Group Singlets as Many-Body Scars

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Quantum Connections

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Quantum Many-Body Scars

- Over the past few years have been an active area in Condensed Matter Physics. Several reviews Serbyn, Abanin, Papic; Moudgalya, Bernevig, Regnault; Chandran, Iadecola, Khemani, Moessner
- Scars do not thermalize with the rest of the states and constitute a violation of the Eigenstate Thermalization Hypothesis.
- The Hilbert space breaks up into two sectors

$$\mathcal{H} = \mathcal{H}_{\mathrm{therm}} \oplus \mathcal{H}_{\mathrm{scar}}$$

- Schematic equidistant scar spectrum for a special scarred Hamiltonian: Serbyn et al.; Schecter and Iadecola

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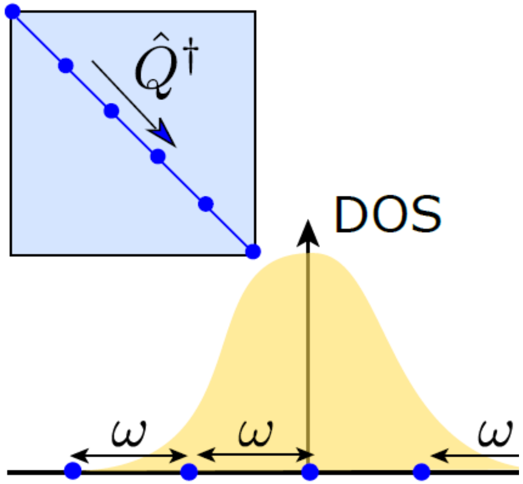
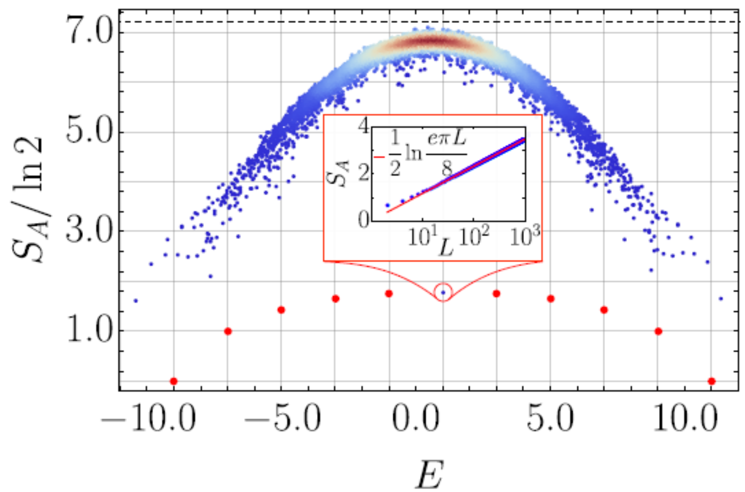




- The scars are characterized by lower entanglement entropy than the typical states.
- In a number of models, the scar sector is invariant under a "large" group whose rank is proportional to the number of lattice sites.

Pakrouski, Pallegar, Popov, IRK, PRL 125 (2020) 230602

Melonic O(N)^3 Tensor Model

- Quantum Mechanics of N^3 Majorana fermions
  IRK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

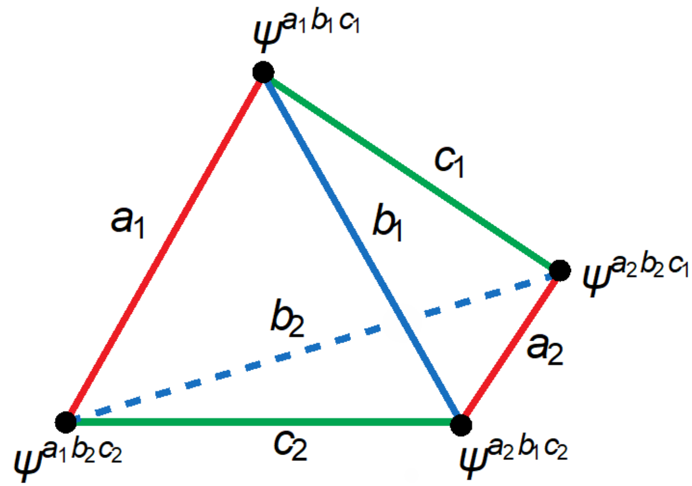
- Has O(N)$_a$xO(N)$_b$xO(N)$_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

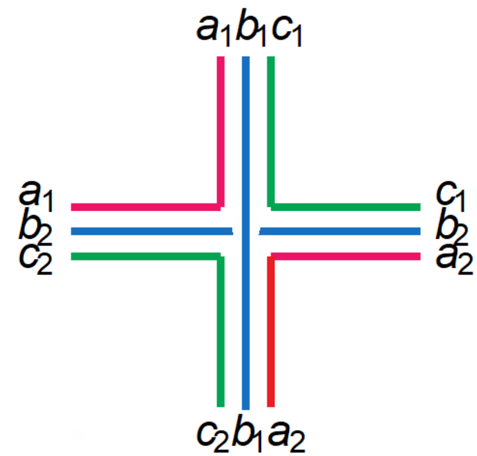
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}] , \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}] , \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.



- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

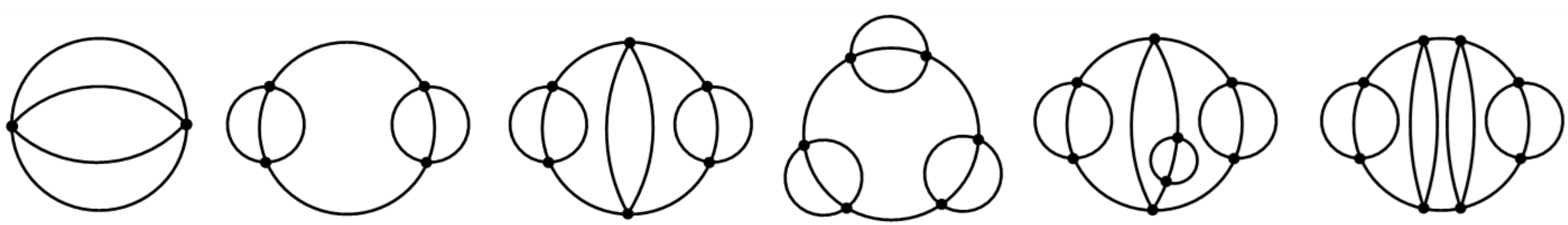
- No SO(N)^3 invariant states for odd N.
- Their number grows very rapidly for even N IRK, Milekhin, Popov, Tarnopolsky

N	# singlet states
2	2
4	36
6	595354780

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- Large N dynamics in the singlet sector is similar to SYK. Same melonic Schwinger-Dyson eqns.



The Hamiltonian

- Convenient to introduce operator basis which breaks the third O(N) to U(N/2)

$$\bar{c}_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} + i\psi^{ab(2k+1)}\right), \quad c_{abk} = \frac{1}{\sqrt{2}}\left(\psi^{ab(2k)} - i\psi^{ab(2k+1)}\right) ,$$

$$\{c_{abk}, c_{a'b'k'}\} = \{\bar{c}_{abk}, \bar{c}_{a'b'k'}\} = 0, \quad \{\bar{c}_{abk}, c_{a'b'k'}\} = \delta_{aa'}\delta_{bb'}\delta_{kk'} ,$$

$$a, b = 0, 1, \ldots, N-1, \text{ and } k = 0, \ldots, \tfrac{1}{2}N - 1$$

- The Hamiltonian couples N/2 sets of N^2 dof

$$H = 2\left(\bar{c}_{abk}\bar{c}_{ab'k'}c_{a'bk'}c_{a'b'k} - \bar{c}_{abk}\bar{c}_{a'bk'}c_{ab'k'}c_{a'b'k}\right)$$

- The Cartan generators of U(N/2) are

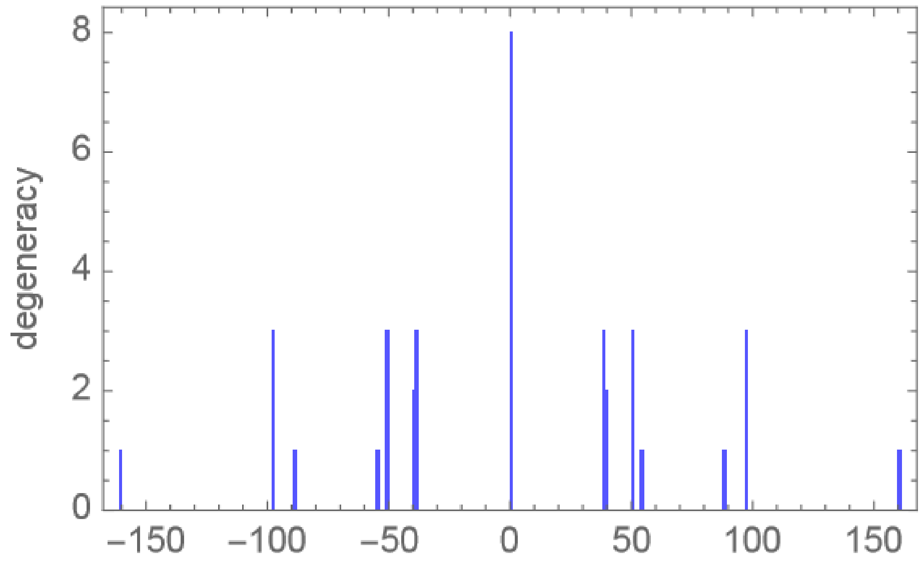
$$Q_k = \sum_{a,b} \frac{1}{2}[\bar{c}_{abk}, c_{abk}] \ , \qquad k = 0, \ldots, \frac{1}{2}N - 1$$

- For the oscillator vaccuum

$$c_{abk} \, |\text{vac}\rangle = 0 \ , \qquad Q_k \, |\text{vac}\rangle = -\frac{N^2}{2} \, |\text{vac}\rangle$$

- The $SO(N)^3$ invariant states appear in the sector where all these charges vanish: each set of N^2 qubits is at half filling.
- This reduces the number of states but it still grows rapidly. For N=4 there are 165636900, while for N=6 over 7.47 * 10^29

Singlet Energies for N=4



- For N=6, over 595 million states packed into energy interval <1932. The singlet gaps should be tiny. Pakrouski, IRK, Popov, Tarnopolsky
- To find the spectrum need a 108 qubit quantum computer. Requires a large number of gates.

