu-Goldstone boson.

Including effects of small quark masses, and strangeness, does not improve the situation.

(6b) Why We Don't Get $U_A(1)$

Quantum Anomaly at Work

There is an *anomaly* in the axial baryon number current:



 $\partial_{\mu}(\bar{u}\gamma_{5}\gamma^{\mu}u + \bar{d}\gamma_{5}\gamma^{\mu}d) \equiv \partial_{\mu}j^{5\mu} \propto \mathrm{Tr} \ \epsilon^{\alpha\beta\gamma\delta}G_{\alpha\beta}G_{\gamma\delta}$

First reaction: So what? There's a modified, conserved current:

Second thought: K^{μ} is gauge dependent; it might be singular.

Central point (looking at the contribution from infinity):

You can have field configurations with finite weight (finite $\int \operatorname{Tr} G^{\mu\nu}G_{\mu\nu}$) for which $\int K^{\mu}$ diverges. This can occur if $G_{\mu\nu} \rightarrow o$ due to cancellations between $\partial_{\mu}A_{\nu}$ and $[A_{\mu}, A_{\nu}]$ that do not occur in K^{μ} . $G_{\mu\nu} \rightarrow 0$ indicates a pure gauge configuration. This on the verge of being trivial.

But topology saves the day. The potential "singularity" (reflected in K^{μ}) might be locally trivial, but globally nontrivial.

This begins a long and interesting mathematical story. We've already previewed the punch lines: Finite-action configurations can contribute to $\int Tr \, \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta}G_{\gamma\delta}$, and thus (according to the anomaly) spoil conservation of $j^{5\mu}$.

The contributions come in integer multiples of $16\pi^2$. Thus the physical effect of $\Delta L_{Euc.} = i \theta (16\pi^2)^{-1} \int Tr \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta} G_{\gamma\delta}$ is 2π periodic in θ .

(6c) A Pictorial Representation

The 't Hooft Vertex

In topologically non-trivial backgrounds, there are (4d) "zero energy" solutions of the Dirac equation. These must be saturated, in order to get a non-zero answer (since the fermion integral yields a determinant).

The zero modes have one chirality in the imaginary-time past, another in the imaginary-time future.

We can capture all this in a picture worth a thousand words, the 't Hooft vertex:



 $e^{i\theta}$
(6d) T is for Trouble

heta Bites

Given the 't Hooft vertex, we can't remove the overall phase of the quark mass matrix, nor a potential θ term in the QCD/SM Lagrangian.

We can shuffle from one to the other, as also indicated directly by the anomaly equation. These interactions potentially violate T (and P), but do not change flavor! Thus they are poised to contribute directly, and strongly, to electric dipole moments.

Phenomenologically, one deduces $|\theta_{\text{effective}}| \leq 10^{-10} \text{ or so.}$

Why??

(Note: It is *not* required anthropically.)

[recollection]

Grand Conclusion

The idea of T violation solely due to a complex phase in the weak currents is **remarkably** successful.

But its success serves to highlight a deep theoretical question:

Why is the θ term of QCD is so small?



Coincidence? | Think Not!





 $e^{i\theta}$

(7) Axion Foundations

T is for Tantalizing

(7a) Two Models of Quark Masses

Turning θ Dynamical

(Minimal) standard model:





(Note: In either scheme, the observable CKM mixings are complicated functions of the basic $g, h, \langle \phi^{(\mu)} \rangle$ (and Z).)

In the minimal (one Higgs doublet) standard model, the overall phase of the quark mass matrix is a definite function of the parameters g and h, namely Arg Det g Det h. The phase of ϕ cancels.

Thus the smallness of strong P, T violation goes unexplained; it requires "fine tuning".

In the variant model the phase of the quark mass matrix becomes a *dynamical variable*. It is Arg Det g Det $\langle \phi_1 \rangle \langle \phi_2 \rangle$.

This opens a possibility to explain the smallness of strong P, T violation dynamically.

(7b) Weak-Scale Axion Model

A Nice Try

Promoting $\theta_{\rm eff.}$ to a dynamical variable is not enough. We want to make sure that it settles down close to 0!

This requires that the energy associated with the total phase of $\langle \phi_1 \rangle \langle \phi_2 \rangle$ should be determined primarily by $\theta_{\rm eff.}$

To insure that, we impose Peccei-Quinn (PQ) symmetry: The classical Lagrangian should be invariant under $\phi_1 \rightarrow e^{i\sigma}\phi_1; \phi_2 \rightarrow e^{i\sigma}\phi_2.$

This forbids, in particular, terms proportional to $\phi_1 \phi_2$ or $(\phi_1 \phi_2)^2$, which might otherwise have appeared.

Now consider how the total phase of $\phi_1 \phi_2$ affects the vacuum energy:



$$V(\alpha) \sim -\cos \alpha \Lambda_{\rm QCD}^4$$

 $\theta_{\rm eff.}$, the phase that multiplies the 't Hooft vertex in vacuum, is exactly what figures in the energy.

