Di Vecchia – 80 fest

NORDITA

Nordic Institute for Theoretical Physics

Loop calculations in string theory

(i.e. my first collaborations with Paolo)

Alberto Lerda

Università del Piemonte Orientale

and INFN – Torino, Italy

UNIVERSITÀ DEL PIEMONTE ORIENTALE

INFN TORINO

Stockholm, 15th May 2023

Plan of the talk

1. Multi-loop calculations in string theory
   (with Paolo + M. Frau + S. Sciuto + K. Hornfeck + F. Pezzella)

2. Field-theory limit of multi-loop string amplitudes
   (with Paolo + L. Magnea + R. Marotta + R. Russo)

Part 1

1987-1990

N-STRING VERTEX AND LOOP CALCULATION IN THE BOSONIC STRING

P DI VECCHIA
NORDITA, Blegdamsvej 17,2100 Copenhagen Ø, Denmark

M FRAU
Sezione di Torino dell' INFN, Torino, Italy

A LERDA
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

S SCIUTO*
Dipartimento di Fisica dell' Universita' di Napoli, Napoli, Italy and Sezione di Torino dell' INFN, Torino, Italy

Received 29 July 1987

A SIMPLE EXPRESSION FOR THE MULTILOOP AMPLITUDE IN THE BOSONIC STRING

P Di VECCHIA [a], M FRAU [b], A LERDA [c,1] and S SCIUTO [d,b,2]
[a] Nordita, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
[b] Sezione di Torino dell'INFN, I-10125 Turin, Italy
[c] Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA
[d] Dipartimento di Fisica dell' Universita' di Napoli, I-80125 Naples Italy

Received 18 September 1987

BRST INVARIANT OPERATOR FORMALISM FOR THE SUPERSTRING

P. DI VECCHIA, K. HORNFECK [1]
Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

M. FRAU
INFN, Sezione di Torino, Cso M. D'Azeglio 46, I-10125 Turin, Italy
and Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

A. LERDA [2]
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

and

S. SCIUTO [3]
Dipartimento di Scienze Fisiche dell'Università di Napoli, I-80125 Naples, Italy
and INFN, Sezione di Napoli, Mostra d'Oltremare Pad. 19, I-80125 Naples, Italy

Received 8 February 1988

N-STRING, g-LOOP VERTEX FOR THE BOSONIC STRING

P. DI VECCHIA, K. HORNFECK [1]
Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

M. FRAU
INFN, Sezione di Torino, Corso M. D'Azeglio 46, I-10125 Turin, Italy
and Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

A. LERDA [2]
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

and

S. SCIUTO [3]
Dipartimento di Scienze Fisiche dell'Università di Napoli and INFN, Sezione di Napoli, Mostra d'Oltremare Pad 19, I-80125 Naples, Italy

Received 9 March 1988

N-STRING, g-LOOP VERTEX FOR THE FERMIONIC STRING

P. DI VECCHIA, K. HORNFECK [1]
Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

M. FRAU
INFN, Sezione di Torino, Cso M. D'Azeglio 46, I-10125 Torino, Italy
and Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

A. LERDA [2,3]
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

and

S. SCIUTO [4]
Dipartimento di Scienze Fisiche dell'Università di Napoli, and INFN, Sezione di Napoli, Mostra d'Oltremare Pad 19, I-80125 Naples, Italy

Received 10 June 1988

N-POINT g-LOOP VERTEX FOR A FREE BOSONIC THEORY WITH VACUUM CHARGE Q

P. DI VECCHIA and F. PEZZELLA [1]
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

M. FRAU
INFN, Sezione di Torino, Cso M. D'Azeglio 46, I-10125 Torino, Italy

K. HORNFECK
Laboratoire d'Annecy-le-Vieux de Physique des Particules, Chemin de Bellevue BP 110, F-74941 Annecy-le-Vieux Cedex, France

A. LERDA [2]
Center for Theoretical Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

S. SCIUTO [3]
Dipartimento di Scienze Fisiche dell'Università di Napoli and INFN, Sezione di Napoli, Mostra d'Oltremare Pad. 19, I-80125 Napoli, Italy

Received 20 December 1988

N-POINT g-LOOP VERTEX FOR A FREE FERMIONIC THEORY WITH ARBITRARY SPIN

P. Di VECCHIA and F. PEZZELLA*
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

M. FRAU
INFN, Sezione di Torino, Cso M. D'Azeglio 46, I-10125 Torino, Italy

K. HORNFECK
Laboratoire d'Annecy-le-Vieux de Physique des Particules, Chemin de Bellevue BP 110, F-74941 Annecy-le-Vieux Cedex, France

A. LERDA**
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

S. SCIUTO***
Dipartimento di Scienze Fisiche dell' Università di Napoli and INFN, Sezione di Napoli, Mostra d' Oltremare Pad. 19, I-80125 Napoli, Italy

Received 2 August 1989

Part 1

- In the late 80's, after the first "string revolution" following the Green-Schwarz paper, there was great and rapid progress in string theory, mainly centered around performing perturbative computations on various backgrounds using the tools of CFT's.

- Two main approaches were developed:

  - a functional path-integral method (Polyakov; D'Hoker + Phong)

  - an operator formalism

- The operator formalism, in turn, was independently developed in three versions, which turned out to be equivalent:

  1. the "string operator formalism" (Harvard) (Alvarez-Gaumé + Gomez + Moore + Vafa)

  2. the "group theory approach" (CERN) (Neveu + West)

  3. the "sewing procedure" (NORDITA/NBI-Torino-Stony Brook)

     (Paolo + Frau+ AL + Sciuto, + Hornfeck, + Pezzella) + (Napoli group) + (Peteresen + Sidenius)

Part 1

- The operator formalism provides a nice constructive way to derive many geometric objects that appear in multi-loop string amplitudes

- It is very explicit and general, even on Riemann surfaces of higher genus

- Contrarily to what one may naively think, in the operator formalism one can do explicit checks of modular properties (like modular invariance)

- It allows to perform many explicit calculations

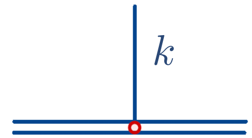
- However, for the superstring in the fermionic sectors it becomes rather involved and unpractical beyond 1-loop

Operator formalism & sewing procedure

- The old operator approach uses as basic ingredients:

Fubini + Veneziano;
Alessandrini + Amati;
Lovelace; Ademollo et al, …

  - Vertex operators:



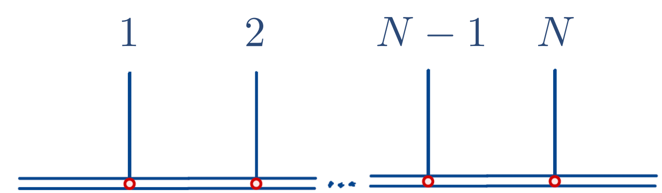
$$V \sim :\partial^{n_1}X\,\partial^{n_2}X\cdots \mathrm{e}^{\mathrm{i}k\cdot X}:$$