From Tensor Models to Scars

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- The eigenstates of H form irreducible representations of the symmetry.

A Fermionic Matrix Model

- For N_3=2 this is a fermionic matrix model with symmetry $O(N_1) \times O(N_2) \times U(1)$

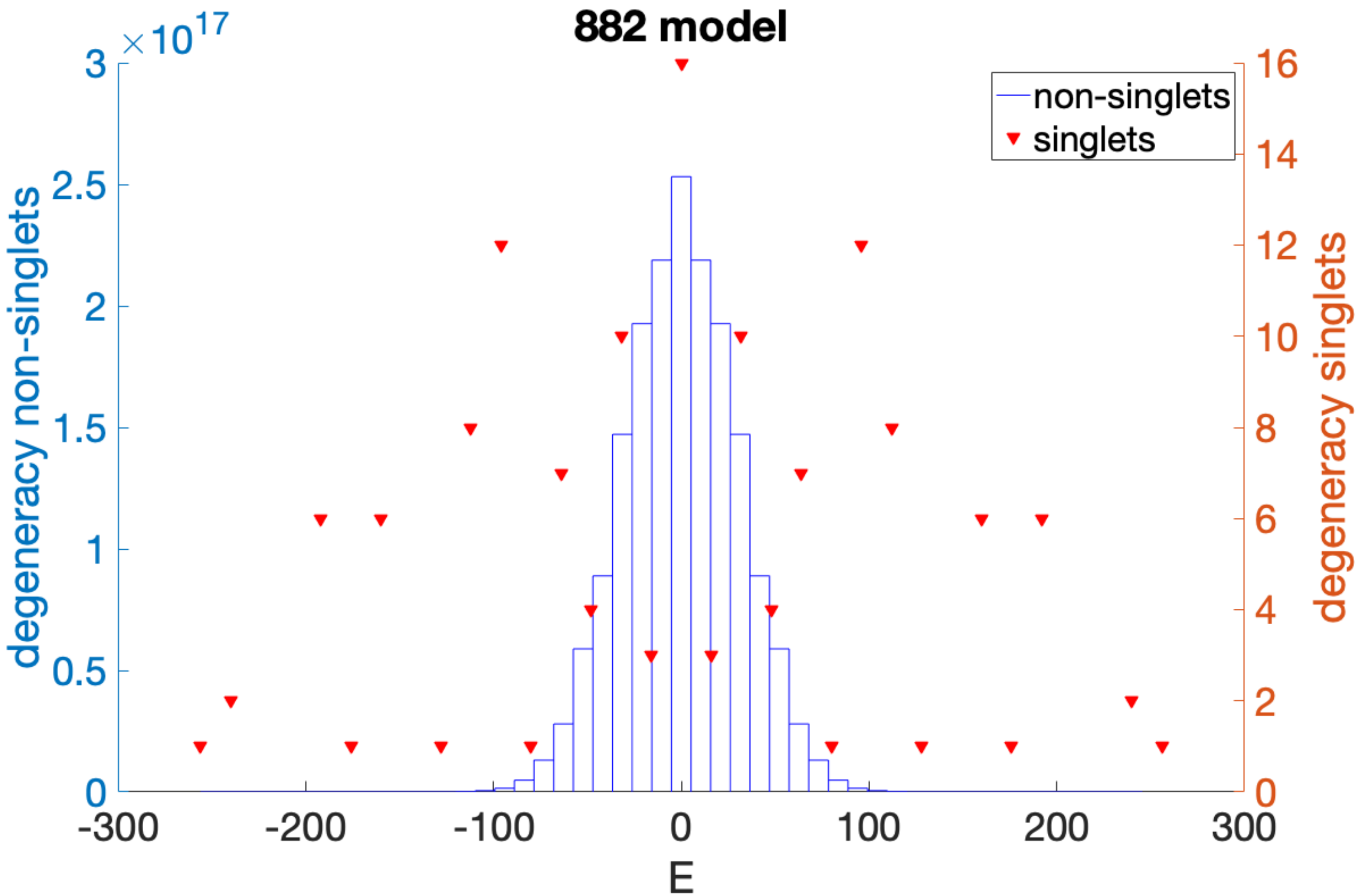
$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} + i \psi^{ab2} \right) , \quad \psi_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} - i \psi^{ab2} \right)$$

$$\{ \bar{\psi}_{ab}, \bar{\psi}_{a'b'} \} = \{ \psi_{ab}, \psi_{a'b'} \} = 0, \quad \{ \bar{\psi}_{ab}, \psi_{a'b'} \} = \delta_{aa'} \delta_{bb'}$$

- Describes qubits on a N_1 x N_2 lattice with non-local couplings. IRK, Milekhin, Popov, Tarnopolsky
- A useful example for studying bounds on eigenvalues of fermionic Hamiltonians. Hastings, O'Donnell

Complete Spectrum

- The $SO(N)^2$ singlets "scar" the histogram.