And the energy is indeed minimized at the symmetry point $\theta_{\text{eff.}} = 0$.

(Note that on general grounds, points of enhanced symmetry have a better chance to be stationary points.) Spontaneous breaking of a (global) symmetry is accompanied by a Nambu-Goldstone boson, with characteristic properties:

It is massless.

It couples gradiently to the symmetry current.

The strength of coupling is inversely proportional to the scale of symmetry breaking.

$$\phi \rightarrow e^{i \alpha} \phi \qquad \langle \phi \rangle \; = \; F$$

$$\mathcal{L} \sim \partial_{\mu} \phi \, \partial^{\mu} \phi^{*} + \mathcal{L}_{\text{int.}} \approx F^{2} \, \partial_{\mu} \alpha \, \partial^{\mu} \alpha + \partial_{\mu} \alpha rac{\delta \mathcal{L}}{\delta \partial_{\mu} \alpha}$$

Noether current

$$F lpha \; \equiv \; b \qquad \qquad {\cal L} \; \sim \; \partial_\mu b \, \partial^\mu b + {1 \over F} \, \partial_\mu b \, j^\mu$$

For axions that basic structure is augmented by *intrinsic* breaking through the anomalies (i.e., roughly, the 't Hooft vertex).

This gives rise to a non-zero mass, and to non-derivative couplings.



$$V(\alpha) \sim -\cos \alpha \Lambda_{\rm QCD}^4$$

$$m_a^2 \sim rac{1}{F^2} rac{d^2 V(lpha)}{d lpha^2} \sim rac{\Lambda_{
m QCD}^4}{F^2}$$

$$\mathcal{L}_{\rm kin} = \frac{1}{2} \left(g^{\mu\nu} \partial_{\mu} a \partial_{\nu} a + m^2 a^2 \right)$$

$$m_a^2 \, \sim \, \frac{(\Lambda_{QCD})^4}{F^2}$$

$$\mathcal{L}_{\text{int}} \sim -\frac{a}{F} \left(c_G \,\alpha_s G_{\mu\nu} \tilde{G}^{\mu\nu} + c_\gamma \,\alpha F_{\mu\nu} \tilde{F}^{\mu\nu} + d_q \sum_q m_q \,\bar{q}\gamma_5 q + d_l \sum_l m_l \,\bar{l}\gamma_5 l + \dots \right)$$

Larger F means smaller mass and weaker interactions.

We will be using very large values of F! ($\sim 10^{12}$ GeV)



(7c) General Axion Models

"Invisible Axions"

In thinking about unification, we now routinely contemplate mass scales well beyond the weak scale.

Could Peccei-Quinn symmetry be broken at a large scale?

It can indeed. For example, let us introduce a standard model singlet complex scalar field ρ that both

transforms non-trivially under PQ symmetry, and

acquires a large vacuum expectation value F.

We can have a colored "Quark" that gets a large mass solely from coupling to the complex field, in the form $\Delta L = g\rho \bar{Q}_L Q_R + h.c.$, where all other terms are invariant under

$$\rho \to e^{i\sigma}\rho, Q_L \to e^{i\sigma}Q_L.$$

In other words, ρ and Q_L have PQ charge 1, while all other fields are neutral.

Many variants are possible. They all predict a similar pattern of axion mass and couplings.

(8) Axion Ferment

A Brief, Biased Introduction

(8a) A Survey of Constraints

- •Axions can provide a new method of energy transport and loss for stars, affecting their evolution.
- •Axions can allow light to "shine through walls".
- •Axions can help rotating black holes spin down.
- The Sun can emit axions.
- Axions are produced abundantly in the big bang




(8b) Axion Cosmology

A (The?) Source of Dark Matter



Direction around \equiv "**Phase**" -> axion

The order parameter field settles down close to the bottom of the well, but there are residual oscillations in its phase.

The residual oscillations can also be considered as a collection of particles, the phase field's quanta. This is the cosmic axion background.

For the residual mass density, one finds $\rho_{\rm axion} \propto \sim F \, \sin^2 \theta_0$

$$egin{aligned} \mathcal{L} &= \sqrt{g}\,(\,rac{1}{2}\,g^{\mu
u}\,\partial_\mu\phi\,\partial_
u\phi\,-\,V(\phi)\,) \ g_{00} &= 1\,; \ \ g_{ij} &= -a(t)^2\delta_{ij}\,; \ \ \sqrt{g} &= a^3 \ (\sqrt{g}\dot{\phi}g^{00})^\cdot &= -\sqrt{g}rac{\delta V}{\delta\phi} \end{aligned}$$



When $3\frac{\dot{a}}{a} \gg m_a$, the field is stuck.