  - Propagators:

$$\mathcal{P} \sim \frac{1}{L_0 - 1}$$

to build amplitudes at tree level:

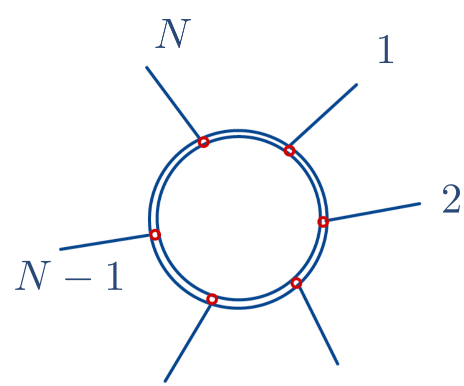
$$A_{\text{tree}}(1,\cdots,N) = \langle V_1\,\mathcal{P}\,V_2\cdots V_{N-1}\,\mathcal{P}\,V_N\rangle = \Big\langle$$



$$\Big\rangle$$

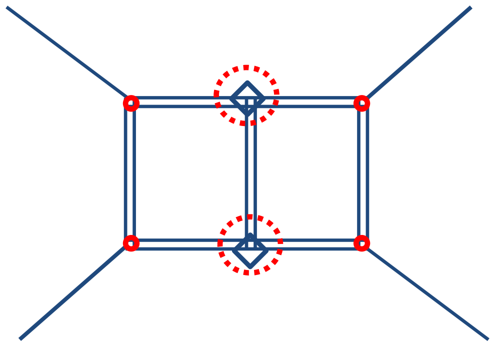
and at 1-loop:

$$A_{1-\text{loop}}(1,\cdots,N) = \text{tr}\Big(V_1\,\mathcal{P}\,V_2\cdots V_{N-1}\,\mathcal{P}\,V_N\Big) =$$

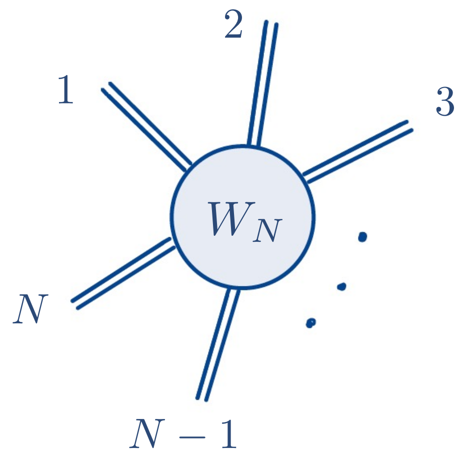


Operator formalism & sewing procedure

- However, this old operator formalism cannot work beyond 1-loop



- The solution is obtained with the construction of the N-Reggeon vertex (or N-string vertex) which provides the coupling at tree level among N arbitrary string states

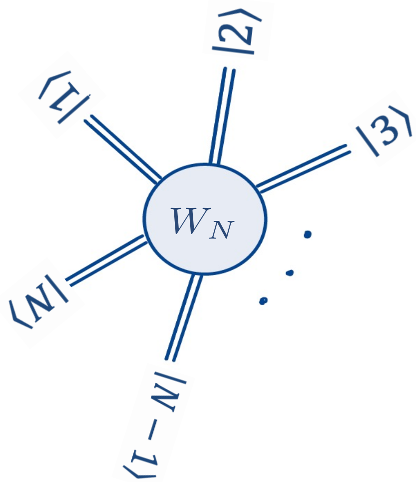


N-Reggeon vertex

- It is a generalization to N strings of the 3-Reggeon vertex Sciuto; Caneschi + Schwimmer + Veneziano

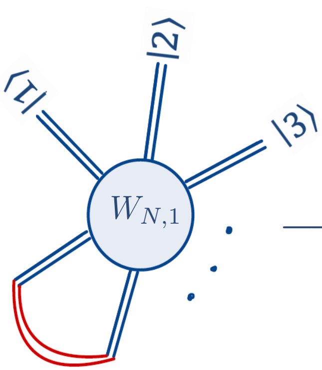
Operator formalism & sewing procedure

- The N-Reggeon vertex can be used to obtain the tree-level amplitude involving N strings by saturating it with N physical string states

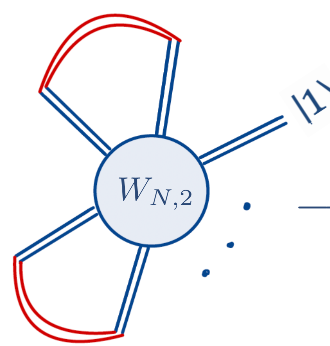


$$= W_N \prod_{i=1}^{N} \left|\text{phys}\right\rangle_i = A_{\text{tree}}(1,2,\cdots,N)$$

- But it can be used also to compute loop amplitudes. Indeed, since the N legs are off-shell one can sew them in pairs and obtain the multiloop N-Reggeon vertex!



$$\longrightarrow \; A_{1-\text{loop}}(1,2,\cdots,N) \quad ;$$



$$\longrightarrow \; A_{2-\text{loop}}(1,2,\cdots,N)$$

Operator formalism & sewing procedure

- This sewing procedure can be made very explicit and precise. But there is a big problem:

How to eliminate the unphysical states from the loops?

- In the old days, this problem was addressed and solved by inserting suitable projections in the propagators, which however make the entire construction quite involved.

- An elegant solution to this problem was obtained in the late 80's with the BRST formalism, introducing ghosts and anti-ghosts. The two basic ingredients are:

1. The BRST invariant 3-Reggeon vertex

Paolo + R. Nakayama + J.L. Petersen + S. Sciuto (1987)

2. The BRST invariant propagator

Paolo + M. Frau + AL + S. Sciuto (1987)

Operator formalism & sewing procedure

- The BRST invariant 3-Reggeon vertex is an extension with ghosts and anti-ghosts of the old Caneschi-Schwimmer-Veneziano (CSV) vertex:

CSV

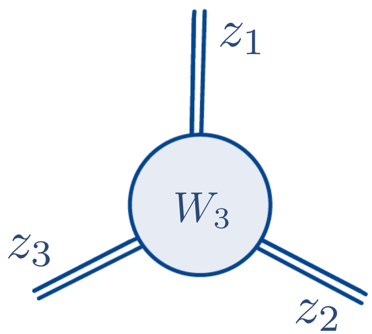
$$W_3 = \left(\prod_{i=1}^{3} {}_i\langle \Omega | \right) \exp\Big[-\frac{1}{2} \sum_{i \neq j = 1}^{3} \sum_{n,m=0}^{\infty} a_n^{(i)} D_{nm}(U_i V_j)\, a_m^{(j)} \Big] \delta(p_1 + p_2 + p_3) \times$$

Paolo et al

$$\times \exp\Big[-\frac{1}{2} \sum_{i \neq j = 1}^{3} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^{(i)} E_{nm}(U_i V_j)\, b_m^{(j)} \Big] \times \text{“anti-ghost } \delta\text{-functions”}$$

where $a_n^{(i)}$ are the orbital oscillators and $c_n^{(i)}$, $b_n^{(i)}$ are the BRST ghost and anti-ghost oscillators.