Towards Hubbard Model

- Can also think of the first index as labeling the lattice site, and the second as labeling spin. When N_2=2, there are two spin states, up and down. The model is beginning to resemble a non-local Hubbard model, but need to add quadratic hopping terms. Pakrouski, Pallegar, Popov, IRK
- Imaginary hopping terms are SO(N) generators

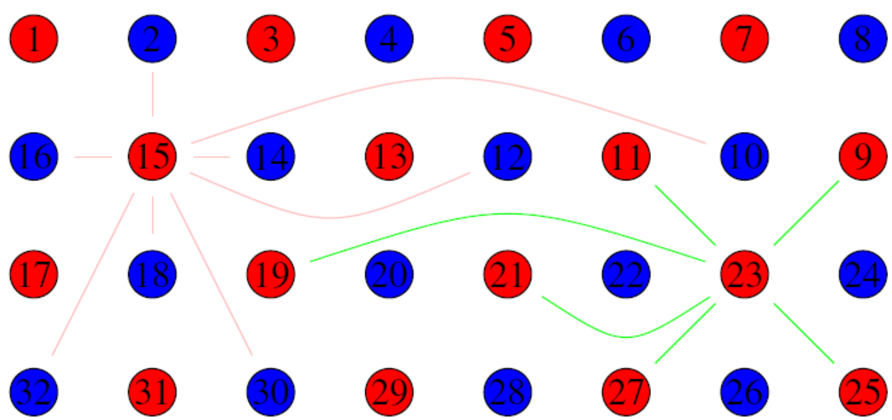
$$T^O_{kl} = i\sum_\sigma (c^\dagger_{k\sigma} c_{l\sigma} - c^\dagger_{l\sigma} c_{k\sigma}) \qquad \sigma = \uparrow, \downarrow$$

- Adding them to H keeps SO(N) singlets as eigenstates but mixes up the non-singlets.

- A simple transformation leads to a model with a real nearest neighbor hopping parameter:

$$H_{nn} = t \sum_{\langle ij \rangle \sigma} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.)$$

- This transformation is possible on a bi-partite lattice



Scars without Pain

- There are Hamiltonians that are not symmetric under a large group G, yet some of their eigenstates are invariant. These are the scars!
- Examples include (deformations of) the Hubbard model

$$T = it \sum_{j=1}^{N-1} \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(c^{\dagger}_{j\sigma} c_{j+1,\sigma} - c^{\dagger}_{j+1,\sigma} c_{j\sigma} \right) - \sum_{j=1}^{N} \left(\mu_{\downarrow} c^{\dagger}_{j\downarrow} c_{j\downarrow} + \mu_{\uparrow} c^{\dagger}_{j\uparrow} c_{j\uparrow} \right)$$

$$V = U \sum_{j=1}^{N} n_{j\uparrow} n_{j\downarrow} = U \sum_{j=1}^{N} c^{\dagger}_{j\uparrow} c_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j\downarrow} \ .$$

- The SO(4) symmetry of the Hubbard model is made manifest by introducing 4 Majorana fermions on each lattice site

$$c_{j\uparrow} = \frac{\psi_j^1 - i\psi_j^2}{\sqrt{2}} \; , \quad c_{j\downarrow} = \frac{\psi_j^3 - i\psi_j^4}{\sqrt{2}}$$

- For special values $\mu_\uparrow = \mu_\downarrow = \frac{U}{2}$

$$H_{Hub} = it \sum_j \sum_{A=1}^{4} \psi_j^A \psi_{j+1}^A - U \sum_j \psi_j^1 \, \psi_j^2 \, \psi_j^3 \, \psi_j^4$$

- Add symmetry breaking terms which annihilate the SO(N) singlets, e.g. TOT terms

$$\widetilde{H}_{\text{int}} = \sum_{\langle j,k \rangle} T_{jk} \left(i \sum_{A<B} r_{AB} \, \psi_j^A \psi_j^B \right) T_{jk}$$

Pseudospin

- The scars are states of maximum pseudospin or spin.
- After transforming to imaginary hopping, the pseudospin $\widetilde{\mathrm{SU}}(2)$ is generated by C.N. Yang, S.C. Zhang

$$\eta^+ = \sum_j c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} = \frac{1}{2} \sum_{j,\sigma,\sigma'} c^\dagger_{j\sigma} c^\dagger_{j\sigma'} \epsilon_{\sigma\sigma'}$$

$$\eta^- = (\eta^+)^\dagger, \quad \eta^3 = \frac{1}{2}(Q - N) \qquad Q = \sum_{i=1}^{N} n_i$$

$$n_{i\uparrow} = c^\dagger_{i\uparrow} c_{i\uparrow}\,, \quad n_{i\downarrow} = c^\dagger_{i\downarrow} c_{i\downarrow}\,, \quad n_i = n_{i\uparrow} + n_{i\downarrow}$$

- It commutes with the rotation group $\mathrm{SU}(2)$ and with the SO(N) that acts on the lattice index.