After entering the adiabatic regime:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2(T(t))\phi = 0$$

$$\phi(t) \, pprox \, A(t) e^{i \int^t d\tau \, m(\tau)}$$
 adiabatic ansatz

$$2\dot{A}m + A\dot{m} + 3\frac{\dot{a}}{a}\dot{A}m = 0$$
 "out of phase" terms

$$(a^3 A^2 m)^{\cdot} = 0$$
 adiabatic invariant



Dark Matter from Axion Strings with Adaptive Mesh Refinement

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Axions are hypothetical particles that may explain the observed dark matter (DM) density and the non-observation of a neutron electric dipole moment. An increasing number of axion laboratory searches are underway worldwide, but these efforts are made difficult by the fact that the axion mass is largely unconstrained. If the axion is generated after inflation there is a unique mass that gives rise to the observed DM abundance; due to nonlinearities and topological defects known as strings, computing this mass accurately has been a challenge for four decades. Recent works, making use of large static lattice simulations, have led to largely disparate predictions for the axion mass, spanning the range from 25 microelectronvolts to over 500 microelectronvolts. In this work we show that adaptive mesh refinement (AMR) simulations are better suited for axion cosmology than the previously-used static lattice simulations because only the string cores require high spatial resolution. Using dedicated AMR simulations we obtain an over three order of magnitude leap in dynamic range and provide evidence that axion strings radiate their energy with a scale-invariant spectrum, to within $\sim 5\%$ precision, leading to a mass prediction in the range (40,180) microelectronvolts.

40 - 180 $\mu eV \approx$ 10 - 45 GHz



(8c) ALPHA Haloscope

New Ideas - Getting There!



$$\mathcal{L} = \kappa a \vec{E} \cdot \vec{B} = \frac{\kappa}{2} a \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

B induces charge

$$\nabla \cdot E = -\kappa \nabla a \cdot B$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{ ADMX, abracadabra,}$$

$$\nabla \cdot B = 0$$

$$\nabla \times B = 0$$

$$\nabla \times B = \frac{\partial E}{\partial t} + \kappa (\dot{a}B + \nabla a \times E)$$

E induces current
(surface Hall effect)

Thus, in the presence of a background magnetic field, an axion field mixes with the photon, and pumps energy into electromagnetic fields.

One can design "antennas" to encourage the pumping, by exploiting *resonance*.

In order to get resonance between the cosmic axion background and the electromagnetic field, we must use a space-dependent medium (modifying k), or impart mass to the photon, or both.

Traditional cavity resonators exploit $k \sim \frac{1}{L}$. Of course, this limits the size!

The α strategy is to exploit plasmon-like response in a wire metamaterial.

This imparts mass to the photon, and relieves the constraint on size.

Thin wire metamaterials

- One of the first metamaterials
- Plasma frequency determined by two factors: effective electron number density and mass
- Wires mutually induct, changing the plasma frequency











Several groups (including α +) have been exploring a different approach, close to but distinct from α .

They propose to use dielectrics, as opposed to metallic wires, in ~ periodic arrays, thus cancelling off the "second k" in (k, k).

Tunable Photonic Crystal Haloscope for High-Mass Axion Searches



FIG. 7. Photo of the photonic crystal cavity and frequency tuning system for demonstration.

Base, Youn, Jong arXiv : 2205.08885



FIG. 5. Auxetic behavior of rotating rigid squares. With the vertices connected together, the relative rotation of the unit cells forms more open or closed structures.

One might also obtain enhancements through use of soft paramagnets, which concentrate B field.



[unused slides follow]

(A more refined analysis should take all forms of spontaneous symmetry breaking into account simultaneously. For example, there is some $a-\pi^{0}$ mixing.)

(8d) Emergent Axions

Synergy of Ideas

Axion Physics In Condensed-Matter Systems

Key points

- 3D insulators can be topologically characterized by the value of their bulk axion field.
- Axion fields introduce additional terms in Maxwell's equations for condensed-matter systems.
- The microscopic expression for the axion field in a crystal is given by the non-Abelian Chern–Simons integral, which depends on the Berry connection matrix of the band structure.
- In strong 3D topological insulators, a half-quantized surface Hall effect appears when the surface states are gapped, together with linear magnetoelectric coupling in their bulk.
- The axion insulator state can be realized in antiferromagnetic insulators without external fields.
- Materials with a non-trivial axion field can be used in dark-matter detectors and non-reciprocal thermal emitters.

Nenno et al.; Nature Reviews Physics 2, 682 (2020)

Quantized Faraday and Kerr rotation and axion electrodynamics of a 3D topological insulator

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Science 02 Dec 2016: Vol. 354, Issue 6316, pp. 1124-1127



(6) Axion Fundamentals

A New Kind of Particle
Photon mass situations:

Superconductor

Plasma

Hot gas-

Metal-

Semiconductor?

Metamaterial



Mobile charges resist electric fields, and make the photon massive



Concept and Prototype



Pathfinder Proposal

(8d) Haloscope Development

New Ideas - Getting There!