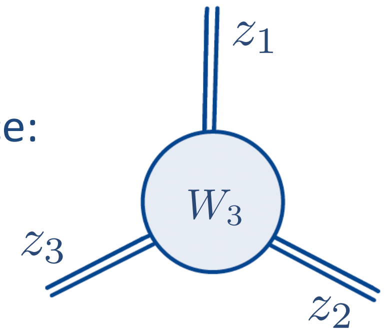
- The “bras” ${}_i\langle \Omega |$ are the SL(2) invariant vacua for the 3 strings



- The coefficients $D_{nm}(U_i V_j)$ and $E_{nm}(U_i V_j)$ are related to the Koba-Nielsen variables and have an interesting geometrical interpretation since they form a representation of the projective group with weight 0 and -1, respectively.

Operator formalism & sewing procedure

- The symbols U_i and V_i denote projective transformations that are related to the choice of local coordinates around the 3 punctures.



- A convenient choice is the symmetric one suggested by C. Lovelace:

$$U_i = \begin{pmatrix} z_{i-1} & z_i & z_{i+1} \\ 0 & \infty & 1 \end{pmatrix} \quad , \quad V_i = \begin{pmatrix} \infty & 0 & 1 \\ z_{i-1} & z_i & z_{i+1} \end{pmatrix}$$

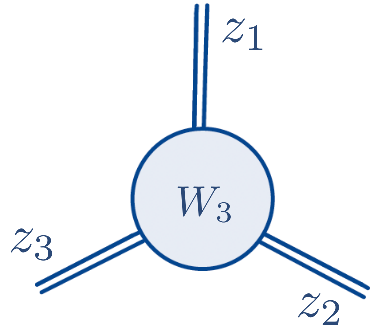
- Once the local coordinates are chosen, the coefficients $D_{nm}(U_i V_j)$ and $E_{nm}(U_i V_j)$ are given by

$$D_{nm}(\gamma(z)) = \frac{1}{m!} \frac{\partial^m}{\partial z^m} \left[\gamma(z)^n\right]\Big|_{z=0} \quad , \quad E_{nm}(\gamma(z)) = \frac{1}{(m+1)!} \frac{\partial^{m+1}}{\partial z^{m+1}} \left[\gamma(z)^{n+1}\right]\Big|_{z=0}$$

- Everything is very explicit!

Operator formalism & sewing procedure

- The 3-Reggeon vertex satisfies the following properties:



  - It is BRST invariant $W_3\,(Q_1 + Q_2 + Q_3) = 0$

  - It is cyclic symmetric $W_3(z_1, z_2, z_3) = W_3(z_2, z_3, z_1) = W_3(z_3, z_1, z_2)$

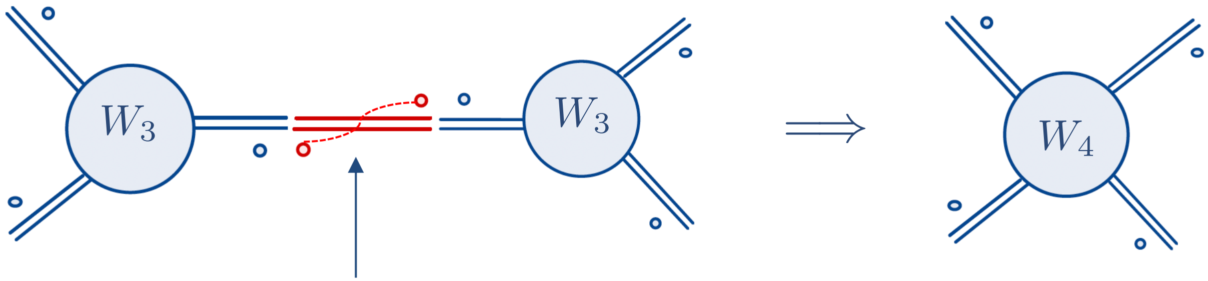
  - It reproduces the dual 3-point functions among 3 arbitrary physical states of the string spectrum at tree level

$$\boxed{W_3\,|\text{phys}\rangle_1\,|\text{phys}\rangle_2\,|\text{phys}\rangle_3 = A_{\text{tree}}(1, 2, 3)}$$

  - At tree level, the addition of ghosts and anti-ghosts is redundant, but it is important in the sewing procedure leading to the multiloop amplitudes.

Operator formalism & sewing procedure

- The second ingredient is the BRST invariant propagator. The basic idea is to sew together several 3-string vertices to obtain vertices with 4, 5, … legs. For example:



twisted BRST invariant propagator

Paolo et al

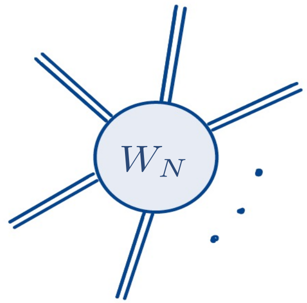
- The twisted BRST invariant propagator is

$$\boxed{\mathcal{P} = (b_0 - b_1) \int_0^1 \frac{dx}{x(1-x)} P(x)} \quad \text{where} \quad P(x) = \begin{pmatrix} 0 & \infty & 1 \\ x & 1 & 0 \end{pmatrix}$$

The "usual" BRST propagator is $\frac{b_0}{L_0} = b_0 \int_0^1 \frac{dx}{x} P(x) \quad \text{where} \quad P(x) = \begin{pmatrix} 0 & \infty & 1 \\ 0 & \infty & x \end{pmatrix}$

Operator formalism & sewing procedure

- By repeatedly sewing vertices with twisted BRST invariant propagators, we obtain the N-Reggeon vertex:



Paolo et al

$$W_N = \left(\frac{1}{dV_{abc}} \int \prod_{i=1}^{N} \frac{dz_i}{(z_{i+1} - z_i)} \; {}_i\langle \Omega | \right) \exp\left[- \frac{1}{2} \sum_{i \neq j = 1}^{N} \sum_{n,m=0}^{\infty} a_n^{(i)} D_{nm}(U_i V_j) \, a_m^{(j)} \right] \delta(p_1 + \cdots + p_N)$$

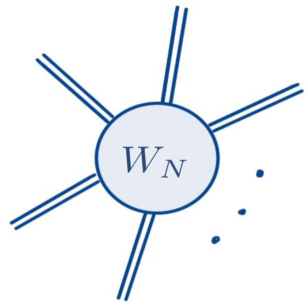
$$\times \exp\left[- \frac{1}{2} \sum_{i \neq j = 1}^{N} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^{(i)} E_{nm}(U_i V_j) \, b_m^{(j)} \right] \times \text{“anti-ghost } \delta\text{-functions”}$$

- Now the coefficients $D_{nm}(U_i V_j)$ and $E_{nm}(U_i V_j)$ are functions of the $(N-3)$ Koba-Nielsen variables that are not fixed, over which one has to integrate to obtain the tree-level N-string amplitude.