Eta-pairing states

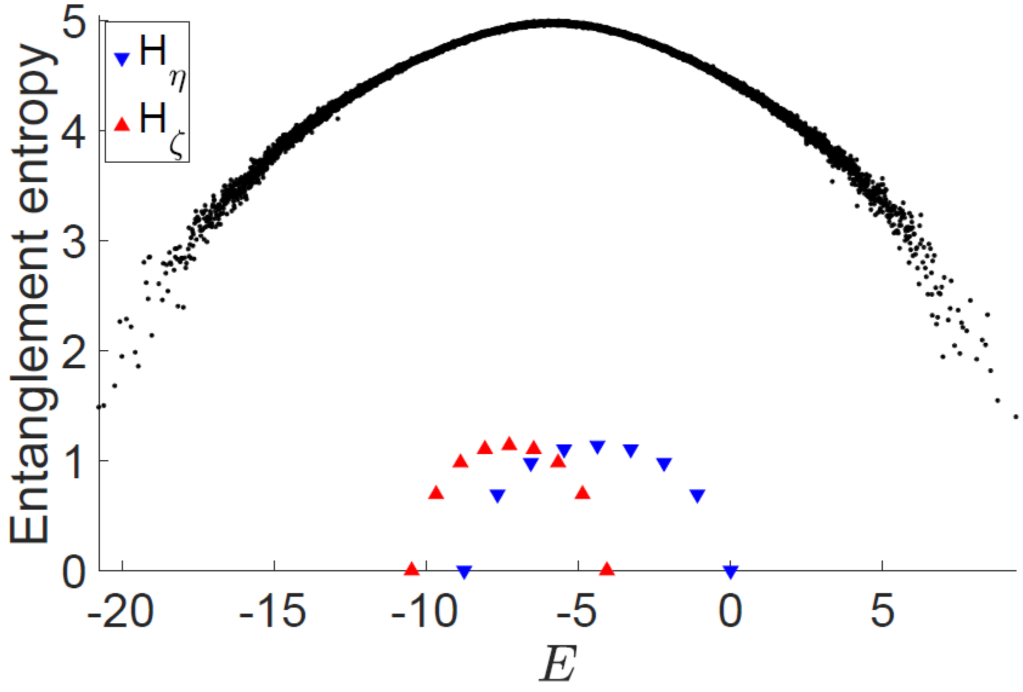
- There are N+1 states that are SU(2) invariant and form a multiplet of pseudospin N/2 Yang, Zhang

$$|n^\eta\rangle = \frac{(\eta)^n}{\sqrt{\frac{N!n!}{(N-n)!}}}|0\rangle \ , \qquad n = 0, \ldots, N$$

- The fact that they are also O(N) invariant was pointed out only recently. Pakrouski, Pallegar, Popov, IRK
- In fact, they are invariant under a bigger group $\widetilde{\mathrm{Sp}}(N)$
- The bi-partite entanglement entropy can be calculated analytically. Vafek, Regnault, Bernevig
- They are highly excited, equally spaced states that play the role of scars in the (deformed) Hubbard model. Mark, Motrunich; Moudgalya, Regnault, Bernevig

Low Entanglement

- The scar states are distinguished by their low entanglement entropy when the system is divided into two parts. For the 6 site chain:



Majorana Scars

- Consider a lattice system with an even number M of Majorana fermions on each lattice site Z. Sun, F. Popov, IRK, K. Pakrouski, arXiv: 2212.11914

$$\psi_j^A, A = 1, 2, \cdots, M \qquad \{\psi_i^A, \psi_j^B\} = \delta^{AB}\delta_{ij}$$

- The generators of SO(N) and SO(M) are

$$T_{ij} = \frac{1}{2}\sum_{A=1}^{M}[\psi_i^A, \psi_j^A], \quad J^{AB} = \frac{1}{2}\sum_{j=1}^{N}[\psi_j^A, \psi_j^B]$$

- Complex fermions

$$c_{j\alpha} = \frac{\psi_j^{2\alpha-1} - i\psi_j^{2\alpha}}{\sqrt{2}} \qquad \text{Cartan}: \quad h_\alpha = \sum_j c_{j\alpha}^\dagger c_{j\alpha} - \frac{N}{2}$$

Scars as SO(N) singlets

- Constructed by acting with

$$\text{Positive roots}: \quad \zeta^\dagger_{\beta\gamma} = \sum_j c^\dagger_{j\beta} c_{j\gamma}, \quad \eta^\dagger_{\beta\gamma} = \sum_j c^\dagger_{j\beta} c^\dagger_{j\gamma}$$

- For M=6 the generalizations of eta-pairing states are explicitly written as Nakagawa, Katsura, Ueda

$$|k_{12}, k_{13}, k_{23}\rangle = C_{\boldsymbol{k}}(N)(\eta^\dagger_{12})^{k_{12}}(\eta^\dagger_{13})^{k_{13}}(\eta^\dagger_{23})^{k_{23}}|0\rangle$$

$$k_T \equiv k_{12}+k_{13}+k_{23} \leq N \qquad C_{\boldsymbol{k}}(N) = \sqrt{\frac{(N-k_T)!}{N!k_{12}!k_{13}!k_{23}!}}$$

- There are $\binom{N+3}{3}$ such eta-states.