- Again, at tree-level the addition of the ghost part is redundant, but it becomes crucial for the multi-loop amplitudes!

Operator formalism & sewing procedure

- The N-Reggeon vertex:



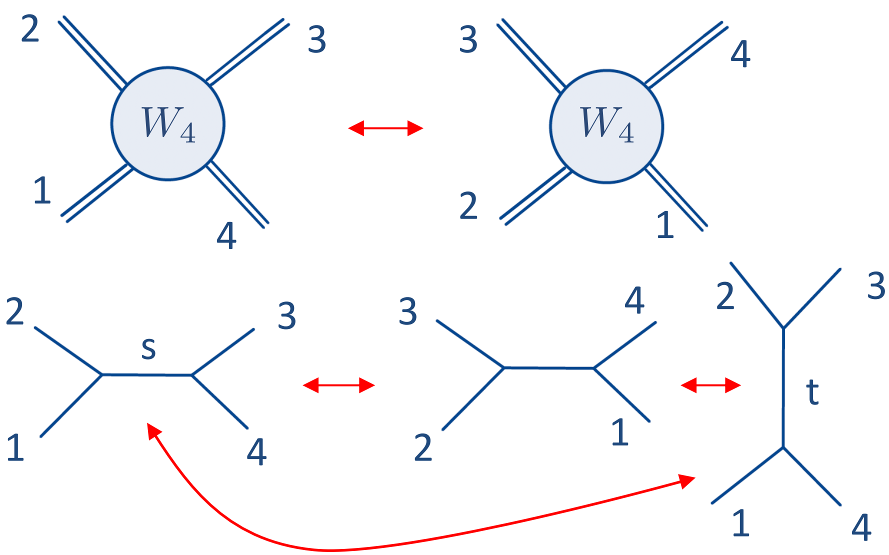
1. is BRST invariant

$$\boxed{W_N\,(Q_1 + Q_2 + \cdots + Q_N) = 0}$$

2. is cyclic symmetric; for example

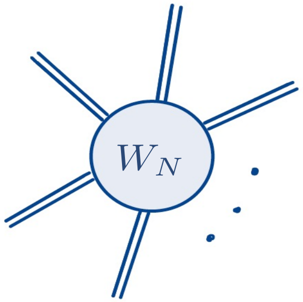
$$\boxed{W_4(1,2,3,4) = W_4(2,3,4,1)}$$

This property is crucial for the (s,t)-duality:



Operator formalism & sewing procedure

- The N-Reggeon vertex:



1. is BRST invariant $$W_N\left(Q_1+Q_2+\cdots+Q_N\right)=0$$

2. is cyclic symmetric; for example $$W_4(1,2,3,4)=W_4(2,3,4,1)$$

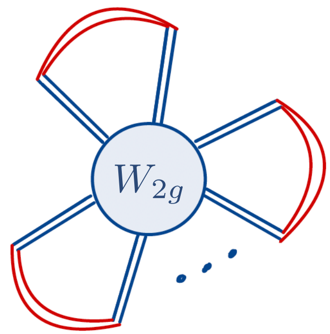
- The BRST invariant N-Reggeon vertex reproduces correctly the N-point amplitudes at tree level for arbitrary physical states

$$W_N\,|\mathrm{phys}\rangle_1\,|\mathrm{phys}\rangle_2\,\cdots|\mathrm{phys}\rangle_N=A_{\mathrm{tree}}(1,2,\cdots,N)$$

- Most importantly, it can be used to obtain multi-loop amplitudes!

Operator formalism & sewing procedure

- By pairwise sewing the legs of a 2g-Reggeon vertex with g twisted BRST invariant propagators, we obtain the g-loop string partition function:



$$\mathcal{Z}_g = \mathrm{Tr}\left[W_{2g} \times \prod_{\mu=1}^{g} \mathcal{P}_\mu \right]$$

Paolo et al

(also Petersen + Sidenius)

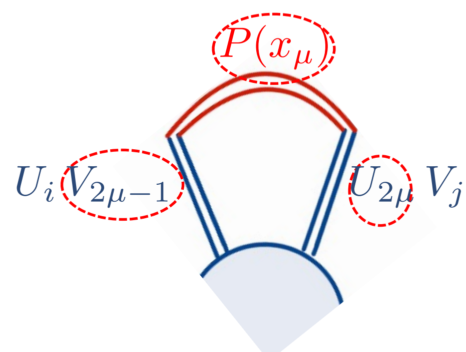
- In $\mathcal{Z}_g$ we have

from the 2g-Reggeon vertex

$$\left(\int \prod_{\mu=1}^{g} \frac{dx_\mu}{x_\mu (1 - x_\mu)} \right) \left(\frac{1}{dV_{abc}} \int \prod_{i=1}^{2g} \frac{dz_i}{(z_{i+1} - z_i)} \right) \longrightarrow \boxed{\text{(3g-3) integration variables}}$$

from the g-propagators

- In each sewing one naturally introduces g new projective transformations



$$\longrightarrow \boxed{S_\mu = V_{2\mu-1}\, P(x_\mu)\, U_{2\mu}}$$

Operator formalism & sewing procedure

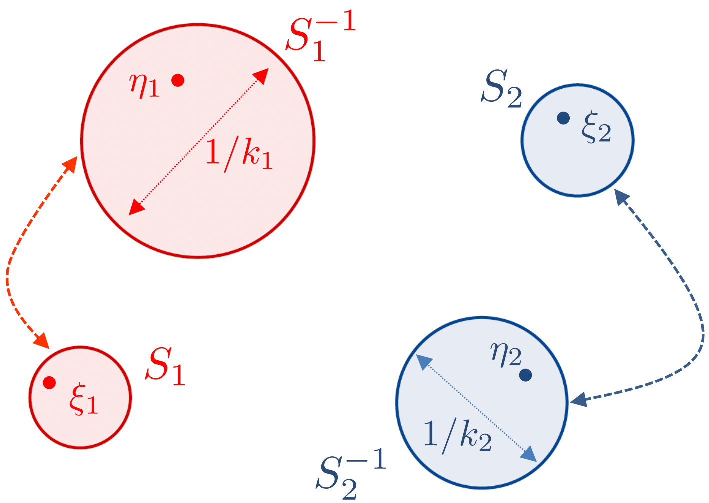
- The g projective transformations $S_\mu = V_{2\mu-1}\, P(x_\mu)\, U_{2\mu}$ generate the so-called Schottky group which can be used to describe a Riemann surface of genus g