- There are also $\binom{N+3}{3}$ zeta-states:

$$|k_{12}, k_{13}, k_{23}\rangle^{\zeta} = C_{\boldsymbol{k}}(N)(\eta_{12}^{\dagger})^{k_{12}}(\zeta_{13}^{\dagger})^{k_{13}}(\zeta_{23}^{\dagger})^{k_{23}}|0^{\zeta}\rangle$$

$$|0^{\zeta}\rangle \equiv c_{1,M/2}^{\dagger}c_{2,M/2}^{\dagger}\cdots c_{N,M/2}^{\dagger}|0\rangle$$

$$k_T \equiv k_{12}+k_{13}+k_{23} \leq N$$

- They are generalizations of the spin N/2 states for M=4 (the usual Hubbard model).
- It is not hard to do the counting of SO(N) invariants for M>6, but the wave functions cannot be written as explicitly.

Entanglement Entropy of Eta States

- Divide the lattice into two disjoint subsets, the first consisting of N_1 sites, and the second of $N_2 = N - N_1$ sites.
- Split the vacuum $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ and eta-operators

$$\eta^{1\dagger}_{\alpha\beta} = \sum_{i=1}^{N_1} c^\dagger_{i\alpha} c^\dagger_{i\beta}, \quad \eta^{2\dagger}_{\alpha\beta} = \sum_{i=N_1+1}^{N} c^\dagger_{i\alpha} c^\dagger_{i\beta}$$

- Each subsystem has its own eta-states

$$|m_{12}, m_{13}, m_{23}\rangle_a = C_{\boldsymbol{m}}(N_a) \prod_{\alpha<\beta} (\eta^{a\dagger}_{\alpha\beta})^{m_{\alpha\beta}} |0\rangle_a$$

- Each eta-state $|\boldsymbol{k}\rangle \equiv |k_{12}, k_{13}, k_{23}\rangle$ may be written

$$|\boldsymbol{k}\rangle = C_{\boldsymbol{k}}(N) \sum_{m_{\alpha\beta}=0}^{k_{\alpha\beta}} \prod_{\alpha<\beta} \binom{k_{\alpha\beta}}{m_{\alpha\beta}} \left(\eta_{\alpha\beta}^{1\dagger}\right)^{m_{\alpha\beta}} \left(\eta_{\alpha\beta}^{2\dagger}\right)^{k_{\alpha\beta}-m_{\alpha\beta}} |0\rangle$$

- The reduced density matrix is

$$\rho_{\Sigma_1}(\boldsymbol{k}) = \sum_{m_{\alpha\beta}=0}^{k_{\alpha\beta}} \lambda_{\boldsymbol{k}}(\boldsymbol{m}) \, |\boldsymbol{m}\rangle_1 \, \langle \boldsymbol{m}|_1$$

$$\lambda_{\boldsymbol{k}}(\boldsymbol{m}) = \frac{C^{N_1}_{m_{12},m_{13},m_{23}} C^{N_2}_{k_{12}-m_{12},k_{13}-m_{13},k_{23}-m_{23}}}{C^{N}_{k_{12},k_{13},k_{23}}}$$

$$C^{N}_{a,b,c} \equiv \frac{N!}{a!\,b!\,c!\,(N-a-b-c)!}$$

from which the Entanglement Entropy follows.