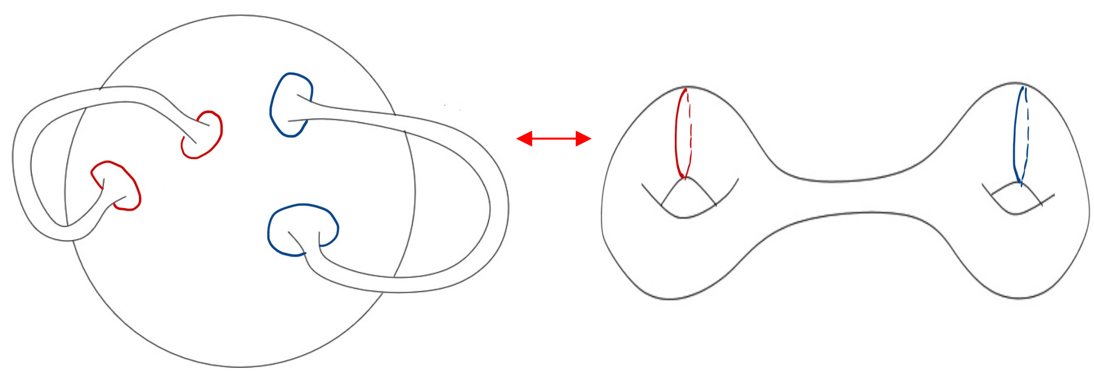
- Any projective transformation, and so also S_μ, can be brought to a canonical form

$$\frac{S_\mu(z)-\eta_\mu}{S_\mu(z)-\xi_\mu} = k_\mu\,\frac{z-\eta_\mu}{z-\xi_\mu} \quad , \qquad |k_\mu| \le 1$$

η_μ and ξ_μ are the attractive and repulsive fixed points, k_μ is the multiplier

- The projective transformations S_μ and S_μ^{-1} define two isometric circles on the complex plane whose boundaries can be identified to form a handle





In this way we build Riemann surfaces!

Operator formalism & sewing procedure

- All geometric objects on the Riemann surface can be explicitly written in this Schottky representation.

- For example, the period matrix is given by

$$2\pi \mathrm{i}\, \tau_{\mu\nu} = \delta_{\mu\nu} \log k_\mu - \sum_\alpha{}' \log \left[\frac{\eta_\nu - T_\alpha(\xi_\mu)}{\eta_\nu - T_\alpha(\eta_\mu)} \, \frac{\xi_\nu - T_\alpha(\eta_\mu)}{\xi_\nu - T_\alpha(\xi_\mu)} \right]$$

where the sum is over all elements of the Schottky group (with some restrictions)

- and the prime form is given by

$$E(z,w) = (z-w) \prod_\alpha{}' \left[\frac{z - T_\alpha(w)}{z - T_\alpha(z)} \, \frac{w - T_\alpha(z)}{w - T_\alpha(w)} \right]$$

where the product is over all elements of the Schottky group (with some restrictions)

- Using these explicit representations, one can check many geometric properties.

Operator formalism & sewing procedure

- Using the sewing procedure, we computed the g-loop string partition function

$$\mathcal{Z}_g = \mathrm{Tr}\left[W_{2g} \times \prod_{\mu=1}^{g} \mathcal{P}_\mu\right] = \cdots$$

Paolo et al

$$= \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} \left[dk_\mu \, d\xi_\mu \, d\eta_\mu \, \frac{(1-k_\mu)^2}{k_\mu^2(\eta_\mu - \xi_\mu)^2}\right] \det\left(\mathrm{Im}\,\tau\right)^{-D/2} \times$$

$$\times \prod_\alpha{}' \left[\prod_{n=1}^{\infty} (1-k_\alpha^n)^{-D} \prod_{n=2}^{\infty} (1-k_\alpha^n)^2\right]$$

from orbital degrees of freedom

from BRST ghosts and anti-ghosts

- This represents the measure of integration on the moduli space at genus g.

- Notice that the inclusion of ghosts and anti-ghosts does not lead to the simple rule $D \to D-2$, as one might have expected.

Operator formalism & sewing procedure

- Using the sewing procedure, we computed the g-loop string partition function

$$\mathcal{Z}_g = \mathrm{Tr}\left[W_{2g} \times \prod_{\mu=1}^{g} \mathcal{P}_\mu\right] = \cdots$$

Paolo et al

$$= \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} \left[dk_\mu \, d\xi_\mu \, d\eta_\mu \, \frac{(1-k_\mu)^2}{k_\mu^2(\eta_\mu - \xi_\mu)^2}\right] \det\left(\mathrm{Im}\,\tau\right)^{-D/2} \times$$

$$\times \prod_\alpha{}' \left[\prod_{n=1}^{\infty} (1-k_\alpha^n)^{-D} \prod_{n=2}^{\infty} (1-k_\alpha^n)^2\right]$$

$$= \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{g} \left[dk_\mu \, d\xi_\mu \, d\eta_\mu \, \frac{1}{k_\mu^2(\eta_\mu - \xi_\mu)^2}\right] \det\left(\mathrm{Im}\,\tau\right)^{-D/2} \times$$

$$\times \prod_\alpha{}' \left[\prod_{n=1}^{\infty} (1-k_\alpha^n)^{-(D-2)}\right] \frac{\prod_{\mu=1}^{g} (1-k_\mu)^2}{\prod_\alpha{}' (1-k_\alpha)^2}$$

This extra factor is crucial for modular invariance (checked numerically by Petersen et al)

proposed by Mandelstam in '85, but then corrected with a mysterious factor Γ in '86.

Operator formalism & sewing procedure

- This result was immediately generalized to include the emission of N arbitrary strings. In this way we obtained the N-Reggeon g-loop vertex which describes the overlap among N strings on a puctured Riemann surface of genus g

$$W_{N;g} \sim \exp\left\{-\sum_{i,j=1}^{N} \oint_{z_i} \frac{dz}{2\pi\mathrm{i}} \oint_{z_j} \frac{dw}{2\pi\mathrm{i}} \left[\frac{1}{2}\partial X(z)\, G^{(X)}(z,w)\, \partial X(w) + c(z)\, G^{(b,c)}(z,w)\, b(w)\right]\right\}$$

where $G^{(X)}(z,w)$ and $G^{(b,c)}(z,w)$ are the Greeen functions on the Riemann surface for the string coordinates and the ghost/anti-ghost system, respectively:

$$G^{(X)}(z,w) = \underbrace{X(z)\,X(w)} \sim \log(z-w) + \cdots$$

$$G^{(b,c)}(z,w) = \underbrace{b(z)\,c(w)} \sim \frac{1}{z-w} + \cdots$$

- This construction was readily extended to the bosonic closed string and to the fermionic string in the NS sector (→ super Schottky group with $(3g-3)$ bosonic moduli + $(2g-2)$ fermionic moduli)

Paolo et al

Operator formalism & sewing procedure

- In the R sector additional subtleties appear. They are related to the presence of fermionic zero-modes in the orbital part and of bosonic zero-modes in the super-ghost part, as well as of $\sqrt{z}$ branch-cut singularities.

- Despite these difficulties, the BRST invariant vertex for the emission of N bosonic and 2M fermionic strings was constructed by Paolo + Hornfeck + Masden + Roland and by Petersen + Sidenius + Tollsten.

- A multiloop fermionic vertex was formally worked out using the sewing procedure also on the fermionic legs by Petersen + Sidenius + Tollsten (up to the zero-mode contribution)

- The general structure of the contribution of the super-ghost system was fully understood and the super-ghost correlation functions were shown by Paolo to be fully equivalent to those derived by Verlinde + Verlinde with a path-integral approach.