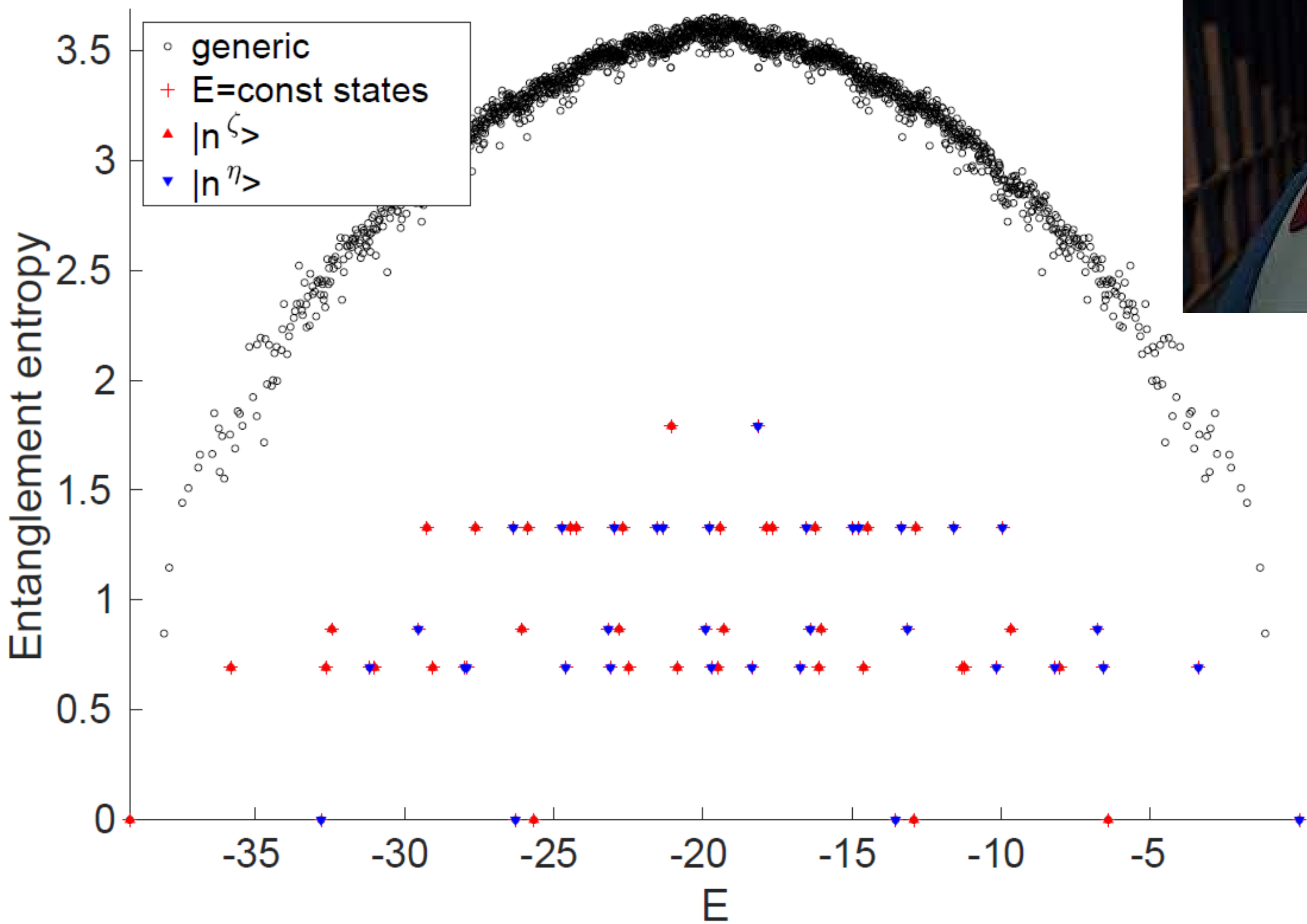
- In the limit of large N, we may replace the sum by an integral

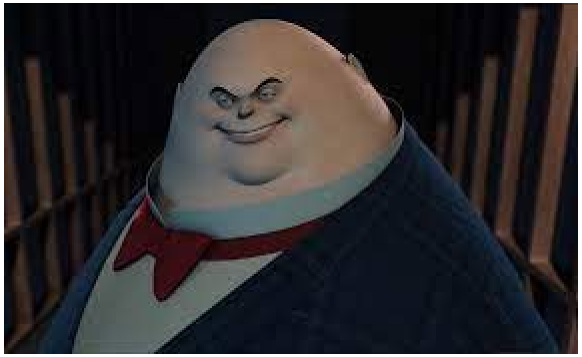
$$S_{\Sigma_1}(\boldsymbol{k}) \approx -\int d^3\boldsymbol{m}\, \lambda_{\boldsymbol{k}}(\boldsymbol{m}) \log \lambda_{\boldsymbol{k}}(\boldsymbol{m}) \sim \frac{3}{2}\log(N_1)$$

- As expected, the EE of scars exhibits a sub-volume growth, but for the other states it has the volume growth.

Spectrum of M=6 with 4 sites

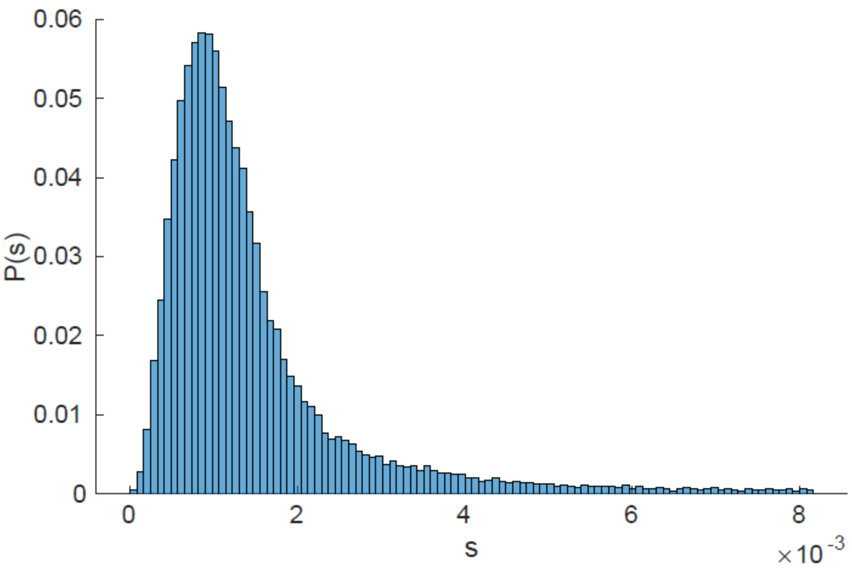
$$H = it\sum_{\langle j,k\rangle} T_{jk} + \sum_{\alpha}\mu_\alpha h_\alpha + U(2i)^{M/2}\sum_j \psi_j^1\psi_j^2\cdots\psi_j^M + \sum_{\langle j,k\rangle} T_{jk}\left(i\sum_{A<B} r_{AB}\,\psi_j^A\psi_j^B\right)T_{jk}$$

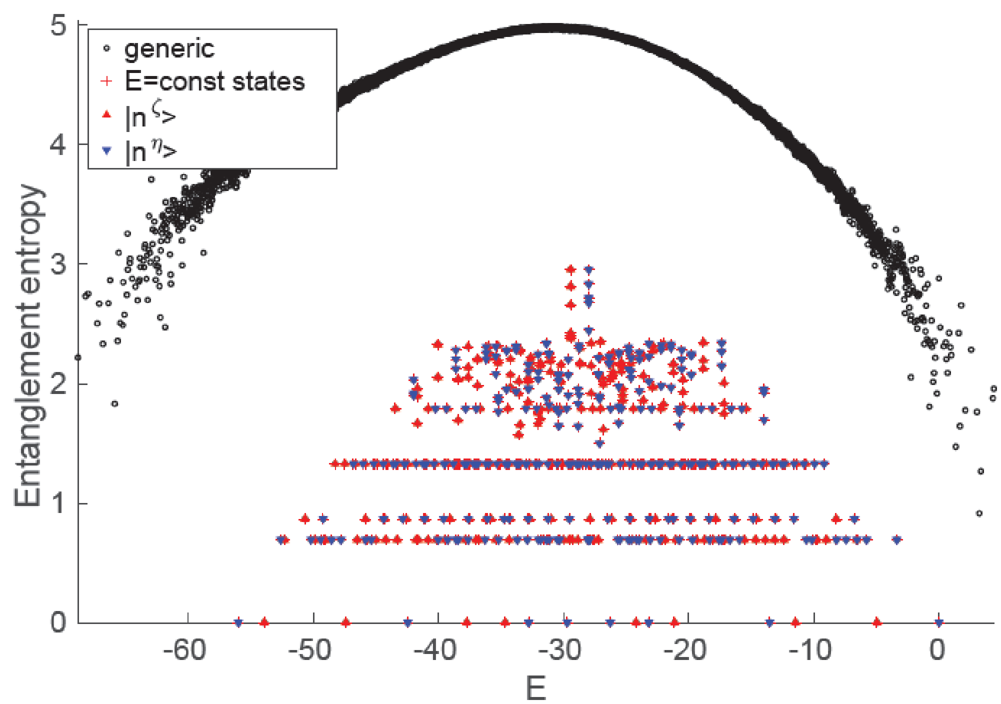




Humpty Dumpty is Frowning

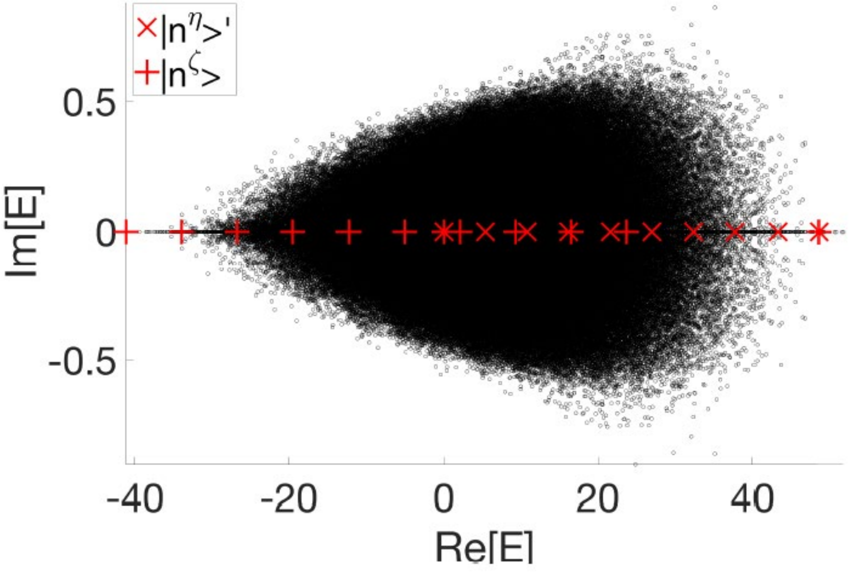
- For the Hubbard model, M=4, the eta-scars are equidistant in energy. For M=6, this is no longer the case. For M=8, there are some degeneracies that typically cannot be lifted by local interactions.





Non-Hermitian Hamiltonians

- The group theoretic approach to scars continues to work when non-Hermitian terms are added to the Hamiltonians, e.g. the tJU model. Pakrouski, Pallegar, Popov, IRK
- The energies of scars continue to be real



Comments

- The many-body scar states, which are invariant under the large Lie group acting on the lattice sites, are decoupled from all the non-singlet states. Only the latter thermalize.
- This decoupling is preserved by the TOT perturbations and may approximately survive some other perturbations.
- While the energies of scars are equidistant in a number of models, this is not generally true.
- The Group singlet approach to scars applies to non-Hermitian Hamiltonians.
- Need a deeper understanding of the general principles behind the scars (see recent work by Moudgalya and Motrunich).