However, a big problem remains unsolved: finding the fundamental domain at g loops and hence the range of integration over the moduli space.

Part 2

1995-1997

Gauge theory renormalizations from the open bosonic string

Paolo Di Vecchia [a,1], Alberto Lerda [b], Lorenzo Magnea [c], Raffaele Marotta [d,2]
[a] *NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[b] *Dipartimento di Fisica Teorica, Università di Salerno, I-84100 Salerno, Italy*
I.N.F.N., Sezione di Napoli, Italy
[c] *Dipartimento di Fisica Teorica, Università di Torino, Via P. Giuria 1, I-10125 Turin, Italy*
I.N.F.N., Sezione di Torino, Italy
[d] *Dipartimento di Scienze Fisiche, Università di Napoli, Mostra D'Oltremare, Pad. 19, I-80125 Napoli, Italy*
I.N.F.N., Sezione di Napoli, Italy

Received 19 March 1995
Editor: L. Alvarez-Gaumé

NUCLEAR PHYSICS B
N·H ELSEVIER
Nuclear Physics B 469 (1996) 235–286

String techniques for the calculation of renormalization constants in field theory

Paolo Di Vecchia [a,1], Lorenzo Magnea [a,2], Alberto Lerda [b,3], Rodolfo Russo [b], Raffaele Marotta [c]
[a] *NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[b] *Dipartimento di Scienze e Tecnologie Avanzate and Dipartimento di Fisica Teorica, Università di Torino, and INFN, Sezione di Torino, Via P.Giuria 1, I-10125 Turin, Italy*
[c] *Dipartimento di Scienze Fisiche, Università di Napoli, and INFN, Sezione di Napoli, Mostra D'Oltremare, Pad. 19, I-80125 Naples, Italy*

Received 9 February 1996; accepted 14 March 1996

N·H ELSEVIER
Nuclear Physics B (Proc. Suppl.) 49 (1996) 85–95
NUCLEAR PHYSICS B PROCEEDINGS SUPPLEMENTS

String-derived renormalization of Yang-Mills theory

Paolo Di Vecchia and Lorenzo Magnea[a*], Alberto Lerda and Rodolfo Russo[b], Raffaele Marotta[c]
[a]NORDITA
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
[b]Dipartimento di Fisica Teorica, Università di Torino
Via P.Giuria 1, I-10125 Turin, Italy
and I.N.F.N., Sezione di Torino
[c]Dipartimento di Scienze Fisiche, Università di Napoli
Mostra D'Oltremare, Pad. 19, I-80125 Napoli, Italy

Two-loop scalar diagrams from string theory

Paolo Di Vecchia [a,1], Lorenzo Magnea [a,2], Alberto Lerda [b,3], Raffaele Marotta [c,4], Rodolfo Russo [d],
[a] *NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
[b] *Dipartimento di Scienze e Tecnologie Avanzate and Dipartimento di Fisica Teorica, Università di Torino, Via P.Giuria 1, I-10125 Torino, Italy*
[c] *Dipartimento di Scienze Fisiche, Università di Napoli, Mostra D'Oltremare, Pad. 19, I-80125 Napoli, Italy*
[d] *Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy and I.N.F.N., Sezione di Torino*

Received 13 August 1996
Editor: L. Alvarez-Gaumé

Part 2

- At each order of string perturbation theory, one does not get the large proliferation of diagrams characteristic of field theories

- It is well known that in the limit of infinite string tension ($\alpha' \to 0$) string theories reduce to non-abelian gauge theories (unified with gravity) order by order in perturbation theory

- This means, in particular, that in the limit $\alpha' \to 0$ one must reproduce, order by order, S-matrix elements and ultraviolet divergences of perturbative non-abelian gauge theories

> Therefore, string theory can be an efficient conceptual and computational tool in different areas of perturbative field theory

(see also Bern + Dixon + Kosower)

- We have applied these ideas to study gluon scattering in Yang-Mills theories.

Paolo et al

Gluon amplitudes

- Using the formalism of the N-Reggeon g-loop vertex we can easily obtain the scattering amplitude for N gluons at g-loops

polarizations

$$A_{N;g}(p_1, \cdots, p_N) = W_{N;g} \, |\varepsilon_1, p_1\rangle_1 \cdots |\varepsilon_N, p_N\rangle_N$$

- More explicitly, we have

$$A_{N;g}(p_1, \cdots, p_N) = \mathcal{N}_{N;g} \int [dm]_{N;g} \left\{ \prod_{i<j=1}^{N} \left[\frac{\exp\left(G^{(X)}(z_i, z_j)\right)}{\sqrt{V_i'(0) V_j'(0)}} \right]^{2\alpha' p_i \cdot p_j} \right.$$
$$\left. \exp\left[\sum_{i\neq j=1}^{N} \left(\sqrt{2\alpha'} p_j \cdot \varepsilon_i \, \partial_{z_i} G^{(X)}(z_i, z_j) + \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \, \partial_{z_i} \partial_{z_j} G^{(X)}(z_i, z_j) \right) \right] \right\}_{\text{m.l.}}$$

where m.l. stands for multilinear in the ε_i's and the integration measure is given by

$$[dm]_{N;g} = \left(\prod_{i=1}^{N} dz_i \right) \mathcal{Z}_g$$

with $\mathcal{Z}_g$ given by the sewing procedure. Also the normalization factor is completely fixed.

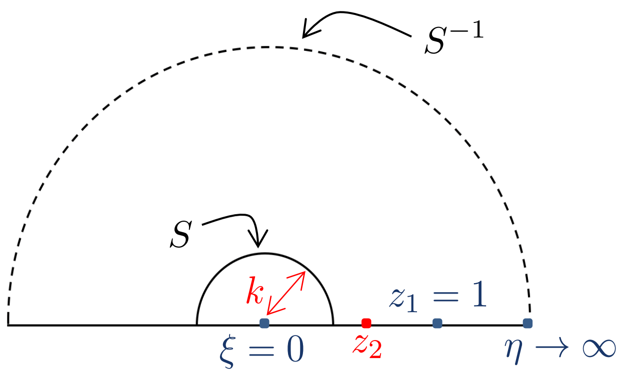
In the field theory limit $\alpha' \to 0$, there are remarkable simplifications.

Gluon amplitudes at 1 loop

- At 1 loop with 2 punctures, the string vertex depends on 5 parameters, 3 of which can be fixed. We thus remain with 2 integration variables

2 punctures: $z_1\ ,\ \ z_2$
1 Schottky generator: $k\ ,\ \ \xi\ ,\ \ \eta$

$$\begin{pmatrix} z_1 & z_2 & \xi & \eta & k \\ \downarrow & & \downarrow & \downarrow & \\ 1 & z_2 & 0 & \infty & k \end{pmatrix}$$



- It is convenient to change variables and define

$$z_2 = \mathrm{e}^{-2\nu}\ ,\ \ k = \mathrm{e}^{-2\tau} \quad (0 \le \nu \le \tau \le \infty)$$

Gluon amplitudes at 1 loop

- At 1 loop with 2 punctures, the string vertex depends on 5 parameters, 3 of which can be fixed. We thus remain with 2 integration variables

2 punctures: $z_1\ ,\ \ z_2$
1 Schottky generator: $k\ ,\ \ \xi\ ,\ \ \eta$

$$\begin{pmatrix} z_1 & z_2 & \xi & \eta & k \\ \downarrow & & \downarrow & \downarrow & \\ 1 & z_2 & 0 & \infty & k \end{pmatrix}$$

- In the new variables, the 2-gluon amplitude at 1 loop is

normalization of the Reggeon vertex

$$A_{2;1}(p_1,p_2) = W_{2;1}|\varepsilon_1,p_1\rangle|\varepsilon_2,p_2\rangle = \cdots = N\,\mathrm{tr}(T^{a_1}T^{a_2})\,\frac{g^2}{(4\pi)^{d/2}}\,(2\alpha')^{2-d/2}$$

$$\times\int_0^\infty d\tau\int_0^\tau d\nu\left[\mathrm{e}^{2\tau}\tau^{-d/2}\prod_{n=1}^{\infty}\left(1-\mathrm{e}^{-2n\tau}\right)^{2-d}\left(-\frac{\varepsilon_1\cdot\varepsilon_2}{2\alpha'}\,\mathrm{e}^{2\alpha' p_1\cdot p_2\,G(\nu)}\,\partial_\nu^2 G(\nu)\right)\right]$$

- The variables τ and ν can be interpreted as the proper-time Schwinger parameters t_1 and t_2 for the Feynman diagrams contributing to the 2-point function

$$\boxed{t_1 = 2\alpha'\tau\ ,\ \ t_2 = 2\alpha'\nu}$$

(Bern + Kosower)

- In the field theory limit $\alpha' \to 0$ the Schwinger parameters remain finite. This means that

$$\boxed{\alpha' \to 0 \quad \Rightarrow \quad \tau \to \infty\ ,\quad \nu \to \infty}$$

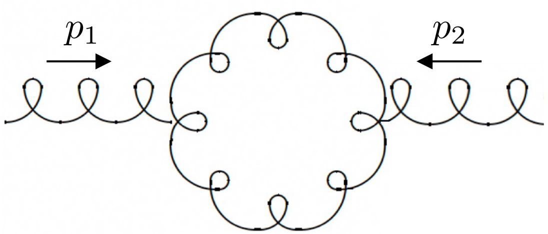
Gluon amplitudes at 1 loop

- So only the corner $\tau \to \infty$, $\nu \to \infty$ of the moduli space contributes to the amplitude in the field theory limit $\alpha' \to 0$

- Discarding the contribution due to the tachyons, the 2-gluon amplitude becomes

$$\boxed{A_{2;1}(p_1,p_2) = -N\delta^{a_1a_2}\frac{g^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{-p_1\cdot p_2}\right)^{\epsilon}\Gamma(\epsilon)\frac{11-7\epsilon}{3-2\epsilon}\,\frac{\Gamma(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,\varepsilon_1\cdot\varepsilon_2\,p_1\cdot p_2}$$

where $d = 4 - 2\epsilon$ and $g \to g\,\mu^{\epsilon}$



We interpret d as the number of uncompactified dimensions, assuming that there are $d' = 26 - d$ compactified dimensions which, in the field theory limit, behave just as d'scalars coupled to the gauge field (which we can safely ignore)

- The amplitude exactly agrees with the gluon polarization, computed with the background field method, in Feynman gauge, with dimensional regularization. The divergence in $d = 4$ can be removed with a standard wave-function renormalization

$$\boxed{Z_A = 1 + \frac{g^2 N}{(4\pi)^2}\frac{11}{3}\frac{1}{\epsilon}}$$

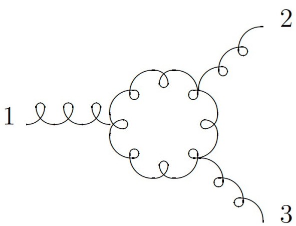
Gluon amplitudes at 1 loop

- To check the consistency of the procedure and verify that gauge invariance is preserved, we computed also the 3- and 4-gluon amplitudes at 1 loop.

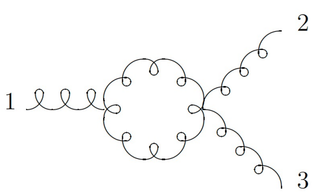
- In the 3-gluon amplitudes, one has to integrate with the string measure over the following moduli

$$k = \mathrm{e}^{-2\tau}\ ,\ z_2 = \mathrm{e}^{-2\nu_2}\ ,\ z_3 = \mathrm{e}^{-2\nu_3}$$

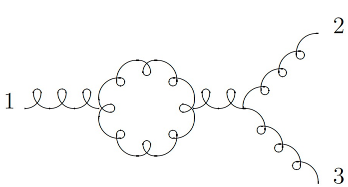
- In the field theory limit $\alpha' \to 0$, we have 3 corners of the moduli space that contribute



$$\begin{cases} \tau \to \infty \\ \nu_2 \to \infty \quad \text{with} \quad \dfrac{\nu_2 - \nu_3}{\tau} \sim O(1) \\ \nu_3 \to \infty \end{cases}$$



$$\begin{cases} \tau \to \infty \\ \nu_2 \to \infty \quad \text{with} \quad \dfrac{\nu_2 - \nu_3}{\tau} \sim O(\tau^{-1}) \\ \nu_3 \to \infty \end{cases}$$



$$\begin{cases} \tau \to \infty \\ \nu_2 \to \infty \quad \text{with} \quad \dfrac{\nu_2 - \nu_3}{\tau} \sim O(\tau^{-2}) \\ \nu_3 \to \infty \end{cases}$$

Adding all 3 contributions leads to

$$\boxed{A_{3;1}^{\mathrm{div}} = \frac{2g^2 N}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A_{3;0}}$$

$$= \left(Z_3^{-1} Z_A^3 - 1\right) A_{3;0}$$

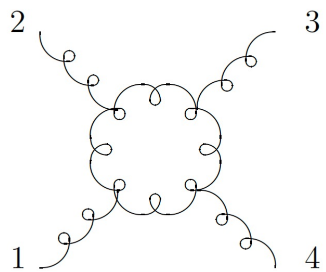
$$\Downarrow$$

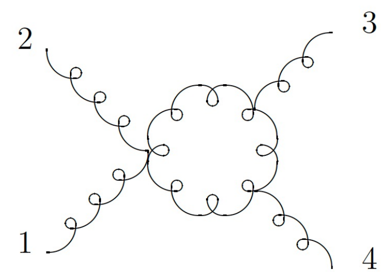
$$\boxed{Z_3 = 1 + \frac{g^2 N}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} = Z_A}$$

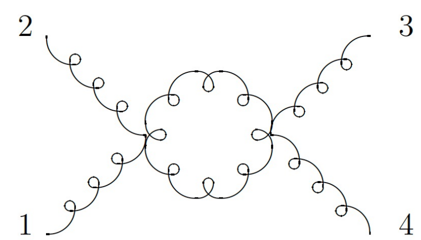
in agreement with the background field method

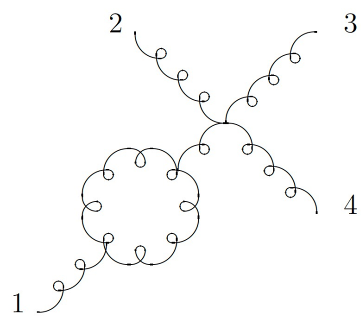
Gluon amplitudes at 1 loop

- The 4-gluon amplitude can be similarly discussed; in this case we have 4 types of contributions corresponding to 4 different corners in the moduli space:









$$\boxed{A_{4;1}^{\text{div}} = \frac{3g^2 N}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} A_{4;0}}$$

$$= \left(Z_4^{-1} Z_A^4 - 1\right) A_{4;0}$$

$$\Rightarrow \quad \boxed{Z_4 = 1 + \frac{g^2 N}{(4\pi)^2} \frac{11}{3} \frac{1}{\epsilon} = Z_A = Z_3}$$

- The Ward identity of the background field method $Z_A = Z_3 = Z_4$ is satisfied!

Scalar amplitudes at 2 loops

- The above analysis has been generalized to scalar amplitudes at 2 loops. Paolo et al
- By shifthing the entire string spectrum (*i.e.* changing the value of the intercept) it is possible to change the tachyon into a scalar with $m^2 > 0$ and obtain consistent results in the limit $\alpha' \to 0$.

J. Scherck

- In this way we can study a massive φ^3 field theory using the bosonic string.
- As an illustration, we consider the vacuum diagrams at 2 loops. The string measure at 2 loops is

$$\mathcal{Z}_2 = \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{2} \left[dk_\mu \, d\xi_\mu \, d\eta_\mu \, \frac{\cancelto{1}{(1-k_\mu)^2}}{k_\mu^2 (\eta_\mu - \xi_\mu)^2} \right] \det \left(\mathrm{Im}\, \tau \right)^{-D/2} \times$$

$$\times \cancelto{1}{\prod_\alpha{}' \left[\prod_{n=1}^{\infty} (1-k_\alpha^n)^{-D} \prod_{n=2}^{\infty} (1-k_\alpha^n)^2 \right]}$$

- In the field theory limit, the Schottky multipliers $k_\mu \to 0$, and thus we have

$$\mathcal{Z}_2 \underset{\alpha' \to 0}{\sim} \int \frac{1}{dV_{abc}} \prod_{\mu=1}^{2} \left[\frac{dk_\mu}{k_\mu^2} \, \frac{d\xi_\mu \, d\eta_\mu}{(\eta_\mu - \xi_\mu)^2} \right] \det \left(\mathrm{Im}\, \tau \right)^{-D/2}$$

Scalar amplitudes at 2 loops

- Fixing $\xi_1 = 1$, $\xi_2 = 0$, $\eta_2 \to \infty$, we remain with

$$\mathcal{Z}_2 = \frac{N^3 g^2 (2\alpha')^{3-d}}{256(4\pi)^d} \int \frac{dk_1}{k_1^2} \int \frac{dk_2}{k_2^2} \int \frac{d\eta_1}{(1-\eta_1)^2} \left[\frac{1}{4} \left(\log k_1 \log k_2 - \log^2 \eta_1 \right) \right]^{-d/2}$$

normalization of the Reggeon vertex

period matrix

- Shifting the string spectrum à la Scherck, one finds

$$\mathcal{Z}_2 = \frac{N^3 g^2 (2\alpha')^{3-d}}{256(4\pi)^d} \int \frac{dk_1}{k_1} \int \frac{dk_2}{k_2} \int \frac{d\eta_1}{(1-\eta_1)} \mathrm{e}^{-m^2\alpha' [\log k_1 + \log k_2 + \log(1-\eta_1)]} \times \left[\frac{1}{4} \left(\log k_1 \log k_2 - \log^2 \eta_1 \right) \right]^{-d/2}$$

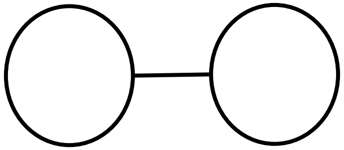
- There are 2 corners in the moduli space that contribute to the amplitude in the field theory limit:

$$i) \quad \begin{cases} k_1 \to 0 \\ k_2 \to 0 \\ \eta_1 \to 1 \end{cases} \qquad ii) \quad \begin{cases} k_1 \to 0 \\ k_2 \to 0 \\ \eta_1 \to 0 \end{cases}$$

Scalar amplitudes at 2 loops

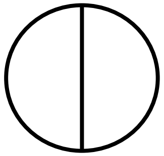
- In region *i)* after introducing the Schwinger proper times, we get

$$\mathcal{Z}_2\Big|_{i)} = \frac{N^3 g^2}{32(4\pi)^d} \frac{1}{2!} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^\infty dt_1 \, \mathrm{e}^{-m^2(t_1+t_2+t_3)} \, (t_1 t_2)^{-d/2}$$



- In region *ii)* after introducing the Schwinger proper times, we get

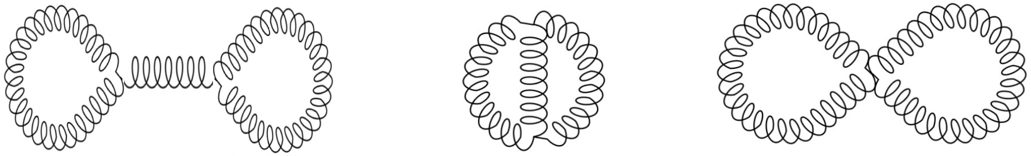
$$\mathcal{Z}_2\Big|_{ii)} = \frac{N^3 g^2}{32(4\pi)^d} \frac{1}{3!} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^\infty dt_1 \, \mathrm{e}^{-m^2(t_1+t_2+t_3)} \, (t_1 t_2 + t_2 t_3 + t_3 t_1)^{-d/2}$$



- A single string amplitude has generated all vacuum diagrams of the ϕ^3 theory at 2 loops, including the correct simmetry factors! Nice connection with the world-line formalism

Schubert et al

- This 2-loop analysis has been later extended to gluon amplitudes by L. Magnea + R. Russo + S. Playle + S. Sciuto



Concluding remarks

- The operator formalism is very concrete and explicit

- Many features of perturbative string theory can be explored in this way:

geometric structure of scattering amplitudes

field theory limit

- It also provides a solid basis for studying some non-perturbative aspects of string theory

(see Marialuisa's talk)

- I was extremely fortunate to have had the opportunity to learn these things by working with Paolo

Grazie Paolo!

not only for the many things you have taught me, but also for the way you have taught them to me and for your friendship over all these years!

Happy Birthday!!!